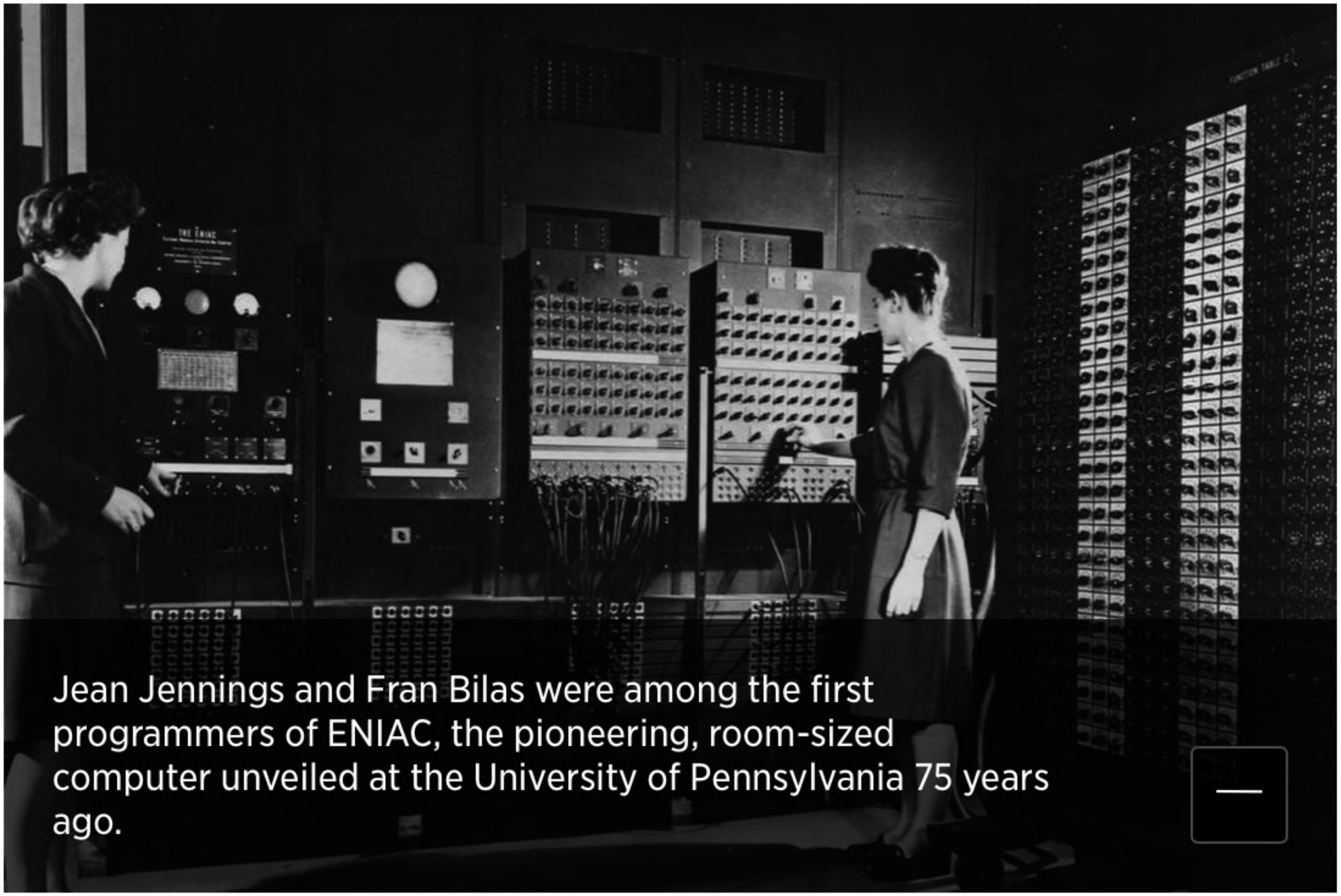
Math 551 2/11/21

Today:

1. 1^{st} order ($err = O(h)$) FD approximation to $f'(x_0)$
2. MATLAB Code
3. Latex commands to include graphics and displaying tables
4. Derive and analyze an $O(h^2)$ FD approximation to $f'(x_0)$

ENIAC: 75 yrs ago at UPenn this week.



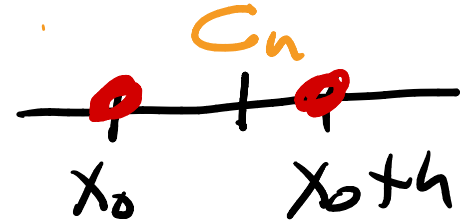
Jean Jennings and Fran Bilas were among the first programmers of ENIAC, the pioneering, room-sized computer unveiled at the University of Pennsylvania 75 years ago.

Electronic NUMERICAL INTEGRATOR and Computer

Solving DEs

FD: 1st Order

2-pt stencil



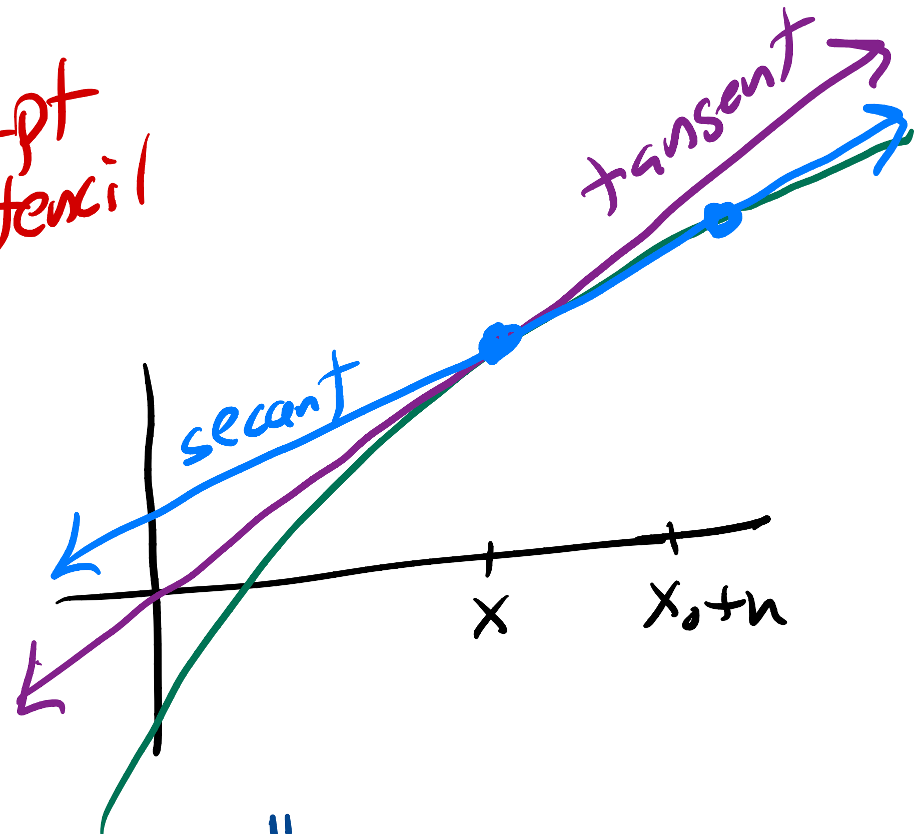
we showed

$$f'(x_0) \underset{\text{equality}}{=} -\frac{1}{h} f(x_0) + \frac{1}{h} f(x_0+h) - \frac{h}{2} f''(c_h)$$

Approximation

$\{x_0, x_0+h\}$ 2pt stencil

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$



Error

$$\frac{E(h)}{h} = \underbrace{\frac{-\frac{h}{2} f''(c_h)}{h}}_{-f''(c_h)/2} = -\frac{f''(c_h)}{2}$$

Ex! $f(x) = e^x \qquad x_0 = 1$

$$\frac{E(h)}{h} = \frac{-\frac{h}{2} e^{c_h}}{h} = \frac{-e^{c_h}}{h}$$

$c_h \xrightarrow{h \to 0} x_0$

$$\Rightarrow \lim_{h \to 0} \frac{E(h)}{h} = \lim_{h \to 0} \frac{-e}{2}$$

$$= \frac{-e^1}{2}$$

Theory should be VERIFIED on machine

h "small"

$$E(h) = ch \qquad c \approx \left| -\frac{1}{2} f''(x_0) \right|$$

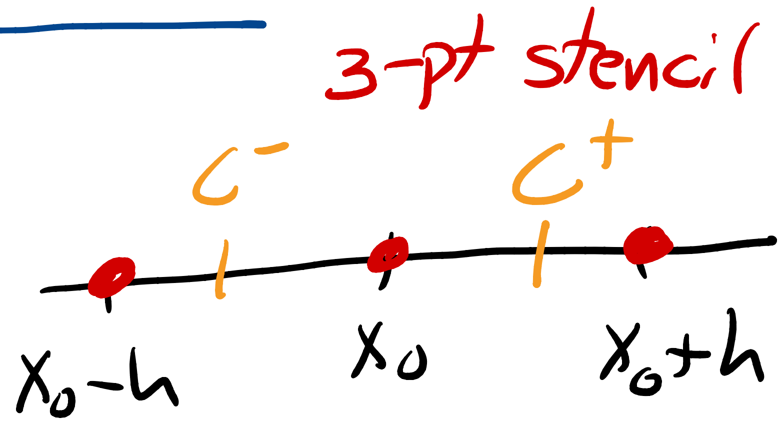
$$\log E = \log(ch) = \log(c) + \log(h)$$

$$\underbrace{\log E} = \underbrace{\log(h)} + \tilde{C}$$

$$\tilde{E} = \underbrace{\tilde{h} + \tilde{C}}$$

Linear in $\tilde{h}$ with slope 1.

2^{nd} FD Approximation to $f'(x_0)$



We want

$$f'(x_0) \approx a\,\underbrace{f(x_0-h)} + b\,\underbrace{f(x_0)} + c\,\underbrace{f(x_0+h)}$$

Taylor Expansion

$$\begin{aligned}
a\,\Big[\; f(x_0-h) &= f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f^{(3)}(c^-) \;\Big] \\
b\,\Big[\; f(x_0) &= f(x_0) \;\Big] \\
c\,\Big[\; f(x_0+h) &= f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f^{(3)}(c^+) \;\Big]
\end{aligned}$$

ADD

$$a f(x_0-h) + b f(x_0) + c f(x_0+h) = (a+b+c) f(x_0) + \left[-ah + ch\right] f'(x_0) + \left[a\frac{h^2}{2} + c\frac{h^2}{2}\right] f''(x_0) + \text{Error}$$

$\underline{a, b, c}$:

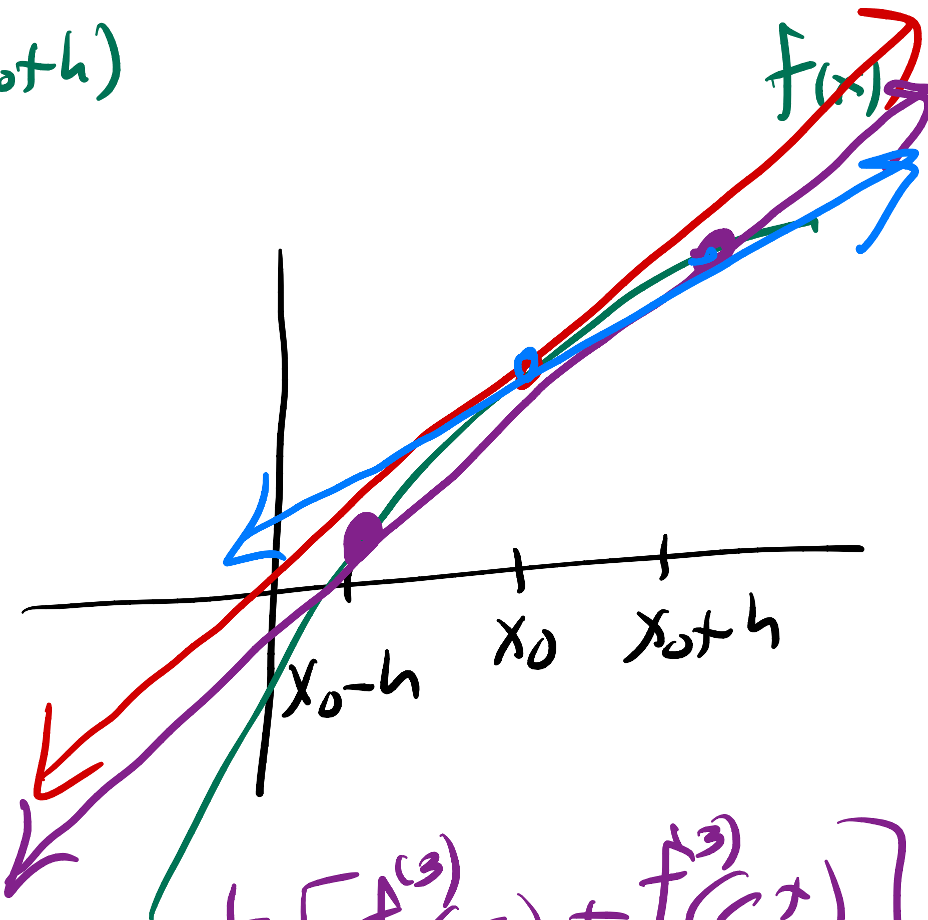
$$\left.\begin{aligned} a+b+c &= 0 \\ -ah+ch &= 1 \\ \frac{ah^2}{2}+\frac{ch^2}{2} &= 0 \end{aligned}\right\}$$

$a = -\frac{1}{2h}$

$b = 0$, $c = \frac{1}{2h}$

$$f'(x_0) \approx -\frac{1}{2h} f(x_0-h) + \frac{1}{2h} f(x_0+h)$$

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$



$$Err = \frac{h^2}{12} f^{(3)}(c^-) + \frac{h^2}{12} f^{(3)}(c_+)$$

$$\frac{Err}{h^2} = \frac{\cancel{h^2}\frac{1}{12}\left[f^{(3)}(c^-) + f^{(3)}(c+)\right]}{\cancel{h^2}} \rightarrow \frac{1}{12}\left[f^{(3)}(c^-) + f^{(3)}(c^+)\right]$$

$$\underbrace{\lim_{h\to 0} \frac{Err}{h^2}}_{\text{3rd Column!}} = \frac{1}{12}\left[f^{(3)}(c^-) + f^{(3)}(c^+)\right]$$

$$c^-, c^+ \to x_0$$

$$\approx \frac{1}{12}\left[2 f^{(3)}(c)\right]$$

$$= \frac{1}{6} f^{(3)}(x_0)$$

$$\frac{Err}{h^2} \to$$