Convex Reflexive Lattice Polygons and the Number 12

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A Bridge Between Two Lands

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Combinatorics

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Combinatorics

polytope
Combinatorics

polytope

|  

polygon (dim. 2)
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Combinatorics

polytope

| polygon (dim. 2)

Algebraic Geometry

toric varieties

toric surface (dim. 2)
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Combinatorics

polytope

| polygon (dim. 2)

Algebraic Geometry

toric varieties
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Combinatorics
polytope
| polygon (dim. 2)

Algebraic Geometry
toric varieties
| toric surface (dim. 2)
Introduction
First, an overview of our adventure...
Convex Polygons
Convex Polygons

Examples.

![Convex Polygons Examples](image)
Convex Polygons

Examples.

Nonexample.
Lattice Polygons

A lattice polygon is a polygon in $\mathbb{R}^2$ which has vertices in $\mathbb{Z}^2$. 
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Lattice Polygons

\[ \mathbb{Z}^2 \subseteq \mathbb{R}^2 \]
Example.
Nonexample.
A polygon $P$ is reflexive $\Rightarrow$ the only interior lattice point of $P$ is $(0,0)$.

Example.

Nonexample.
A polygon $P$ is reflexive
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Example.
Reflexive Polygons

A polygon $P$ is reflexive $\Rightarrow$ the only interior lattice point of $P$ is $(0,0)$.

Example.

Nonexample.
So far we have:
So far we have:

\[ (-1, 0) = P_3 \]

\[ P_1 = (1, -1) = P_4 \]

\[ P_2 = (0, 1) \]
Good News!

There are exactly 16 equivalence classes of convex reflexive lattice polygons in $\mathbb{R}^2$.
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Good News!

There are exactly 16 equivalence classes of convex reflexive lattice polygons in \( \mathbb{R}^2 \):

\[
\begin{align*}
3 & \quad 4a & \quad 4b & \quad 4c \\
5a & \quad 5b & \quad 6a & \quad 6b \\
6c & \quad 6d & \quad 7a & \quad 7b \\
8a & \quad 8b & \quad 8c & \quad 9
\end{align*}
\]
Given a convex reflexive lattice polygon $P$ with vertices $p_1, p_2, \ldots, p_n$, we define the dual of $P$ to be the polygon with vertices $q_i$ where

$$q_i = p_{i+1} - p_i$$
Dual of a Polygon

Given a convex reflexive lattice polygon $P$ with vertices $p_1, p_2, \ldots, p_n$, we define the dual of $P$ to be the polygon with vertices $q_i$ where

$$q_i = p_{i+1} - p_i$$

Note: Let $p_{n+1} = p_1$
Example 1: Dual of a Polygon

\[ q_i = p_{i+1} - p_i \text{ for } i = 1, \ldots, n, \text{ and } p_{n+1} = p_1.\]
Example 1: Dual of a Polygon

\[ q_i = p_{i+1} - p_i \] for \( i = 1, \ldots, n \), and \( p_{n+1} = p_1 \).

Example.

\begin{tikzpicture}
  \coordinate (A) at (-1,0);
  \coordinate (B) at (0,1);
  \coordinate (C) at (1,-1);
  \coordinate (D) at (2,-1);
  \coordinate (E) at (1,0);

  \draw (A) -- (B) -- (C) -- (D) -- (E) -- cycle;

  \draw[fill] (A) circle (2pt);
  \draw[fill] (B) circle (2pt);
  \draw[fill] (C) circle (2pt);
  \draw[fill] (D) circle (2pt);
  \draw[fill] (E) circle (2pt);

  \node at (-1,0) [left] {\(-1,0\) = \(p_3\)};
  \node at (0,1) [above] {\(p_2 = (0,1)\)};
  \node at (1,-1) [right] {\(p_1 = (1,-1) = p_4\)};
\end{tikzpicture}
Example 1: Dual of a Polygon

\[ q_i = p_{i+1} - p_i \text{ for } i = 1, \ldots, n, \text{ and } p_{n+1} = p_1. \]

Example.

\[ q_1 = p_2 - p_1 = (0, 1) - (1, -1) = (-1, 2) \]
\[ q_2 = p_3 - p_2 = (-1, 0) - (0, 1) = (-1, -1) \]
\[ q_2 = p_4 - p_3 = (1, -1) - (-1, 0) = (2, -1) \]
Example 1: Dual of a Polygon

\[ q_1 = (-1, 2) \]
\[ q_2 = (-1, -1) \]
\[ q_3 = (2, -1) \]
Example 1: Dual of a Polygon

\[ q_1 = (-1, 2) \]
\[ q_2 = (-1, -1) \]
\[ q_3 = (2, -1) \]
Example 1: Dual of a Polygon

So we have found that the following polygons are dual:
Example 1: Dual of a Polygon

So we have found that the following polygons are dual:
Example 2: Dual of a Polygon

\[ q_1 = p_2 - p_1 = (0, 1) - (1, 0) = (-1, 1) \]

\[ q_2 = p_3 - p_2 = (-1, -1) - (0, 1) = (-1, -2) \]

\[ q_3 = p_4 - p_3 = (0, -1) - (-1, -1) = (1, 0) \]

\[ q_4 = p_5 - p_4 = (1, 0) - (0, -1) = (1, 0) \]

\[ q_5 = p_6 - p_5 = (2, -1) - (-1, -1) = (1, 0) \]

\[ q_6 = p_7 - p_6 = (1, 0) - (2, -1) = (-1, 1) \]
Example 2: Dual of a Polygon

\[ q_1 = p_2 - p_1 = (0, 1) - (1, 0) = (-1, 1) \]
\[ q_2 = p_3 - p_2 = (-1, -1) - (0, 1) = (-1, -2) \]
\[ q_3 = p_4 - p_3 = (0, -1) - (-1, -1) = (1, 0) \]
\[ q_4 = p_5 - p_4 = (1, -1) - (0, -1) = (1, 0) \]
\[ q_5 = p_6 - p_5 = (2, -1) - (1, -1) = (1, 0) \]
\[ q_6 = p_7 - p_6 = (1, 0) - (2, -1) = (-1, 1) \]
Example 2: Dual of a Polygon

\[ q_1 = (-1, 1) \]
\[ q_2 = (-1, -2) \]
\[ q_3 = (1, 0) \]
\[ q_4 = (1, 0) \]
\[ q_5 = (1, 0) \]
\[ q_6 = (-1, 1) \]
Example 2: Dual of a Polygon

\[ q_1 = (-1, 1) \]
\[ q_2 = (-1, -2) \]
\[ q_3 = (1, 0) \]
\[ q_4 = (1, 0) \]
\[ q_5 = (1, 0) \]
\[ q_6 = (-1, 1) \]
Example 2: Dual of a Polygon

So we have found that this polygon is self-dual:
Example 2: Dual of a Polygon

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Example 2: Dual of a Polygon

So we have found that this polygon is **self-dual**:

Fact: The dual of a convex reflexive lattice polygon is also a convex reflexive lattice polygon!
Example 2: Dual of a Polygon

So we have found that this polygon is self-dual:

Fact: The dual of a convex reflexive lattice polygon is also a convex reflexive lattice polygon!

Exercise: Try all of them!
Let $\partial P$ denote the number of boundary lattice points of polygon $P$. 

Examples.
Boundary Lattice Points

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Examples.

1. $\partial P = 5$

2. $\partial P = 8$
Main Theorem. Let $P$ be a convex reflexive lattice polygon and let $P^\circ$ be its dual. Then $\partial P + \partial P^\circ = 12$. 
Main Theorem. Let $P$ be a convex reflexive lattice polygon and let $P^\circ$ be its dual. Then

$$\partial P + \partial P^\circ = 12$$
Example.
Example.

\[ \partial P + \partial P^\circ = 3 + 9 = 12 \]
Example.
Example.

\[ 5a + 7a = 5 + 7 = 12 \]
Example.

\[ 5a + 7a = 5 + 7 = 12 \]

Exercise: Verify the formula holds for all 16 polygons!
Why?
Let’s turn to algebraic geometry!
Why?

Let’s turn to algebraic geometry!

convex reflexive lattice polygon $P \leftrightarrow$ toric surface $X$
General idea:
General idea:

polygon $P$
General idea:

polygon $P \rightarrow$ fan $\Sigma$
General idea:

polygon $P$ $\rightarrow$ fan $\Sigma$ $\rightarrow$ Toric Surface $X$
Example: Polygon to Fan
Example: Polygon to Fan
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Example: Polygon to Fan

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Example: Polygon to Fan

\[ \begin{align*}
\chi &= 0 \\
y &= 0 \\
y &= -\chi
\end{align*} \]
Example: Polygon to Fan
Example: Polygon to Fan
Example: Polygon to Fan

\[
\begin{array}{c}
\bullet (-1,-1) \\
\bullet (1,0) \\
\bullet (0,1)
\end{array}
\]
Example: Polygon to Fan
Example: Polygon to Fan
Example: Polygon to Fan
Example: Polygon to Fan

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The regions $\sigma_1, \sigma_2, \text{ and } \sigma_3$ are two-dimensional cones. A fan $\Sigma$ is a union of cones.
The regions $\sigma_1, \sigma_2, \text{ and } \sigma_3$ are two-dimensional cones.
The regions $\sigma_1$, $\sigma_2$, and $\sigma_3$ are two-dimensional cones. A fan $\Sigma$ is a union of cones.
Exercise: Find the associated fan for all 16 polygons!
Example: Fan to Toric Surface
Example: Fan to Toric Surface
Example: Fan to Toric Surface

\[ \Sigma \]

\[ \sigma_1 \]
\[ \sigma_2 \]
\[ \sigma_3 \]

\[ \mathbb{P}^2 \]
Example: Fan to Toric Surface

\[ \Sigma \]

\[ \sigma_1, \sigma_2, \sigma_3 \]

\[ \mathbb{P}^2 \]
Example: Fan to Toric Surface

\[ N \]

\[ \sigma_1 \quad \sigma_2 \quad \sigma_3 \]

\[ e_1 \quad e_2 \quad -e_1 - e_2 \]
Example: Fan to Toric Surface
Example: Fan to Toric Surface

\[ \mathbb{P}^2 \]
Example: Fan to Toric Surface

From these fans, we can read what the topological properties of the corresponding toric surface are.
convex reflexive lattice polygon $P$ $\leftrightarrow$ toric surface $X$
convex reflexive lattice polygon $P \iff$ toric surface $X$

$P$ is reflexive
convex reflexive lattice polygon $P \iff$ toric surface $X$

$P$ is reflexive $\iff$ 0 is the only integral lattice point of $P$
convex reflexive lattice polygon $P \leftrightarrow$ toric surface $X$

$P$ is reflexive $\implies 0$ is the only integral lattice point of $P$
$\implies X$ is a smooth toric surface
convex reflexive lattice polygon $P \iff$ toric surface $X$

$P$ is reflexive $\implies$ 0 is the only integral lattice point of $P$
$\implies$ $X$ is a smooth toric surface

smooth:
convex reflexive lattice polygon $P \leftrightarrow$ toric surface $X$

$P$ is reflexive $\implies$ 0 is the only integral lattice point of $P$
$\implies X$ is a smooth toric surface

smooth: it’s a nice property 😊
convex reflexive lattice polygon $P \iff$ toric surface $X$

$P$ is reflexive $\implies$ 0 is the only integral lattice point of $P$
$\implies$ $X$ is a smooth toric surface

smooth: it’s a nice property 😊

It lets us use Noether’s Formula!
Noether’s Formula

For a smooth projective surface $X$, we have the following:

$$\chi(O_X) = K_X \cdot K_X + e(X)$$

$K_X$ = canonical divisor of $X$

$e(X)$ = topological Euler Characteristic of $X$

In our case (smooth toric surface):

$$\chi(O_X) = 1$$

Noether’s Formula.
Noether’s Formula. For a smooth projective surface $X$, we have the following:
Noether’s Formula. For a smooth projective surface $X$, we have the following:

$$\chi(\mathcal{O}_X) = \frac{K_X \cdot K_X + e(X)}{12}$$
Noether’s Formula. For a smooth projective surface $X$, we have the following:

$$\chi(O_X) = \frac{K_X \cdot K_X + e(X)}{12}$$

$K_X = \text{canonical divisor of } X$

$e(X) = \text{topological Euler Characteristic of } X$
Noether’s Formula. For a smooth projective surface $X$, we have the following:

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In our case (smooth toric surface): $\chi(\mathcal{O}_X) = 1$
Noether’s Formula. For a smooth projective surface $X$, we have the following:

$$\chi(\mathcal{O}_X) = \frac{K_X \cdot K_X + e(X)}{12}$$

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$e(X) = \text{topological Euler Characteristic of } X$

In our case (smooth toric surface): $\chi(\mathcal{O}_X) = 1$
Noether’s Formula. For a smooth projective surface $X$, we have the following:

$$
\chi(\mathcal{O}_X) = \frac{K_X \cdot K_X + e(X)}{12}
$$

$K_X = \text{canonical divisor of } X$

$e(X) = \text{topological Euler Characteristic of } X$

In our case (smooth toric surface): $\chi(\mathcal{O}_X) = 1$

Noether’s Formula.

$$
K_X \cdot K_X + e(X) = 12
$$
Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]
Noether’s Formula

Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:
Noether’s Formula

Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:

\[ K_X \cdot K_X = \partial P \]
Noether’s Formula

Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:

\[ K_X \cdot K_X = \partial P \]
\[ e(X) = \partial P^\circ \]
Noether’s Formula

Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:

\[ K_X \cdot K_X = \partial P \]
\[ e(X) = \partial P^o \]

Noether’s Formula.
Noether’s Formula

Noether’s Formula.

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:

\[ K_X \cdot K_X = \partial P \]
\[ e(X) = \partial P^\circ \]

Noether’s Formula.

\[ \partial P + \partial P^\circ = 12 \]
Noether’s Formula

\[ K_X \cdot K_X + e(X) = 12 \]

2 Fun Facts:

\[ K_X \cdot K_X = \partial P \]
\[ e(X) = \partial P^\circ \]

Noether’s Formula.

\[ \partial P + \partial P^\circ = 12 \]

The Main Theorem!
The Big Result

The Main Theorem is another way of stating Noether’s Formula for 2-dimensional smooth toric varieties!
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