Chapter 8: Probability

$\Pr(S) = 1$

Discrete Random Variable X

Possible outcomes (Events E)

Ex: 1 die

$S = \{1, 2, 3, 4, 5, 6\}$ } set of all outcomes is SAMPLE SPACE

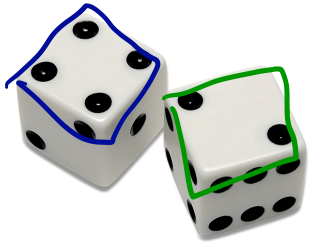
$E = \{3\}$

$$\Pr(E) = \Pr(3) = \frac{\#\text{ ways to get a 3}}{\text{total \# of outcomes}} = \frac{1}{6}$$

E = even # $= \{2, 4, 6\}$

$$\Pr(E) = \frac{\#\text{ of ways to get } E}{\text{total \# of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Ex: a pair of dice



E = sum of the die

6 choices * 6 choices

36 outcomes

E	# of E	Pr(E)
2	1	1/36
3	2	2/36
4	3	3/36
5	4	4/36
6	(2,4) (5,1) (4,2) (1,5) (3,3) 5	5/36
7	6	6/36
8	5	5/36
9	4	4/36
10	3	3/36
11	2	2/36
12	1	1/36
	Total	1

Ex: 5 card hand from a deck of 52 cards

How many are there:

$$\frac{\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48}}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2{,}598{,}960$$

$$\Pr(\text{Any given}) = \frac{1}{2{,}598{,}960}$$

$$\Pr(\text{Royal Flush}) = \frac{4}{2{,}598{,}960} \approx 0.00000153$$

$$\Pr(\text{4 of a kind}) = \frac{13 \cdot 48}{2{,}598{,}960}$$

$\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underbrace{\quad}_{48}$

In general: S = sample space
= set of all possible outcomes

E = Event

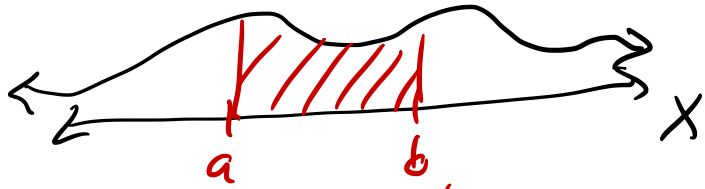
$$Pr(E) = \frac{\#\text{ of times } E \text{ occurs}}{\text{total \# of outcomes}}$$

Continuous Random Variable X

Probability Distribution Function $p(x)$

1) $p(x) \geq 0$

2) $\int_{-\infty}^{\infty} p(x) = 1$



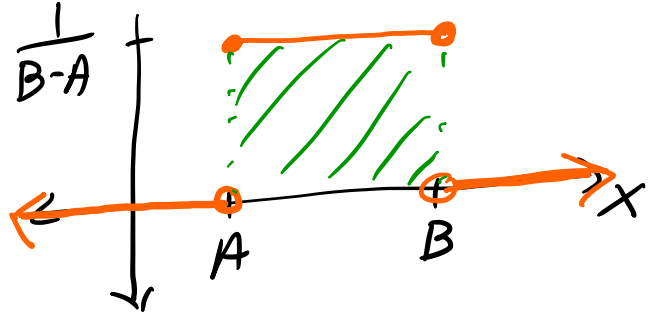
$$Pr(a \leq X \leq b) = \frac{\int_a^b p(x)\,dx}{\int_{-\infty}^{\infty} p(x)\,dx}$$

$$= \int_a^b p(x)\,dx$$

Examples of PDF

Uniform Distribution

$$p(x) = \begin{cases} 0 & x < A \\ \frac{1}{B-A} & A \le x \le B \\ 0 & x > B \end{cases}$$



Is $p(x)$ a PDF? Yes!!

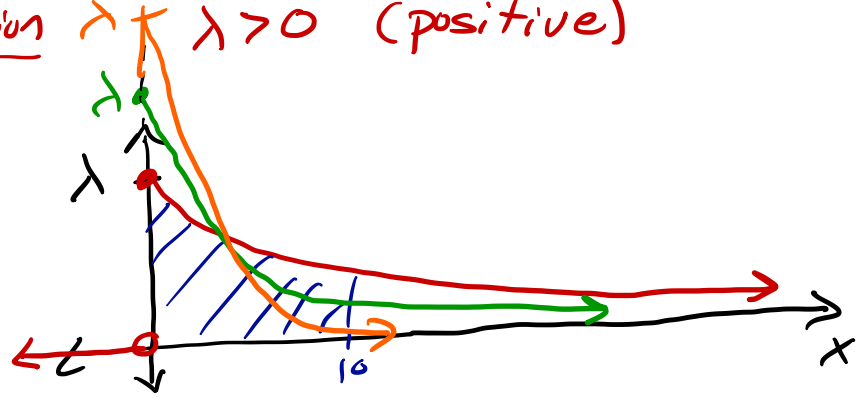
1) $p(x) \ge 0$ Yes!

2) $\int_{-\infty}^{\infty} p(x)\,dx = \int_{A}^{B} p(x)\,dx = \underbrace{(B-A)}_{\text{width}} \underbrace{\frac{1}{B-A}}_{\text{height}} = 1$ Yes

Exponential Distribution

$\lambda > 0$ (positive)

$$p(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$



$$p(0) = \lambda \cdot e^{-\lambda \cdot (0)}$$
$$= \lambda \cdot e^{0} = \lambda$$

Is $p(x)$ a PDF?

① $p(x) \geq 0$? Yes!

② $\int_{-\infty}^{\infty} p(x)\,dx = \int_{0}^{\infty} p(x)\,dx \stackrel{?}{=} 1$

$$\int_0^\infty p(x)\,dx = 1\ ?$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

$$\int_0^\infty p(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda\left[\int_0^\infty e^{-\lambda x}\,dx\right] = \lambda\cdot\frac{1}{\lambda} = 1$$

Yes!

$$\underbrace{\int_0^\infty e^{-\lambda x}\,dx}_{\text{Improper}} = \lim_{b\to\infty}\underbrace{\int_0^b e^{-\lambda x}\,dx}_{\text{defined}} = \lim_{b\to\infty}\left[-\frac{1}{\lambda}e^{-\lambda x}\Big|_0^b\right]$$

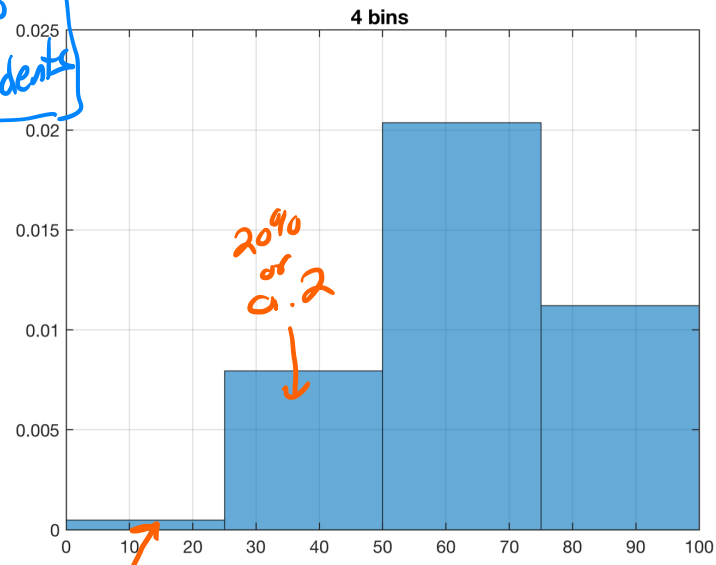
$$= -\frac{1}{\lambda}\lim_{b\to\infty}\left[e^{-\lambda\cdot b} - e^{-\lambda\cdot 0}\right] =$$

$$= -\frac{1}{\lambda}\lim_{b\to\infty}\left[\frac{1}{e^{\lambda b}} - 1\right] = -\frac{1}{\lambda}\left[0 - 1\right] = \frac{1}{\lambda}$$

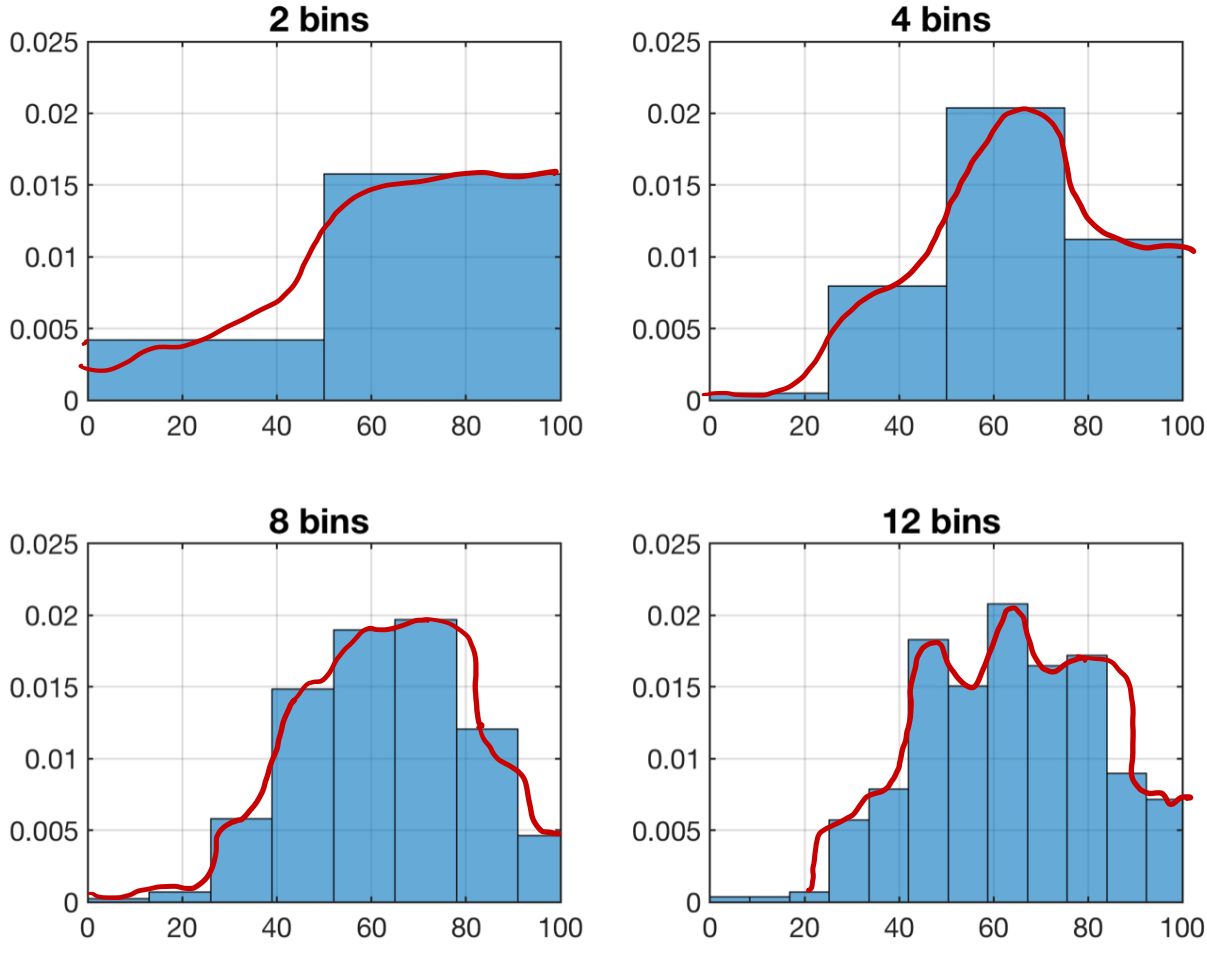
Smoothing out a Histogram

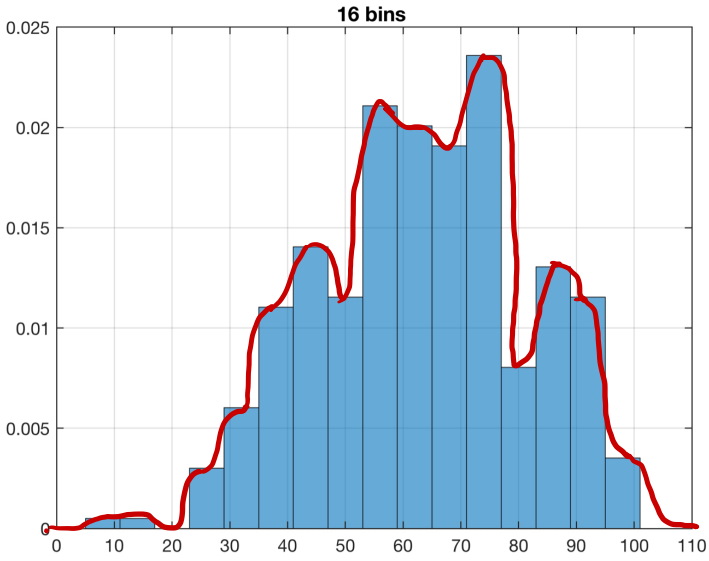
DISCRETE to CONTINUOUS

Exam Scores
332 Math 128 Students



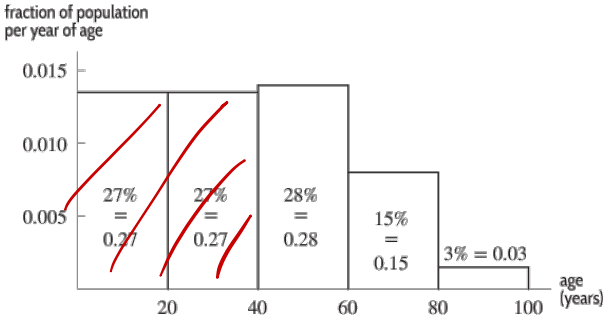
score	% of students
0-25	$25 \cdot 0.0004 = 0.01$
25-50	$25 \cdot 0.008 = 0.2$
50-75	$25 \cdot 0.0204 = 0.51$
75-100	$25 \cdot 0.0112 = 0.28$
	1

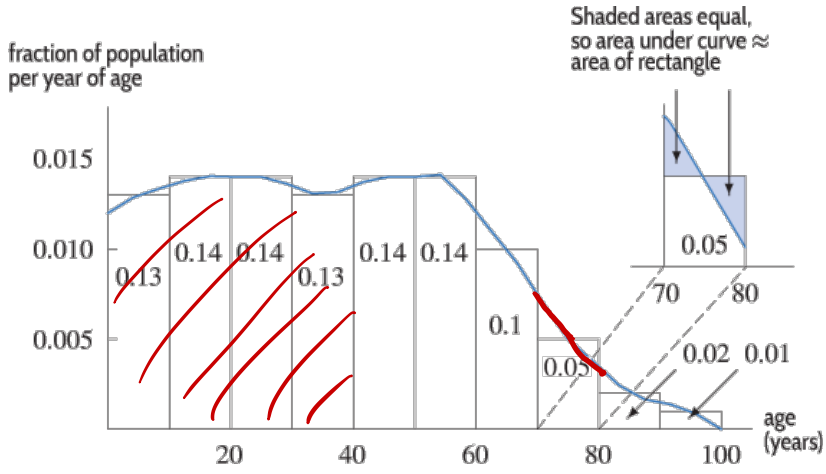

2 bins
4 bins
8 bins
12 bins
0.025
0.02
0.015
0.01
0.005
0
0
20
40
60
80
100



Estimate the PDF $P(x)$

Distribution of US ages





what % of the population is ≤ 40?

$\approx 54\%$

Summary: Probability {Density / Distribution} Function PDF

$p(x)$ is a PDF if

(1) $p(x) \geq 0$

(2) $\int_{-\infty}^{\infty} p(x)\,dx = 1$

$\Rightarrow \Pr(a \leq x \leq b) = \int_a^b p(x)\,dx$

$\overset{FTC}{=} P(b) - P(a)$

where $P'(x) = p(x)$

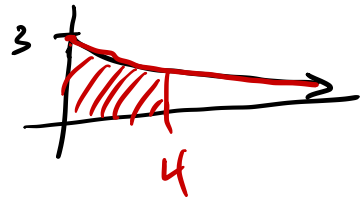
CDF

$P(x)$ is called the Cumulative Distribution Function

Ex: Which of the following makes the most sense as a model of a PDF representing the time (starting at $t=0$) that a person walks into a store?

a) ~~$p(t) = \begin{cases} \cos(t) & 0 \le t \le 2\pi \\ e^{t-2\pi} & t > 2\pi \end{cases}$~~

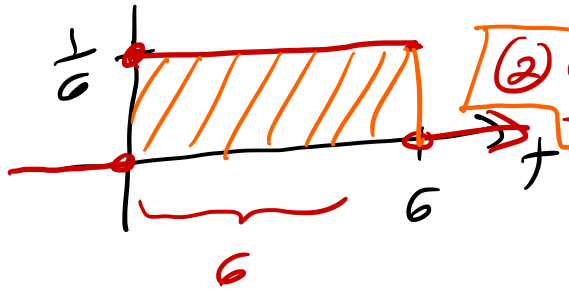
b) $p(t) = 3e^{-3t}, \quad t \ge 0$



c) ~~$p(t) = e^{-3t}, \quad t \ge 0$~~

d) $p(t) = \frac{1}{6}, \quad 0 \le t \le 6$

Uniform PDF $A=0$ $B=6$



① $p(t) \ge 0$ Yes!

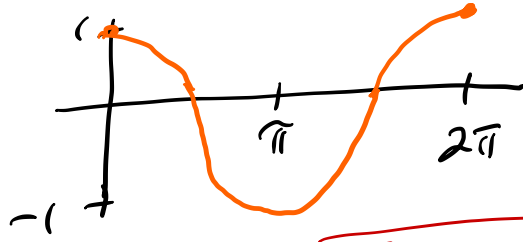
② $\int_{-\infty}^{\infty} p(t)\,dt = 1$ Yes!

a) $p(t) = \begin{cases} \cos t & 0 \le t \le 2\pi \\ e^{t-2\pi} & t > 2\pi \end{cases}$

$p(t)$ is negative for so values of t,

NOT PDF

Is it a PDF?

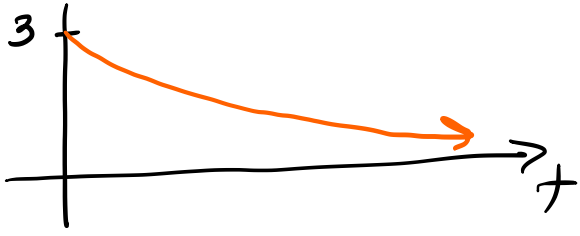


Exponential Distribution

$$p(t) = \begin{cases} \lambda e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

(a) $p(t) = 3e^{-3t}, \quad t \ge 0$

$\lambda = 3$



Yes, it is a PDF!

c) $p(t) = \begin{cases} 0 & t < 0 \\ e^{-3t} & t \geq 0 \end{cases}$

① $p(t) \geq 0$

② $\int_{-\infty}^{\infty} p(t)\,dt = 1$? No! not a PDF

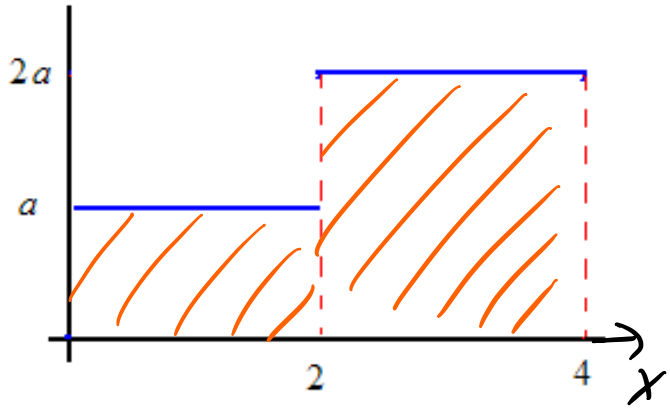
why $\int_{-\infty}^{\infty} p(t)dt = \int_{0}^{\infty} p(t)\,dt$

$= \int_{0}^{\infty} e^{-3t}dt = 1/3 \neq 1$

$1 = \int_{0}^{\infty} 3e^{-3t}dt = 3\boxed{\int_{0}^{\infty} e^{-3t}dt} = 1/3$

Ex: Find a so that the graph is that of a PDF.

$$p(x) = \begin{cases} 0 & x < 0 \\ a & 0 \le x < 2 \\ 2a & 2 \le x \le 4 \\ 0 & x > 4 \end{cases}$$

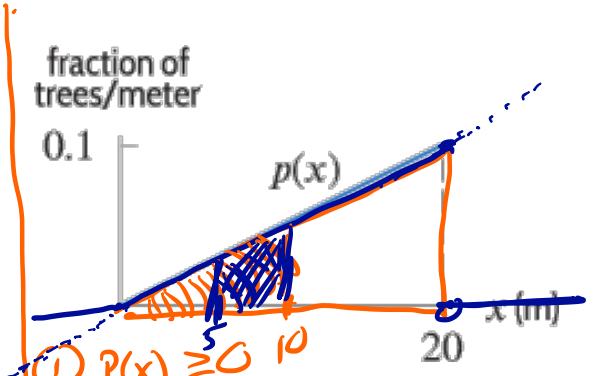


① $p(x) \ge 0$

② $\int_{-\infty}^{\infty} p(x)\,dx = 1$

$$1 = \int_0^4 p(x)\,dx = (2-0)\cdot a + (4-2)2a = 2a + 4a = 6a \Rightarrow 6a = 1 \Rightarrow \boxed{a = 1/6}$$

Ex: The distribution of heights x (meters) of trees is represented by $p(x)$.



① $p(x) \geq 0$

② $\int_0^{20} p(x)\,dx = \frac{1}{2} \cdot 20 \cdot \frac{1}{10} = 1$

(a) What fraction of trees are 10 meters or less?

$$\Pr(0 \leq x \leq 10) = P(x \leq 10)$$

$$= \int_0^{10} p(x)\,dx = \int_0^{10} \frac{1}{200} x\,dx = \frac{1}{200} \cdot \frac{1}{2} x^2 \Big|_0^{10}$$

$$= \frac{1}{400} x^2 \Big|_0^{10} = \frac{1}{400}\left[10^2 - 0^2\right] = \frac{100}{400} = \boxed{\frac{1}{4}}$$

(b) What fraction is between 5 and 10 meters?

$$\Pr(5 \leq x \leq 10) = \int_5^{10} p(x)\,dx = \int_5^{10} \frac{1}{200} x\,dx$$

$$= \frac{1}{400} x^2 \Big|_5^{10} = \frac{1}{400}\left[10^2 - 5^2\right] = \frac{75}{400}$$

$$= \frac{15}{60} = \boxed{\frac{3}{16}} =$$

What is $p(x)$?

$p(x) = mx + b$ ($b = 0$)

$= mx$

$m = \frac{\Delta y}{\Delta x} = \frac{\frac{1}{10}}{20} = \frac{1}{200}$

$$p(x) = \frac{1}{200} x$$

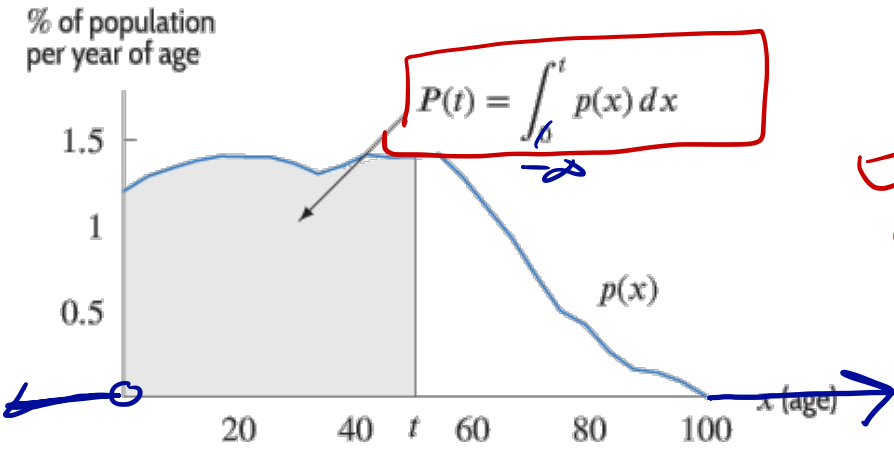
10/23

HW 8.1 due 10/30 PDF

HW 8.2 due 11/1 CDF

Read 8.3 } mean, median

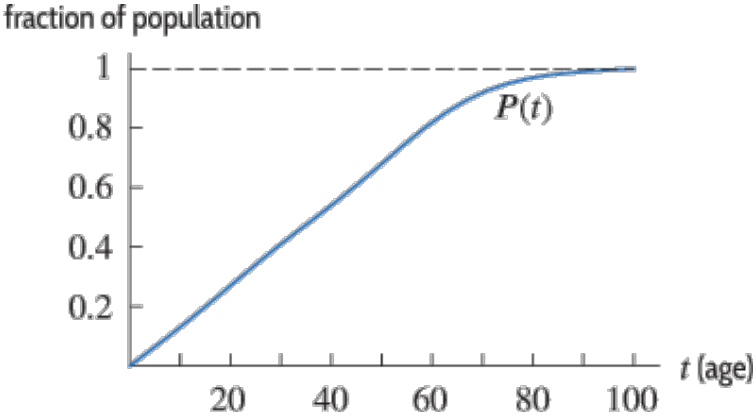
Cumulative Distribution Function : CDF



$$P(t) = \int_0^t p(x)\,dx$$

$P(t)$: CDF

$p(x)$: PDF



In General

$$P(t) = \int_{-\infty}^{t} p(x)\,dx$$

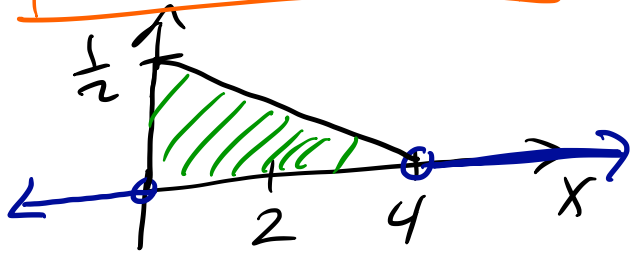
Ex: Suppose a PDF is given by

$$p(x) = \begin{cases} 0 & x < 0 \\ -\frac{1}{8}x + \frac{1}{2} & 0 \le x \le 4 \\ 0 & x > 4 \end{cases}$$

$p(0) = -\frac{1}{8}(0) + \frac{1}{2} = \frac{1}{2}$

$p(4) = -\frac{1}{8}(4) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$

Find the CDF



$$\int_{-\infty}^{\infty} p(x)\,dx = \int_{0}^{4} p(x)\,dx = \int_{0}^{4} -\frac{1}{8}x + \frac{1}{2}\,dx = \text{Area of Triangle}$$

$$= \frac{1}{2}(4-0)\cdot\frac{1}{2}$$

$$= \boxed{1}$$

CDF

$$P(t) = \int_{-\infty}^{t} p(x)\,dx = \int_{0}^{t} p(x)\,dx$$

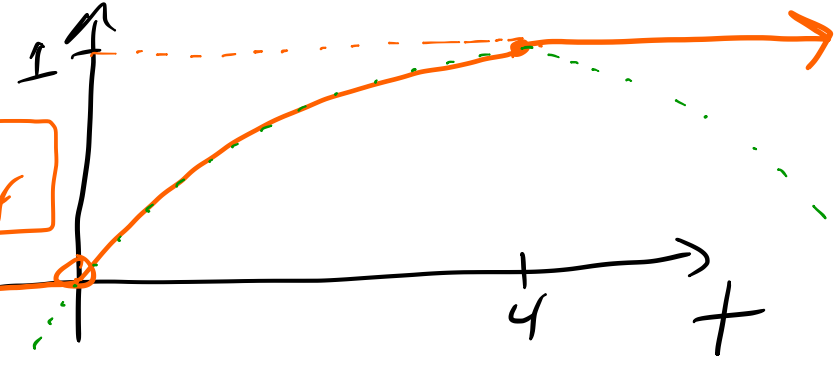
$$= \int_{0}^{t} -\frac{1}{8}x + \frac{1}{2}\,dx = -\frac{1}{16}x^2 + \frac{1}{2}x \Big|_{0}^{t}$$

$$= \left(-\frac{1}{16}\cdot t^2 + \frac{1}{2}t\right) - \left(-\frac{1}{16}\cdot 0^2 + \frac{1}{2}\cdot 0\right)$$

$$= -\frac{1}{16}t^2 + \frac{1}{2}t$$

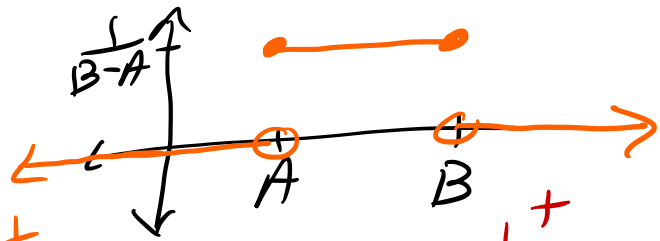
$$P(t) = \begin{cases} 0 & t < 0 \\ -\frac{1}{16}t^2 + \frac{1}{2}t & 0 \le t \le 4 \\ 1 & x > 4 \end{cases}$$

CDF



Ex: Find the CDF of a UNIFORM distribution.

$$p(x) = \begin{cases} 0 & x < A \\ \frac{1}{B-A} & A \le x \le B \\ 0 & x > B \end{cases}$$



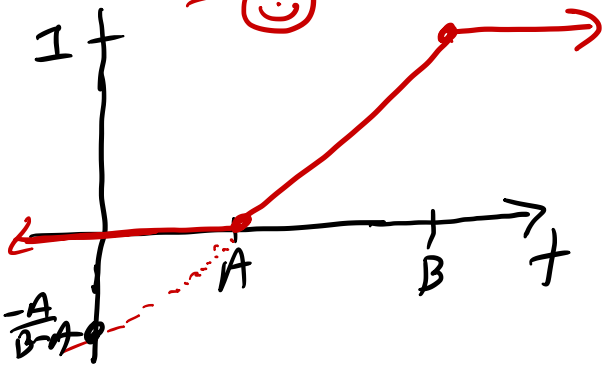
CDF: $P(t) = \int_{-\infty}^{t} p(x)\,dx = \int_{A}^{t} \frac{1}{B-A}\,dx = \frac{1}{B-A} \cdot x \Big|_{A}^{t}$

$P(A) = \frac{1}{B-A} \cdot A - \frac{A}{B-A}$

$= \frac{1}{B-A}\left[t - A\right] = \boxed{\frac{1}{B-A}\,t - \frac{A}{B-A}}$ *

$= $ ☺

$m = \frac{1}{B-A} \quad b = -\frac{A}{B-A}$



$$= \begin{cases} 0 & t < A \\ mt + b & A \le t \le B \\ 1 & t > B \end{cases}$$

Ex: Find the CDF of an EXPONENTIAL distribution

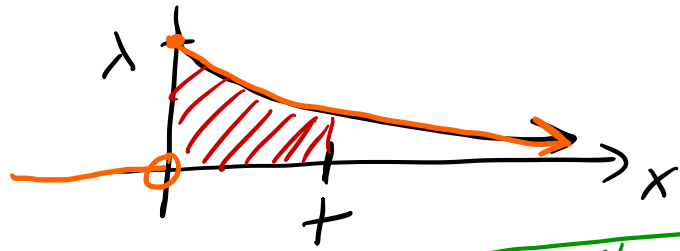
PDF

$$P(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$\lambda > 0$

$P(0) = \lambda \cdot e^{-\lambda \cdot 0}$

$= \lambda \cdot e^{0} = \lambda$



Recall

$$\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$$

CDF $P(t) = \int_{-\infty}^{t} P(x)\,dx = \int_{0}^{t} \lambda e^{-\lambda x}\,dx$

$k = -\lambda$ $= \lambda \int_{0}^{t} e^{-\lambda x}\,dx = \lambda \left[\frac{1}{-\lambda} e^{-\lambda x} \Big|_{0}^{t} \right]$

$= \frac{\lambda}{-\lambda} \left[e^{-\lambda t} - e^{-\lambda \cdot 0} \right] = -1 \left[e^{-\lambda t} - 1 \right]$

$= 1 - e^{-\lambda t}$

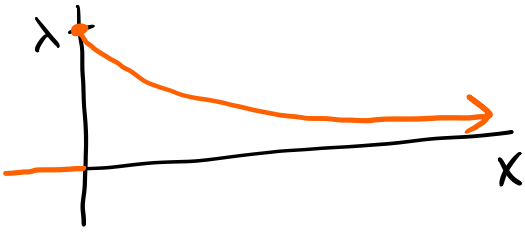
$$P(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t} & t \geq 0 \end{cases}$$

Check: $P(0) = 1 - e^{-\lambda \cdot 0} = 1 - 1$

$= 0$ ☺

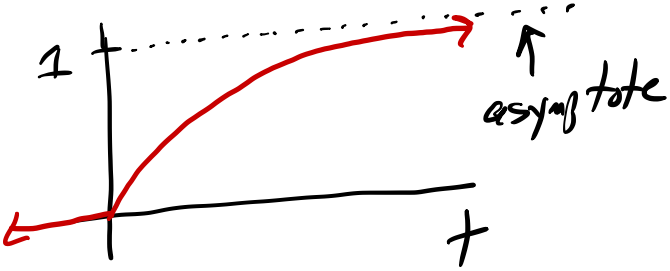
$\lambda > 0$ pdf

$$p(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$



CDF

$$P(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t} & t \geq 0 \end{cases}$$



Always True

- $p(x) \geq 0$
- $\int_{-\infty}^{\infty} p(x)\,dx = 1$
- $P(t) = \int_{-\infty}^{t} p(x)\,dx$
- $P'(t) = p(t)$
- $\lim_{t \to -\infty} P(t) = 0$
- $\lim_{t \to +\infty} P(t) = 1$

$$P(t) = 1 - e^{-\lambda t} = 1 - \frac{1}{e^{\lambda t}}$$

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \left[1 - \frac{1}{e^{\lambda t}} \right]$$

$$= 1 - \text{"}\frac{1}{\infty}\text{"} = 1 - 0 = 1$$

Note: $P'(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right)$

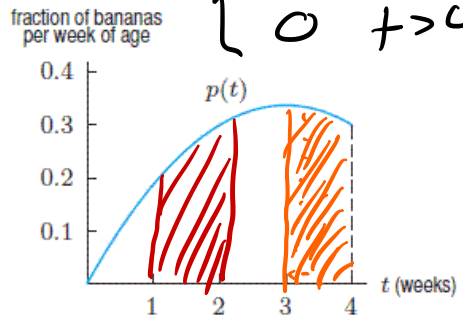
$$= 0 - e^{-\lambda t} \cdot (-\lambda) = \lambda e^{-\lambda t}$$

$$= p(t)$$

Ex: $p(t) = -0.039t^2 + 0.229t$

$$\begin{cases} 0 & t<0 \\ p(t) & 0 \le t \le 4 \\ 0 & t>4 \end{cases}$$

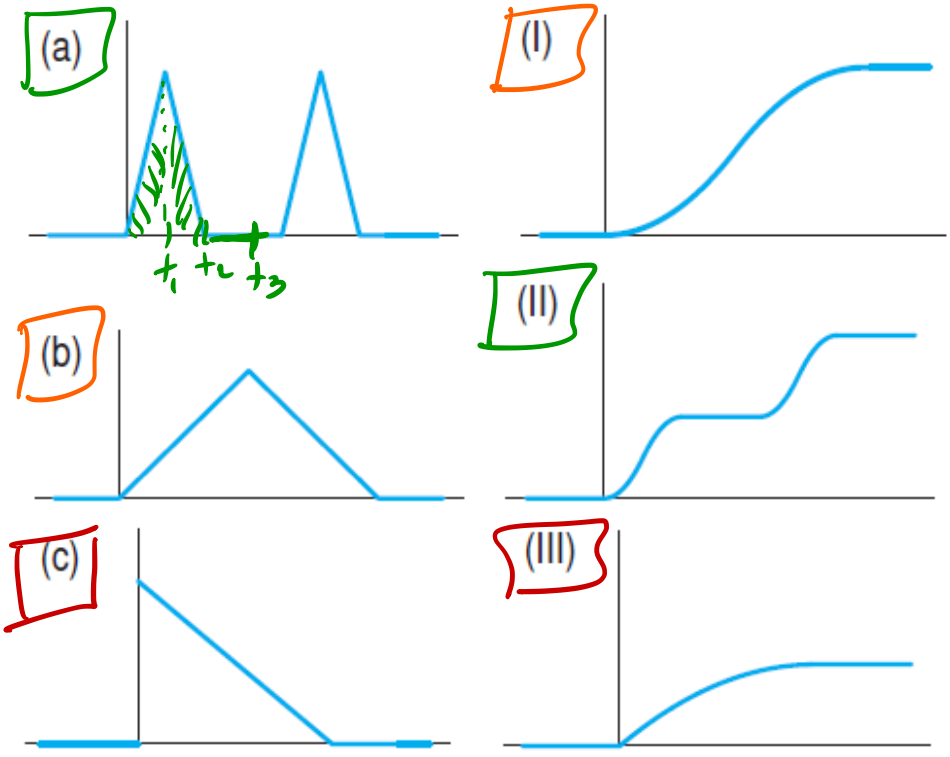
PDF for shelf life of Bananas, where $0 \le t \le 4$ is measured in weeks



a) $Pr(1 \le t \le 2) = \int_1^2 p(t)\,dt$

$= \int_1^2 -0.039t^2 + 0.229t\,dt \approx 0.2525$ $\approx 25\%$

b) $Pr(t \ge 3) = \int_3^\infty p(t)\,dt = \int_3^4 p(t)\,dt$

$= \int_3^4 -0.039t^2 + 0.229t\,dt \approx 0.32$ $\approx 32\%$

Ex: Match PDF with CDF