Math 551 3/16

HW #4 due 3/18

Newton's Method

Follow the tangent line!

$y = f(x)$ and $f'(x)$ exists!

$f(\alpha) = 0$

Given x_0:

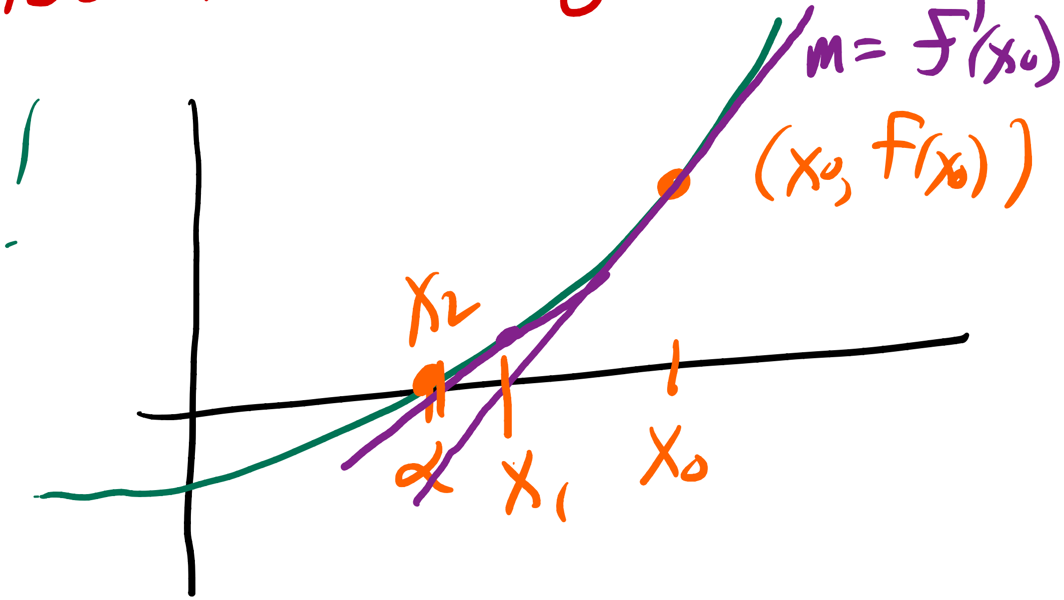
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= g(x_n)$$

where

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Iteration Function



Tangent Line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

x_1?

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Note: ① Newton's Mthd is a Fixed-pt mthd

② Requires $\underline{2}$ function evals per step

Recall Bisection is only $\underline{1}$.

(order of convergence Bisection $= 1$, rate $c = \frac{1}{2}$)

③ The method does not always converge, but when it does, the convergence is generally rapid!

④ Stopping Criteria:

(a) $|f(x_n)| < \varepsilon$

(b) $|x_{n+1} - x_n| < \varepsilon$

(c) set $n = n_{max}$

<u>Thm</u>: (Convergence of Newton's mthd)

Suppose $f \in C^2([a,b])$, $f(\alpha)=0$ for $\alpha \in [a,b]$, and α is a SIMPLE ROOT, i.e. $f'(\alpha) \neq 0$.

Then for x_0 sufficiently close to α, Newton's method converges to α.

<u>pf</u>: The iteration function $g(x) \in C^1([a,b])$ where $g(x) = x - \dfrac{f(x)}{f'(x)}$. Note

$$g'(x) = 1 - \frac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{[f'(x)]^2}$$

Then

$$g'(\alpha) = 1 - \frac{[f'(\alpha)]^2 - f(\alpha) f''(\alpha)}{[f'(\alpha)]^2}$$

$$= 1 - \frac{[f'(\alpha)]^2}{[f'(\alpha)]^2} = 0 \text{ since } f'(\alpha) \neq 0$$

So $|g'(\alpha)| = 0 < 1$ and we showed previously that there is a $\delta > 0$ such that if $x_0 \in (\alpha - \delta, \alpha + \delta)$ then

$$\lim_{n \to \infty} x_n = \alpha \qquad \#$$

Thm: (Order of Convergence of Newton's Method)
Under the same assumptions as the previous,
Newton's method converges with order
at least $p = 2$.

pf: Since $g'(\alpha) = 0$, by a previous
theorem the order of convergence is

$$p \geq 2.$$

#

Ex: $F(x) = x^2 - 3 = 0, \quad \alpha = \sqrt{3}$

simple root α $\quad F(x)$ $\quad \boxed{g'(\alpha) = 1 - \frac{1}{s}}$

Defn: α is a simple root, or a root of order $\underline{1}$, if $f(\alpha) = 0$ but $f'(\alpha) \neq 0$.

Note: If α is a root of order s of $f(x)$ then $F(x) = (x-\alpha)^s \cdot g(x)$

for some $g(x)$. $\qquad g(x) = \dfrac{F(x)}{(x-\alpha)^s}$