Math 551 2/4/21

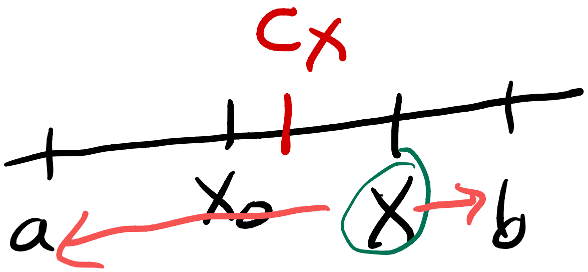
To Do:

1. Install MATLAB (see link on syllabus)
2. Install $\boxed{\text{Latex}}$ or XeTex
3. Take a look at HW #1

Thm: (Taylor's Theorem with Remainder)

Let $f \in C^{n+1}([a,b])$ ($f, f', f'', \dots, f^{n+1}$ are cont. on $[a,b]$)

for some $n \ge 0$ and let $x, x_0 \in [a,b]$



Then there exist c_x between x and x_0 such that

$$f(x) = \underbrace{P_n(x)}_{\text{nth degree Taylor Poly}} + \underbrace{\boxed{R_n(x)}}_{\text{Remainder or Error Term}}$$

$\rightarrow$ accounts for everything that truncated

$$= \boxed{\sum_{k=0}^{n} \boxed{\frac{f^{(k)}(x_0)}{k!}} (x-x_0)^k} + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

Note:

(3) $\left|f(x) - P_n(x)\right|_{[a,b]} \leq \left|R_n(x)\right|_{[a,b]}$

(1) Letting $n \to \infty$

$P_n(x)$ is just the Taylor Series

(2) $P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$P_n'(x) = 0 + \frac{f'(x_0)}{1!} \cdot 1\,(x-x_0)^0 + \frac{f''(x_0)}{2!} \cdot 2\,(x-x_0)^1 + \cdots$

$P_n(x_0) = f(x_0)$

$P_n'(x_0) = f'(x_0)$

$P_n''(x_0) = f''(x_0)$

Fact: $P_n(x)$ is the UNIQUE Poly of degree $\leq n$ such that

$$P_n^{(k)}(x_0) = f^{(k)}(x_0)$$

$\underline{\text{Ex}}$: $f(x) = \cos x$, $\boxed{x_0 = 0}$ Find $P_0(x)$ through $\underline{P_3}(x)$

$f(x) = \cos x$ $\quad f'(x) = -\sin x$ $\quad f''(x) = -\cos x$ $\quad f^{(3)}(x) = \sin x$

$f(0) = 1$ $\quad f'(0) = 0$ $\quad f''(0) = -1$ $\quad f^{(3)}(0) = 0$

$$P_0(x) \overset{\text{defn}}{=} \sum_{k=0}^{0} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \frac{f^{(0)}(0)}{0!}(x-0)^0 = \frac{1}{1} \cdot 1 = 1$$

$$P_1(x) \overset{\text{defn}}{=} \sum_{k=0}^{1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \boxed{\frac{f^{(0)}(0)}{0!} x^0} + \frac{f^{(1)}(0)}{1!} x^1$$

$$= P_0(x) + \frac{0}{1} x^1 = 1 + 0 = 1$$

$$P_2(x) \overset{\text{defn}}{=} \sum_{k=0}^{2} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = P_1(x) + \frac{f^{(2)}(0)}{2!}(x-0)^2$$

$$= 1 - \frac{1}{2} x^2$$

$\underline{\text{Note}}$: $P_3(x) = P_2(x)$ since $f^{(3)}(0) = 0$

$P_0(x) = 1$

$P_1(x) = 1$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_3(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_5(x) = P_4(x)$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6$

$\cos(-x) = \cos(x)$

Recall the Taylor Series

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(-x)^{2k}}{(2k)!}$$

- only Even powers of x

$\Rightarrow \cos x$ is an EVEN

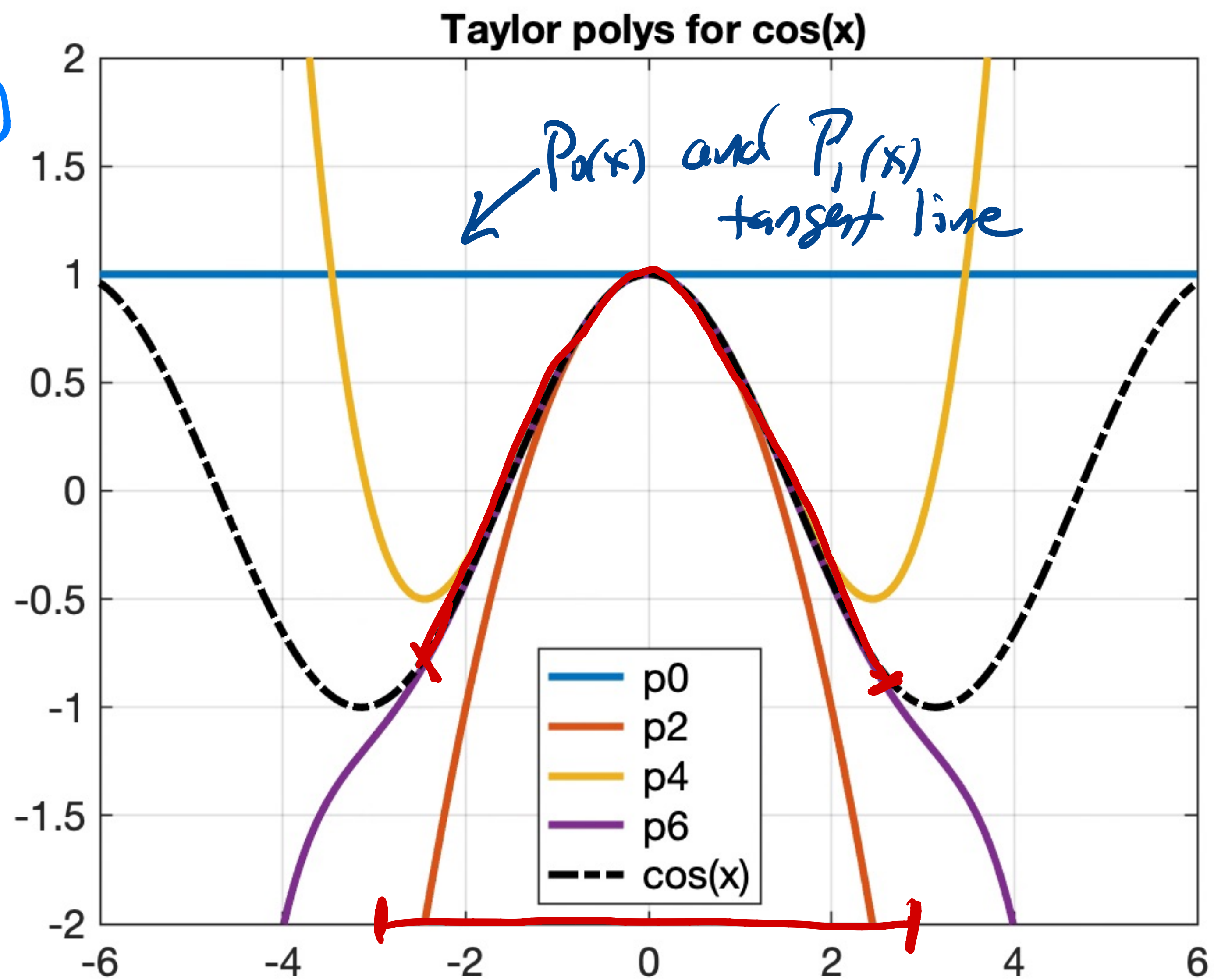
$$-\frac{d}{dx}(\cos x) = \sin x$$

Even	Odd
$f(-x) = f(x)$	$f(-x) = -f(x)$

$P_0(0) = 1 = \cos(0)$

$P_2(0) = 1 = \cos(0)$

$P_n(0) = \cos(0)$



Ex: $f(x) = e^x \quad x_0 = 0 \Rightarrow f^{(k)}(0) = 1$

Find $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$

ans: $P_0(x) = f(x_0) = 1$

$$P_1(x) = P_0(x) + \frac{f^{(1)}(0)}{1!} x$$

$$= 1 + x$$

$$P_2(x) = P_1(x) + \frac{f^{(2)}(0)}{2!} x^2$$

$$= 1 + x + \frac{1}{2} x^2$$

$\vdots$

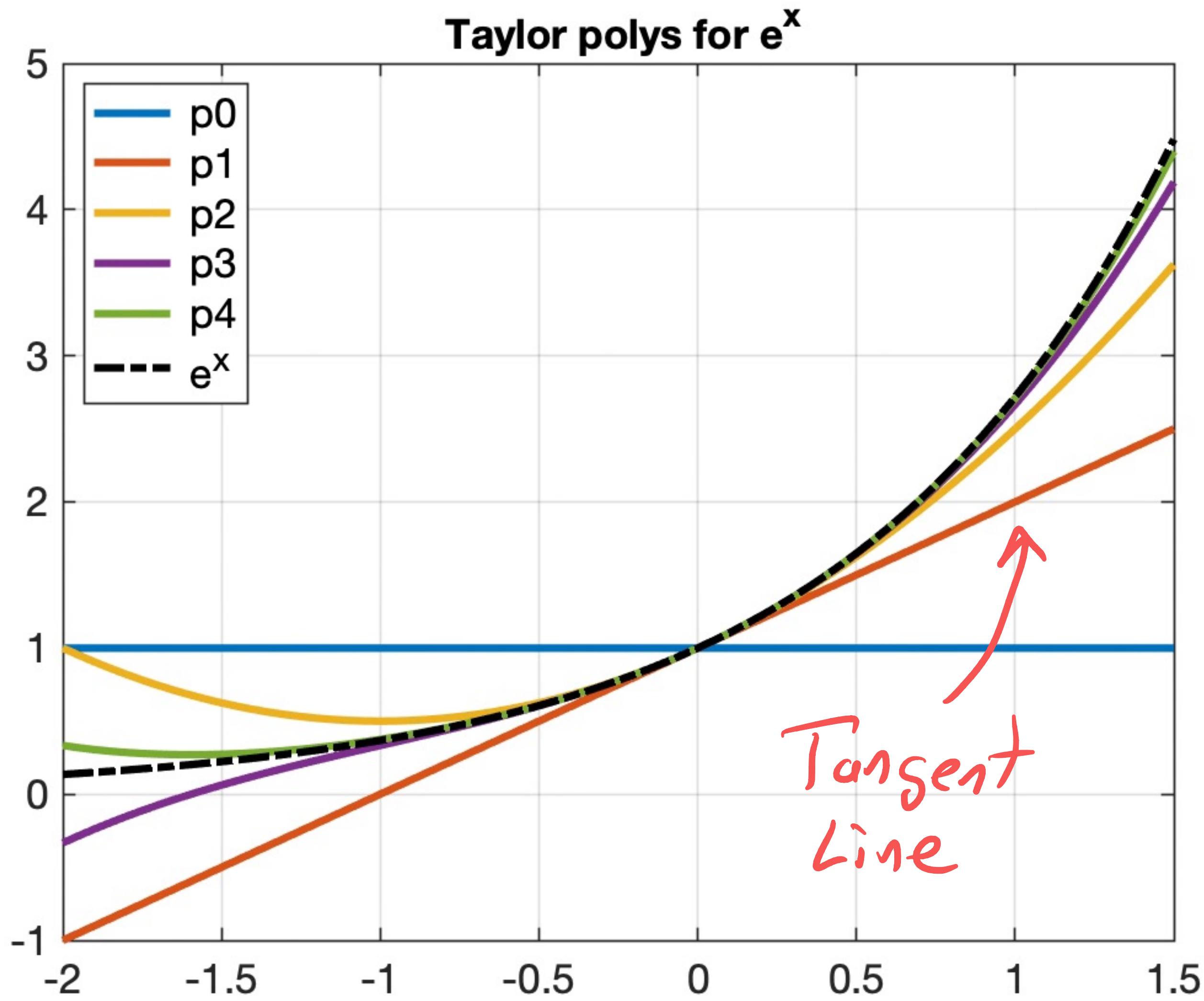
$$P_n(x) = 1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n$$

Note: Taylor Series of e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So $P_n(x)$ is just the Taylor Series, truncated at the n^{th} term

We see that $P_n(x)$ is a better and better approx. over a larger and larger interval around $x_0 = 0$ as $n \to \infty$.



Fun with Taylor Series (∞-degree poly)

$$\frac{d}{dx}\left(e^x\right) \stackrel{?}{=} e^x \qquad e^{x+h} = e^x \cdot e^h$$

$$\frac{d}{dx}\left(e^x\right) \stackrel{\text{defn}}{=} \lim_{h\to 0} \frac{e^{(x+h)} - e^x}{h} = \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= e^x \boxed{\lim_{h\to 0} \frac{e^h - 1}{h}} = 1$$

Differentiate TERM by Term (not always!)

Alternative:

$$\left(e^x\right)' = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right)$$

$$= 0 + 1 + \frac{1}{2}\cdot 2x^1 + \frac{1}{6}3x^2 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = e^x$$