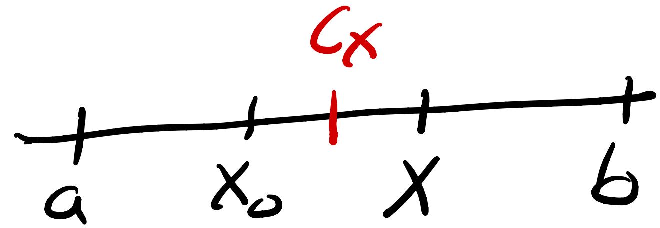
Math 551 2/9/21

Today:

1. Taylor's Thm ("Fundamental" Thm of Numerical Analysis)
2. Derive a Finite Difference (FD) approximation to $f'(x_0)$, with an Error Term!
3. MATLAB code to implement our method from 2.

Taylor's Thm: $f \in C^{n+1}([a,b])$ for some $n \geq 0$, and let $x_0, x \in [a,b]$. Then there exists c_x between x_0 and x such that



$$f(x) = \underbrace{P_n(x)}_{\text{n}^{\text{th}}\text{ degree Taylor Poly}} + \underbrace{R_n(x)}_{\text{Remainder or Error Term}}$$

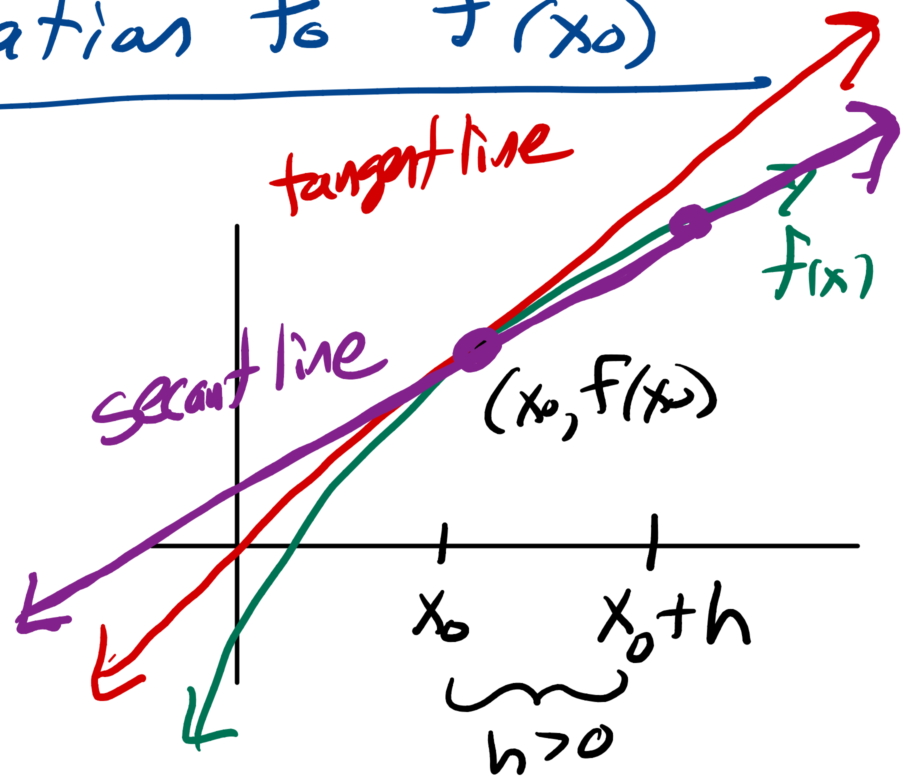
$$= \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k} + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

$$P_n(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x-x_0)^1 + \frac{f^{(11)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Poly of degree <u>AT MOST</u> n.

Finite Difference Approximation to $f'(x_0)$

$y = f(x)$, x_0



$m_{tan} \approx m_{sec}$

$$\boxed{f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Definition of the Derivative

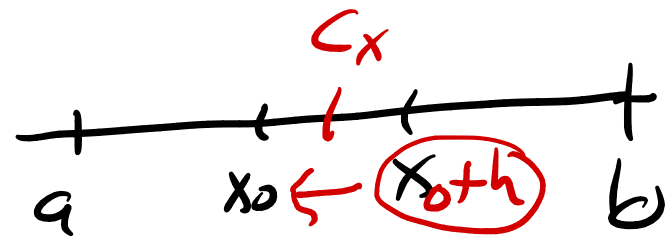
Approximation is then:

$$\boxed{f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h} \quad \text{for } h \text{ "small"}}$$

Q: How GOOD/ACCURATE is this approximation?

Derivation of FD Approximation to $f'(x_0)$

Taylor's Thm with $n=1$



$h \to 0$

$c_x \to x_0$

$$\underset{x_0+h}{f(x)} = P_1(x) + R_1(x)$$

$$= \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{P_1(x)} + \underbrace{\frac{f''(c_x)}{2!}(x-x_0)^2}_{R_1(x)}$$

$$a\,[\;f(x_0) = f(x_0)\;]$$

$$b\,[\;f(x_0+h) = f(x_0) + f'(x_0)\,h + \frac{f''(c_x)}{2!}h^2\;]$$

$+$

$$a\,f(x_0) + b\,f(x_0+h) = \underbrace{(a+b)}_{0} f(x_0) + \underbrace{b}_{1}h\,f'(x_0) + \frac{bh^2}{2} f''(c_x)$$

$$\frac{\frac{1}{h}\cdot h^2}{2} = \frac{h}{2}$$

Choose a, b

$$\Rightarrow \left.\begin{aligned} a + b &= 0 \\ bh &= 1 \end{aligned}\right\} \quad \begin{aligned} a &= -1/h \\ b &= 1/h \end{aligned}$$

Method of Undetermined Coefficients

$$-\frac{1}{h} f(x_0) + \frac{1}{h} f(x_0+h) = f'(x_0) + \frac{h}{2} f''(c_x)$$

or
$$f'(x_0) = \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\text{FD Approximation}} \underbrace{- \frac{h}{2} f''(c_x)}_{\text{Error}}$$

1st order Approximation

or

$$\underbrace{\boxed{E(h) = -\frac{h}{2} f''(c_x)}}_{\text{Behaves as } C \cdot h^1}$$

$$O(h):\ (h)^1 \to \left(\frac{h}{2}\right)^1 = \frac{1}{2} h$$

$$O(h^2):\ (h)^2 \to \left(\frac{h}{2}\right)^2 = \frac{1}{4} h^2$$

Ex: $f(x) = e^x \quad x_0 = 1$

$$f'(1) \approx \frac{f(1+h) - f(1)}{h}$$

or

$$f'(1) \approx \frac{e^{1+h} - e^1}{h}$$

$$\begin{array}{c} f'(1) \\ \parallel \\ e^1 \end{array}$$