Numerical Methods:
Are more than Mathematics

Math:
Accuracy
Stability
Convergence
Consistency

Physics:
Conservation
Spurious Modes
Wave propagation
Maximum/minimum
Constraints

Mimetic methods mimic the physics.
Relationship

Finite Difference

Finite Element

Finite Volume

Meshless

SOM

Edge/Face

Natural Neighbors

Staggered

Mimetic Methods

All Numerical Methods
Mimetic Advection \( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = 0 \)

1D Advection: Change in mesh size (3x more mesh on the right side)

Central

Box Method

Spurious Wave Reflection
Eigenvalues: Theory

Vector Laplacian

$$\nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = \lambda \mathbf{v}$$

$$\lambda = m^2 + n^2$$

$$1, 1, 2, 4, 4, 5, 5, 8, \ldots$$

Linear FE

$$\mathbf{v}_{node}$$

Nedelec FE

$$\mathbf{u} = \mathbf{v} \cdot \mathbf{t}_{edge}$$
Eigenvectors: Practice

\[ \nabla \times \nabla \times \mathbf{v} - \nabla (\nabla \cdot \mathbf{v}) = f \]

Arnold

Linear FE

Nedelec FE
Mimetic Surface Tension

Diagram showing a circular pattern.
Overview

Use Exact Discretization

Separate **Discretization** from **Approximation**

- Do **ALL** discretization exactly.
- This means that the calculus and the physics remain exact.

- **Numerical approximation in material laws.**
- Which are engineering approximations already.
- Numerical approximation goes with physical approximation.
Discrete Calculus: Part 1

Exact Discretization

\[ \frac{\partial a}{\partial t} + \nabla \cdot b = 0 \]

\[ [A \ B] \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \vec{r} \]

Infinite Dimensional     Finite Dimensional

Partial Differential Eqn.    Matrix Problem

Basic unknowns are integral quantities. Collect infinite data into finite groups.
Solution requires Approximation

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{pmatrix}
\vec{a} \\
\vec{b}
\end{pmatrix}
=
\vec{r}
\Rightarrow
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{pmatrix}
\vec{a} \\
\vec{b}
\end{pmatrix}
=
\begin{pmatrix}
\vec{r} \\
0
\end{pmatrix}
\]

Underdetermined \quad Unique Square

Relate discrete unknowns to each other. This relation is a material law. Also related to interpolation. Also related to discrete inner products
Example: Heat Eqn

\[ \frac{\partial (\rho cT)}{\partial t} = \nabla \cdot k \nabla T \]

Components of the Physical Equation

\[ \frac{\partial i}{\partial t} + \nabla \cdot \mathbf{q} = 0 \quad \text{Conservation of Energy} \]
\[ \mathbf{g} = \nabla T \quad \text{Definition of Gradient} \]

Material Approximation

\[ \mathbf{q} = -k \mathbf{g} \quad \text{Fourier’s Law} \]
\[ i = \rho cT \quad \text{Perfectly Caloric Material} \]
Exact Discretization

Perfect representation of Physics and Calculus

\[ \int_{\tilde{c}} idV |^{n+1} - \int_{\tilde{c}} idV |^{n} + \sum_{\tilde{f}} \int_{\tilde{f}} dt \int_{\tilde{f}} \mathbf{q} \cdot \mathbf{n} dA_{\tilde{f}} = 0 \]

\[ \int_{e} \mathbf{g} \cdot d\mathbf{l} = T_{n2} - T_{n1} \]

\[ I^{n+1}_{\tilde{c}} - I^{n}_{\tilde{c}} + \mathbf{D} \mathbf{Q}_{\tilde{f}} = 0 \]

\[ g_{e} = \mathbf{G} T_{n} \]
Solution

Numerical Approximation of Constitutive Eqns.

\[ Q_{\tilde{f}} = -M_1 g_e \]

\[ I_{\tilde{c}} = M_2 T_n \]

\[ Q_{\tilde{f}} = -k \frac{A_{\tilde{f}}}{L_e} g_e \]

\[ I_{\tilde{c}} = \rho c V_{\tilde{c}} T_n \]

Approximate

Dependent on Mesh
Dual Mesh Viewpoint

\[ I_{\tilde{c}}^{n+1} - I_{\tilde{c}}^{n} + DQ_{\tilde{f}} = 0 \]

\[ I_{\tilde{c}}^{n+1} = \int_{\tilde{c}} idV \bigg|_{n+1} \]

\[ Q_{\tilde{f}} = \int dt \int_{\tilde{f}} q \cdot n dA_{\tilde{f}} \]

\[ l_{\tilde{c}} = \rho c V_{\tilde{c}} T_{n} \]

\[ g_{e} = GT_{n} \]

\[ \int_{e} g \cdot dI = g_{e} \]

\[ Q_{\tilde{f}} = -k \frac{A_{\tilde{f}}}{L_{e}} g_{e} \]

Dual Mesh

Primary Mesh
Properties

Conservation of Energy
Entropy Production
Maximum Principal

Any continuous principle for the PDE ...

All errors appear as imperfect material properties.
Variations

Choice of the Dual Mesh.

Dual or Primary
Node centered pressure.
Cell centered pressure.

Choice of interpolation.
polynomial reconstruction in cells.
reconstruction in dual cells
weighted interpolations (FE).
Results

Log scale
References


Summary

- Numerical Methods are changing.
- Exact Discretization Approx Solution.
- Works on all types of PDEs
Questions
Navier-Stokes Results