Math 551 2/25

HW 2

- Due by 9PM today in Moodle
- Due by 9PM TOMORROW in Gradescope

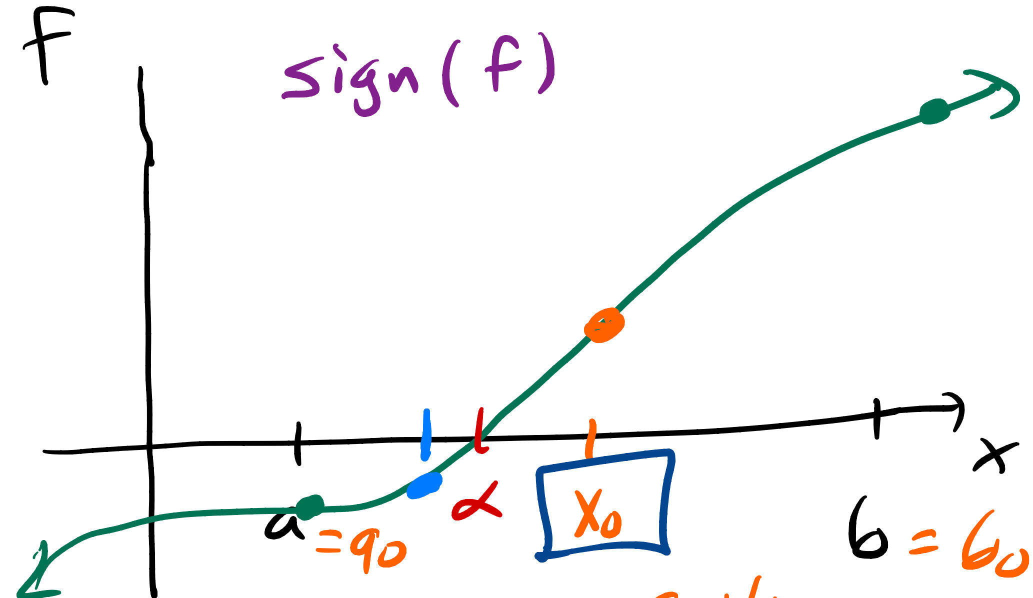
HW 3

- Due 3/4

Bisection:

1. $f \in C([a,b])$
2. $f(a) \cdot f(b) < 0$

By IVT $f(\alpha) = 0$
where $\alpha \in (a,b)$



$$x_0 = \frac{a_0 + b_0}{2}$$

$n=0$: a_0 —— α —— b_0

$\boxed{x_1} = (a_1 + b_1)/2$

$n=1$: a_1 —— α —— $b_1 \; (= x_0)$

$n=2$: $a_2 \; (= x_1)$ —— α —— $b_2 \; (= b_1)$

So at each step
$\alpha \in [a_n, b_n]$
and
$$|b_n - a_n| = \frac{1}{2} |b_{n-1} - a_{n-1}|$$

Ex: $f(x) = x^2 - 3$, $[a,b] = [1,2]$, $\alpha = \sqrt{3}$

$\sqrt{3} = 1.73205\ldots$

not monotone!

n	a_n	b_n	x_n	$f(x_n)$	$\lvert\alpha - x_n\rvert$
0	1^-	2^+	1.5^-	-0.75	0.2321
1	1.5^-	2^+	1.75^+	0.0625	0.0179
2	1.5^-	1.75^+	1.625^-	-0.3594	0.1071
3	1.625^-	1.75^+	1.6875^-	-0.1523	0.0446
4	1.6875^-	1.75^+	1.71875^-	-0.0459	0.0133
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	α	α	α	0	☺

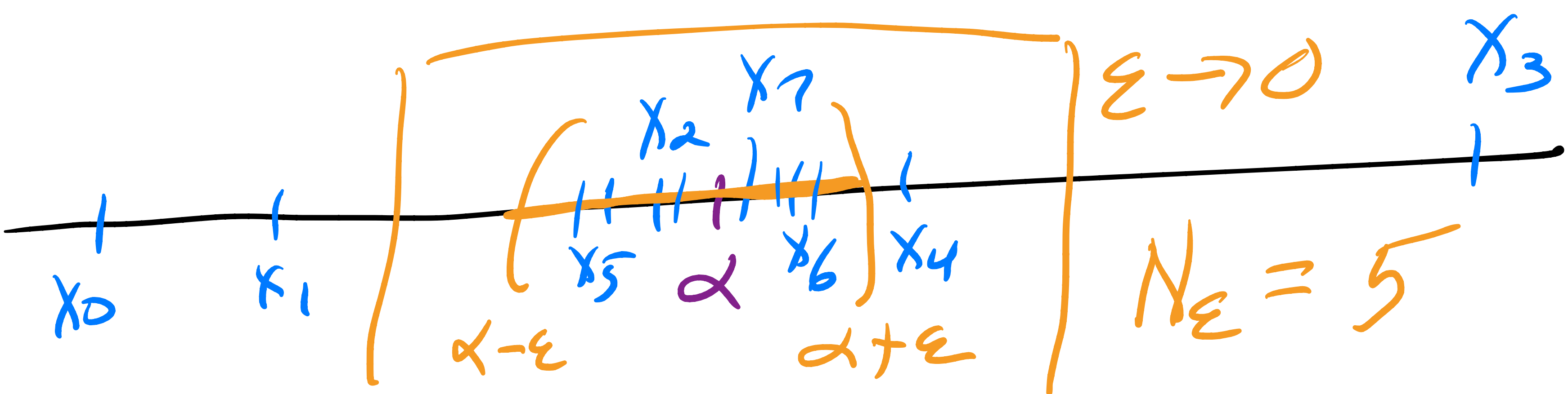
Defn (Convergence of a Sequence of Numbers)

A sequence $\{x_n\}_{n=0}^{\infty}$ CONVERGES to α if

$$\boxed{\lim_{n\to\infty} x_n = \alpha}$$

or, given $\varepsilon > 0$ there is an $N_\varepsilon \geq 0$ such that

$$|\alpha - x_n| < \varepsilon \quad \text{for } n \geq N_\varepsilon.$$



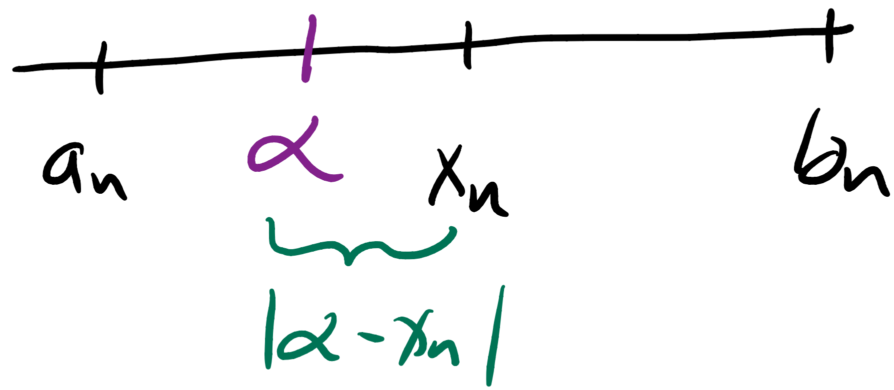
Thm: (Error Estimate for Bisection)

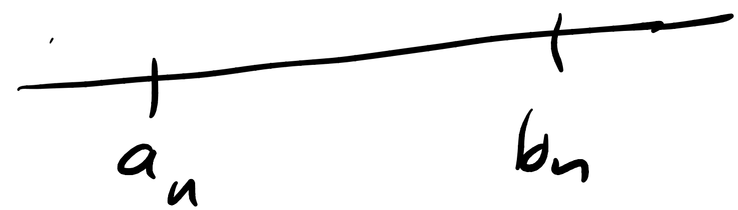
Suppose $f \in C([a,b])$, $f(a) \cdot f(b) < 0$, and $\alpha \in (a,b)$ such that $f(\alpha) = 0$. If $\{x_n\}_{n=0}^{\infty}$ is the sequence generated by the algorithm then

$$\boxed{|\alpha - x_n| \leq \frac{b-a}{2^{n+1}}}$$

For all $n \geq 0$.

Pf:





$[a_n, b_n]$

$[a_{n-1}, b_{n-1}]$

$$
\begin{aligned}
|\alpha - x_n| &\le \frac{1}{2}|b_n - a_n| \\
&= \frac{1}{2}\cdot\frac{1}{2}|b_{n-1} - a_{n-1}| \\
&= \left(\frac{1}{2}\right)^2 |b_{n-1} - a_{n-1}| \\
&= \left(\frac{1}{2}\right)^3 |b_{n-2} - a_{n-2}| \\
&\ \ \vdots \\
&= \frac{1}{2^{n+1}}|b_0 - a_0| = \frac{b-a}{2^{n+1}}
\end{aligned}
$$

#

So given a tol

$$|\alpha - x_n| \le \boxed{\frac{b-a}{2^{n+1}} \le tol}$$

$$\Rightarrow \quad 2^{n+1} \ge \frac{b-a}{tol}$$

Ceiling func

$\lceil 1.7 \rceil = 2$

$\lceil 1 \rceil = 1$

$$\Rightarrow \quad n+1 \ge \log_2\left(\frac{b-a}{tol}\right)$$

$$\Rightarrow \quad n \ge \left\lceil \log_2\left(\frac{b-a}{tol}\right) - 1 \right\rceil$$