Math 551 2/23

- HW #2 due Th 2/25
- HW #3 due Th 3/4
- Look over 3.1 - 3.4

Floating Point Systems

For any β, $b_1 \neq 0$

Double-Precision (64-bits IEEE)

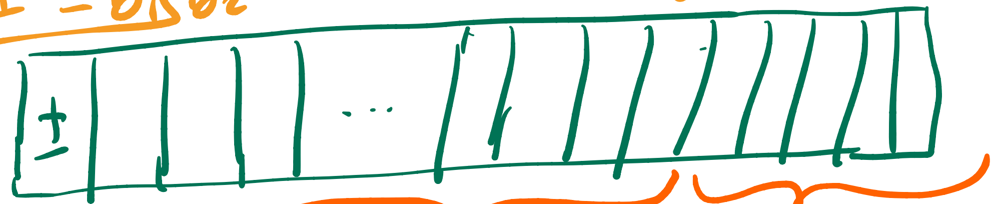
$\beta = 2$

$\{0, 1\}$

$b_1 = 1$

$\boxed{1 = b_1}\ b_2 \qquad b_{52}$

$fl(x) =$



sign bit

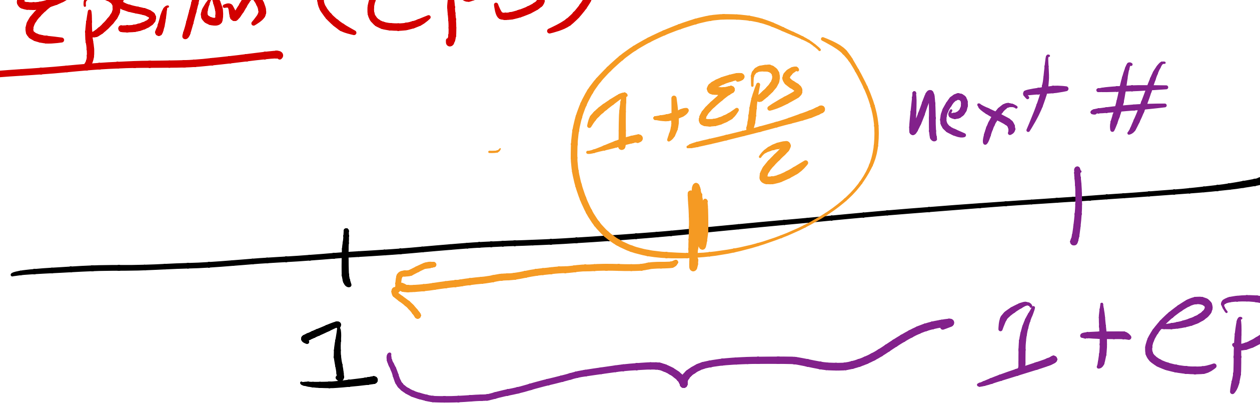
mantissa 52 bits

exponent 11 bits

$b_2 b_3 b_4 \cdots b_{53}$

NOT REPRESENTABLE!

Machine Epsilon (eps)



eps?

$$fl(1) = +(0.10000\cdots00)_2 \times 2^1 = 1$$

$$fl(1) + eps = +(0.\underbrace{1}_{b_1}00\cdots\cdots\underbrace{01}_{b_{52}})_2 \times 2^1$$

$$\text{so } eps = +(0.000\cdots\underbrace{001}_{2^{-52}})_2 \times 2^1$$

$$= (+)(2^{-52})(2) = \cancel{2^{-51}}$$

Wrong!

MATLAB

NOT STORED

$$fl(1) + eps = +(0.\boxed{1}0 \cdots 0\underbrace{1}_{b_{53}})_2 \times 2^1$$

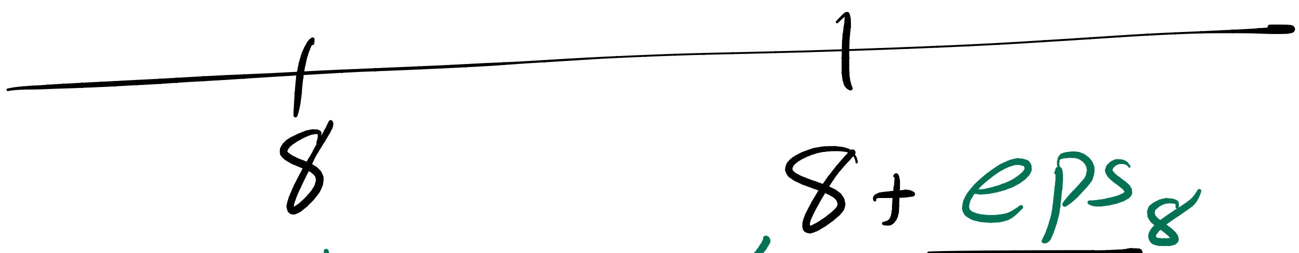
$$\Rightarrow eps = +(2^{-53}) \cdot 2 = +2^{-52}$$

as we saw in MATLAB

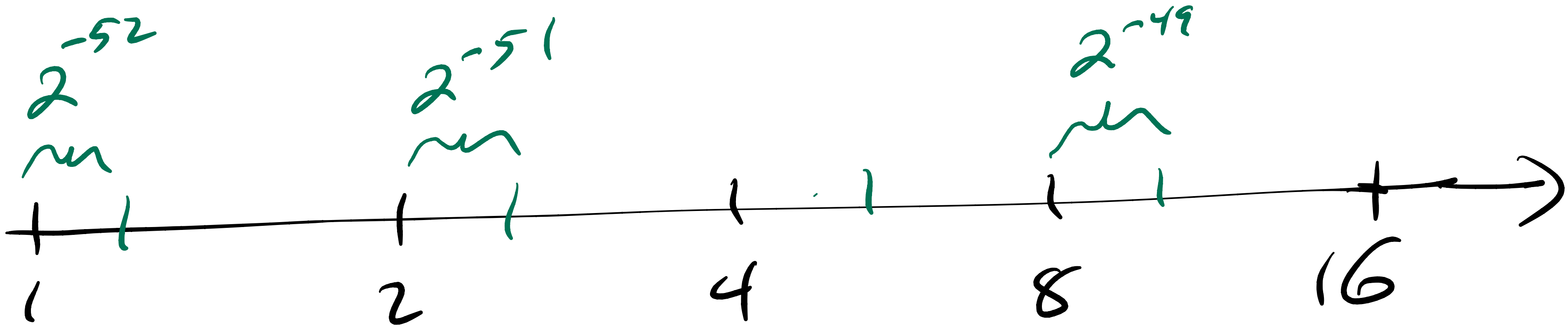
$$fl(8) + eps_8 = +(\underbrace{0.\overset{2^{-1}}{1}\overset{2^{-2}}{0}\cdots 00\overset{2^{-52}}{0}\overset{2^{-53}}{1}}_{1/2})_2 \times \underbrace{2^4}_{4}$$

$$eps_8 = +(0.00\cdots001)_2 \times 2^4$$

$$2^{-53} \cdot 2^4 = 2^{-49}$$



$$eps_8 = 2^{-49}$$


2^{-52}
2^{-51}
2^{-49}
1
2
4
8
16

Roots of Nonlinear Equations $\quad F: \mathbb{R} \to \mathbb{R}$

<u>Defn</u>: Given $f(x)$, α is called a ROOT or ZERO of $f(x)$ if $f(\alpha) = 0$.

<u>Ex</u>: $f(x) = x^2 - 3x + 2 = (x-2)(x-1) = 0$

$$\Rightarrow \quad \alpha = 1, 2$$

<u>Ex</u>: $F(x) = 2 - e^x = 0$

$$\Rightarrow \quad e^x = 2$$

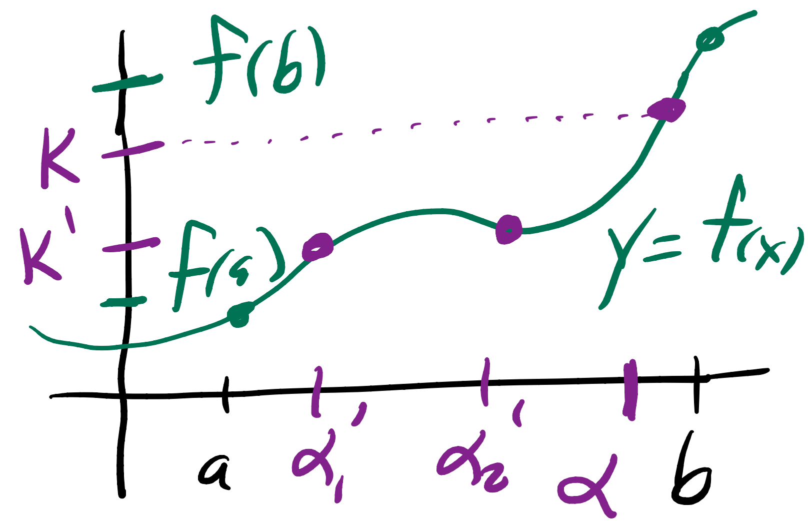
$$\Rightarrow \quad \alpha = \ln(2) = 0.69314\ldots$$

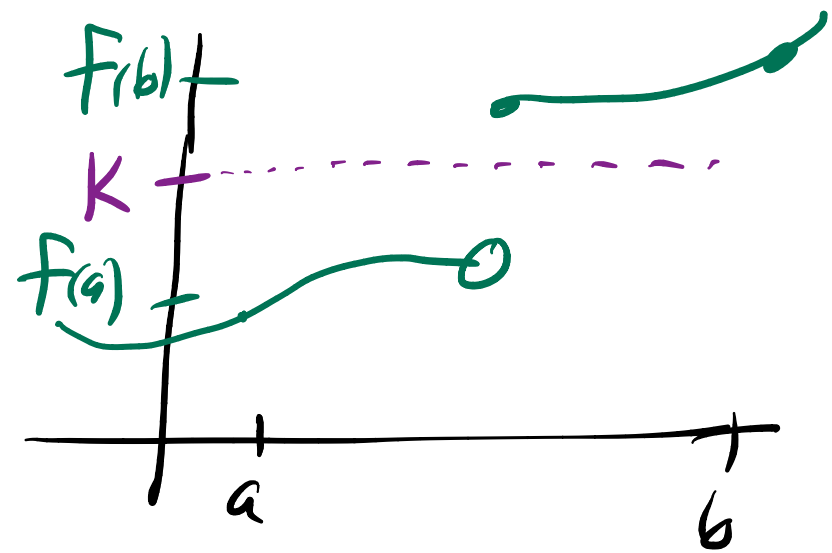
<u>Question</u>: Given $f(x)$ how do we determine the roots? Algorithm?

<u>Thm</u> (Intermediate Value Theorem) IVT

Suppose $f \in C([a,b])$ and K is any value between $f(a)$ and $f(b)$. Then there exist $\alpha \in (a,b)$ such that

$$f(\alpha) = K.$$



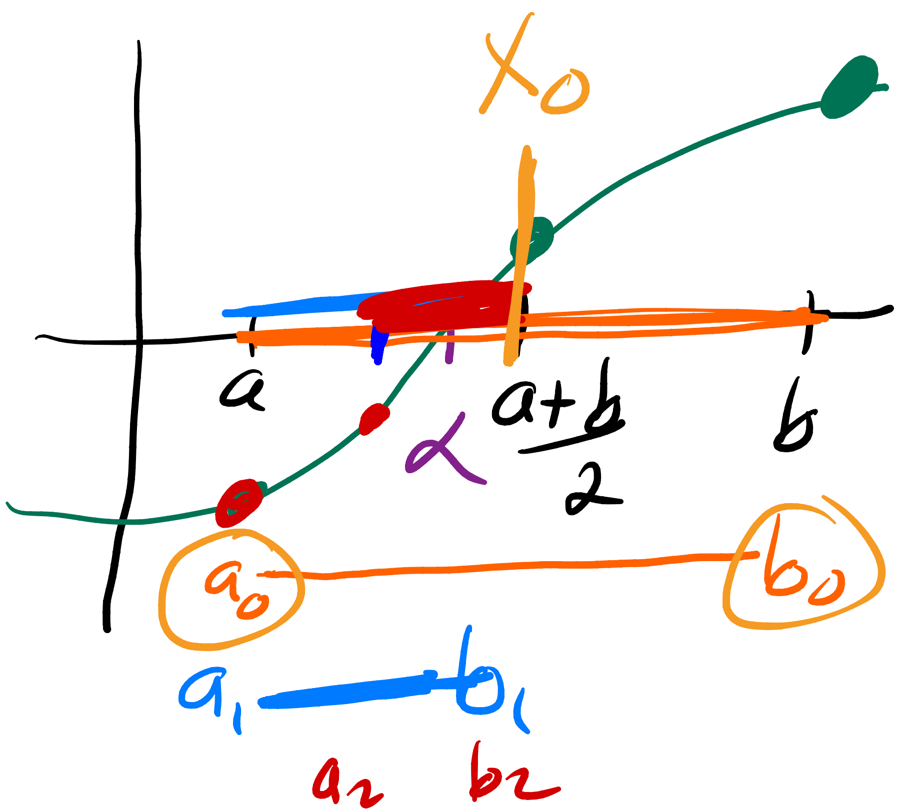


Algorithm:

① $f \in C([a,b])$ $\Longrightarrow$ IVT with $k=0$ the IVT tells us there exists $\alpha \in (a,b)$ such

$$f(\alpha) = 0.$$

② $\underbrace{f(a) \cdot f(b) < 0}_{f(a) \text{ and } f(b) \text{ opposite sign}}$



Check the sign of $f\left(\frac{a+b}{2}\right)$

$$\alpha \in \begin{matrix} (a_0, b_0) \\ (a_1, b_1) \\ (a_2, b_2) \end{matrix}$$

Bisection

Given $F \in C([a,b])$

$f(a) \cdot f(b) < 0$

Divide

and

Conquer

Algorithm loop

$a_0 = a$

$b_0 = b$

$n = 0$

$x_n = \boxed{(a_n + b_n)/2}$ +, /

if $f(a_n) \cdot f(x_n) < 0$

Function eval

$a_{n+1} = a_n$

$b_{n+1} = x_n$

else

$a_{n+1} = x_n$

$b_{n+1} = b_n$

end

$n = n + 1$