A main goal of birational geometry is to classify algebraic varieties up to birational isomorphism. Convex geometry is a useful tool for understanding algebraic varieties. More specifically, given a complex projective variety $X$, there is a (closed) cone of curves of $X$, denoted by $\text{Curv}(X)$, which is the closure of the cone of positive linear combinations of curve classes in $H_2(X, \mathbb{R})$. The dual of the cone of curves is called the nef cone. These cones contain information about the variety. The cone theorem, which is a result in the minimal model program, describes the structure of the cone of curves of a variety.

The three main types of varieties are Fano varieties (i.e., $-K_X$ is ample), Calabi-Yau varieties (i.e., $K_X = 0$), and varieties with ample canonical bundle (i.e., $K_X$ is ample). Fano varieties behave nicely in many ways. For instance, it has been proven (Kollár, Miyaoka, and Mori, 1992) that in any dimension, there are finitely many diffeomorphism types of smooth Fano varieties. Moreover, for any Fano variety, the cone of curves and the nef cone are rational polyhedral, meaning that the cone is generated by a finite number of rational points. For varieties in general, this is not the case - the cone of curves of an arbitrary variety might have countably many extremal rays, or it might be round (for a cone $\sigma \subset V \simeq \mathbb{R}^n$, a ray $\rho = \mathbb{R}_{\geq 0} \cdot v$ for some nonzero $v \in V$ is an extremal ray if there exists a hyperplane $H \subset V$ such that $\rho = \sigma \cap H$).

At the other extreme, the classification of varieties which have ample canonical bundle is more complex. For example, in dimension one, varieties with ample canonical bundle are curves of genus $g \geq 2$, and there are infinitely many diffeomorphism types of these varieties.

The Calabi-Yau varieties lie in the middle. There are only finitely many topological types of smooth Calabi-Yau surfaces of dimension $n$ when $n$ is at most two. But if the dimension is greater than two, then this is an open question. The Kawamata-Morrison cone conjecture states that the action of the automorphism group of a Calabi-Yau variety $X$ on the nef cone of $X$ has a rational polyhedral fundamental domain. This conjecture has been proven for Calabi-Yau surfaces and for all three-dimensional Calabi-Yau fiber spaces, but it is still an open question for Calabi-Yau 3-folds.

In some cases, it is useful to consider pairs $(X, \Delta)$, where $X$ is a variety and $\Delta$ is a divisor (a formal linear combination of codimension one subvarieties $\Delta_i$ with positive coefficients $a_i$). We say that a pair $(X, \Delta)$ is klt (Kawamata log terminal) if it has mild singularities, which we will not describe here in a precise sense. For example, if $X$ is smooth and the codimension one subvarieties $\Delta_i$ are smooth and intersect transversely (i.e., $\sum \Delta_i$ is a normal crossing divisor), then $(X, \Delta)$ is klt if and only if the coefficients are strictly less than one.

All of the results dealing with smooth projective varieties that arise from minimal model theory can be generalized to the case of klt pairs. For instance, the generalization of Mori’s cone theorem for smooth Fano varieties to klt Fano pairs is as follows:

**Theorem.** If $(X, \Delta)$ is a klt Fano pair (meaning $-(K_X + \Delta)$ is ample), then the cone of curves of $X$ is rational polyhedral.

A significance of this theorem is that it has the same conclusion as the cone theorem for smooth Fano varieties. In this way, the restatement for pairs is a generalization - for smooth varieties $X$ that are not Fano, the cone of curves of $X$ could still be rational polyhedral, as long as we can find some $\Delta$ such that $(X, \Delta)$ is klt Fano.
Similarly, we may generalize the cone theorem for Calabi-Yau varieties to a statement for pairs. A pair \((X, \Delta)\) is said to be Calabi-Yau if \(K_X + \Delta = 0\). This conjecture has been proven in dimension 2 (Totaro, 2010):

**Theorem.** Let \((X, \Delta)\) be a klt Calabi-Yau pair of dimension two. Then \(\text{Aut}(X, \Delta)\) acts on the nef cone with a rational polyhedral fundamental domain.

We are studying what we call the Kawamata-Morrison-Totaro cone conjecture, which is related to but different from Totaro’s version. We are studying pairs \((X, D)\), where \(X\) is a smooth projective surface, \(D = D_0 + D_1 + \cdots + D_{n-1}\) is a normal crossing divisor on \(X\) with coefficients equal to one, and \(K_X + D = 0\). We assume that \((X, D)\) corresponds to a general point in the moduli space of pairs of a given topological type. The conjecture asserts that the monodromy group acts with a rational polyhedral fundamental domain on the nef cone of \(X\).

Equivalently, there is a distinguished pair \((X_e, D_e)\) in the moduli space, which can be characterized as follows. By a theorem of Deligne, the cohomology of any algebraic variety has a canonical mixed Hodge structure which is a generalization of the Hodge structure on a smooth projective variety. The pair \((X_e, D_e)\) is the **unique** pair \((X, D)\) for which the mixed Hodge structure on \(H^2(U)\) (where \(U = X \setminus D\)) is a direct sum of Hodge structures. Then, under these conditions, the Kawamata-Morrison-Totaro conjecture asserts that the automorphism group of \((X_e, D_e)\) acts with a rational polyhedral fundamental domain on the nef cone of \(X_e\).