Notes on Weyl modules for semisimple algebraic groups

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Over many decades of development, there has been some evolution in the language and notation used for *Weyl modules* in the theory of semisimple algebraic groups in prime characteristic. Here we survey this evolution briefly, in the hope of clarifying what goes on in the literature. We also discuss briefly the related notion of *tilting module*. Since our emphasis is on rational representations, it is convenient to fix an arbitrary algebraically closed field K of characteristic $p > 0$ and take G to be a connected semisimple algebraic group over K which is moreover simply connected. (Similar ideas apply to a connected reductive group whose derived group is simply connected, but with added bookkeeping needed for characters of the central torus.)

We fix a pair of opposite Borel subgroups B^+ and B^- intersecting in a maximal torus T and corresponding to respective choices of positive and negative roots, while $W = N_G(T)/T$ is the Weyl group. Though notation for root systems and related concepts differs widely in the literature, we take Φ to be the root system and $X := X(T)$ the character group of T, identifiable in our simply connected case with the full lattice of integral weights. Relative to the positive roots Φ^+ we get a cone of dominant integral weights in X, denoted X^+. Since G is simply connected, the half-sum of positive roots ρ (= sum of fundamental dominant weights) lies in X^+.

1 Weyl modules

1.1

First we recall some early history. An important milestone was Chevalley's uniform proof (1955) that every semisimple Lie algebra over $\mathbb{C}$ has a $\mathbb{Z}$-form with fairly simple structure constants. Partly in order to bring algebraic groups into the picture, Kostant (1965) showed how to obtain from a Chevalley basis of the Lie algebra a "divided power" $\mathbb{Z}$-form of its universal enveloping algebra. By incorporating certain integers $k!$ into the denominators, one is able to construct formal exponentials corresponding to nilpotent elements of the Lie algebra; these in turn allow one to construct (adjoint) "Chevalley groups".

This theme was developed more fully for all isogeny types including simply connected groups by Steinberg [13] in his 1967–68 Yale lectures (see §12), then written up in Borel's lectures [3]. In particular, any choice of an "admissible lattice" in a simple module in characteristic 0 permits reduction modulo p to get a (usually non-simple) rational G-module in characteristic p.

In particular, choice of a *minimal* admissible lattice leads to a *cyclic* G-module with a unique simple quotient: this G-module is generated by a "maximal" or "highest weight" vector v^+ of weight λ relative to T. At first such objects were given *ad hoc* names such as "standard cyclic" or left unnamed. In one notational scheme the original module in characteristic 0 is written V_λ or $V(\lambda)$ and a bar is placed over this notation to denote the reduction modulo p. See for example the treatments in [7] and in the last chapter of [8].

1.2

The name *Weyl module* first occurs in the work of Carter and Lusztig [4, 3.2] for the case of a special linear group G. They actually start with the general linear group, which though not semisimple is initially more natural for the characteristic p analogue of Schur–Weyl duality. Here they construct directly a copy of the module $\overline{V}_\lambda$ inside the rth tensor power of the natural n-dimensional module $\overline{V}$ for their group, with λ described concretely as a partition of r into at most n parts written in non-increasing order. (Taking T in diagonal form, its characters are conveniently viewed as partitions.) In fact, they first used the dual partition μ to define a module denoted $\overline{V}^\mu$, which later (in 3.7) they renamed $\overline{V}_\lambda$.

In this setting the term "Weyl module" is consistent with Weyl's original construction of a finite dimensional simple module in characteristic 0. Moreover, the formal character and dimension of $\overline{V}_\lambda$ are still given by Weyl's character and dimension formulas even in prime characteristic, though the module structure may get quite complicated. Perhaps with the latter feature in mind, the name "Weyl module" was almost immediately applied to the modules $\overline{V}_\lambda$ obtained by reduction modulo p for any semisimple G, and the simpler notation $V(\lambda)$ became standard. Following the convention popularized by Dixmier, BGG, and others for simple highest weight modules in characteristic 0, the unique simple quotient of $V(\lambda)$ in characteristic p soon got denoted $L(\lambda)$.

Formally, the dual module $V(\lambda)^*$ has "reversed" structure as a G-module; in particular, it has a unique simple submodule $L(\lambda)^*$. Recall the standard fact that $L(\lambda)^* \cong L(\lambda^*)$, where $\lambda^* = -w_\circ\lambda \in X^+$. Here $w_\circ$ is the longest element of W, equal to -1 unless the Lie type is A_n for $n > 1$, D_n for n odd, or E_6.

1.3

The modules $V(\lambda)$ have the advantage of being constructed explicitly by reduction modulo p from known modules in characteristic 0. Moreover, work of Warren Wong (and later Jantzen) on "contravariant forms" led to effective computational methods for the weight multiplicities of $L(\lambda)$ for weights and primes that are not too large. These methods are recursive and therefore potentially lengthy; moreover, they fail to give theoretical insight.

Some results of the author in [7] imitated the infinite dimensional theory in characteristic 0 by carrying over versions of Harish-Chandra's central characters related to "linkage" of weights under the ρ-shifted action of the Weyl group, at first just taken modulo p but later encoded in an affine Weyl group relative to p.

1.4

At first the technical methods were limited to some algebraic and combinatorial manipulation. In particular, it was unclear how to characterize $V(\lambda)$ intrinsically in the category of finite dimensional rational G-modules. Jantzen's algebraic methods in type A_n could show that $V(\lambda)$ plays the role of a universal highest weight module, but such a characterization remained elusive for other types.

2 Induced modules

2.1

There is a more intrinsic way to construct finite dimensional rational G-modules, which works at first uniformly in arbitrary characteristic. This is an algebraic group version of "induction" from a proper subgroup, here taken to be a Borel subgroup B. Such a construction goes back many decades, at least to the 1956–58 Chevalley seminar, and by now it has been well codified in a group scheme setting (for example, in Jantzen's book [10]).

When B is decomposed into a semidirect product TU, with T a maximal torus of G, each character $\lambda \in X$ yields a rational B-module if we make U act trivially on the 1-dimensional space involved. On the other hand, the *flag variety* G/B is a projective variety with natural G-action on the left. Associating to λ a *line bundle* $\mathcal{L}(\lambda)$ on G/B, we get sheaf cohomology groups $H^i(G/B, \mathcal{L}(\lambda))$ which can only be nonzero in the range $i = 0, 1, \ldots, N$ with $N = \dim G/B$. Recall that N can also be characterized as $|\Phi^+|$. Here each cohomology group is a finite dimensional vector space over the underlying field and acquires a natural rational (left) action of G. For brevity, write $H^i(\lambda)$ in place of $H^i(G/B, \mathcal{L}(\lambda))$.

It is known (in all characteristics) that the space of global sections $H^0(\lambda)$ is nonzero if and only if $\lambda \in X^+$. However, to avoid sign complications, the notion of dominance is defined here relative to a fixed Borel subgroup B^+ while the flag variety is defined using the *opposite* Borel subgroup $B := B^-$.

2.2

Following special case treatments by himself and others, Kempf arrived in 1976 at a general cohomology vanishing theorem for the group G in characteristic p: the space $H^i(\lambda)$ is 0 for all $i > 0$ whenever $\lambda \in X^+$. This important result recovers part of the classical theorems of Borel–Weil and Bott in characteristic 0, where Kodaira vanishing comes into play. But examples show that there can be non-vanishing cohomology in multiple degrees for non-dominant weights. Kempf's proof relies heavily on algebraic geometry, in particular the features of Schubert varieties in G/B. Later proofs by Andersen and Haboush were shorter but also geometric in flavor.

Combined with an Euler characteristic argument, Kempf's theorem shows that over K the formal character and dimension of $H^0(\lambda)$ are given by Weyl's formulas. Moreover, $H^0(\lambda)$ has a unique simple submodule, which is isomorphic to $L(\lambda)$. The sought-for universal property of $V(\lambda)$ now follows from Kempf vanishing by an elementary argument, formulated following my suggestions as Satz 1 in Jantzen's 1980 paper [9].

2.3

In Jantzen's 1987 book [10], which was expanded into a second edition in 2003 while retaining the foundational material, he relied heavily on the methods of algebraic geometry and scheme theory. In this spirit, he viewed the "induced" modules $H^0(\lambda)$ as basic objects and redefined Weyl modules as duals of these. To be precise, he *defines* $V(\lambda) := H^0(-w_\circ\lambda)^*$, so in particular the formal characters of $V(\lambda)$ and $H^0(\lambda)$ agree while $L(\lambda)$ is the unique simple quotient of the former but the unique simple submodule of the latter. (See [10, II.2.13, II.5.11] along with his treatment of Kempf's theorem in Chapter II.4.)

This cohomology setting also identifies the renamed $V(\lambda)$ with the highest cohomology group $H^N(w_\circ \cdot \lambda)$ via Serre duality. Here the "dot" action of W is given by $w \cdot \nu := w(\nu + \rho) - \rho$ for all $\nu \in X$.

2.4

Thanks to Kempf's vanishing theorem, the two collections of G-modules $V(\lambda)$ (in the original construction) and $H^0(\lambda)$ with $\lambda \in X^+$ are interchangeable via taking duals (though the dualization is a bit twisted when $w_\circ \neq -1$ in W). The modules in the first collection are not defined intrinsically in characteristic p but their formal characters and dimensions are known in principle. These modules can be constructed and studied concretely by algebraic methods (and to some extent by computer algorithms). The modules in the second collection are defined intrinsically and can be studied using methods of algebraic geometry. But finer details such as their formal characters and dimensions require an indirect comparison with the $V(\lambda)$. Either characterization leaves largely open the problem of determining the multiplicities of composition factors, though Lusztig's conjectures and the theorems of Andersen-Jantzen-Soergel, Fiebig et al. for large p are important guideposts for the eventual theoretical solution.

2.5

While Jantzen settles on the notations $V(\lambda)$ and $H^0(\lambda)$, Donkin proposed a more symmetric-looking substitute: write $\Delta(\lambda)$ in place of $V(\lambda)$ and $\nabla(\lambda)$ in place of $H^0(\lambda)$. The symbols contain a visual reminder that the first module has $L(\lambda)$ as its head while the second has $L(\lambda)$ as its socle. (Even so, the symbols are occasionally reversed in the literature.) Probably the main advantage of these symbols is their neutrality in more axiomatic settings such as *highest weight categories* (defined by Cline–Parshall–Scott), where module categories of varied origin may satisfy the axioms. Here one typically works with "standard" and "costandard" objects such as $\Delta(\lambda)$ and $\nabla(\lambda)$, indexed by a partially ordered set of "weights".

3 Tilting modules

3.1

In the absence of complete reducibility, modular representation theory for G is usually studied in a wider context than that of simple modules: the category $\mathcal{C}$ of all *rational* G-modules. These need not be finite dimensional, though each is locally so in the sense that any vector lies in a finite dimensional G-submodule. Perhaps the most important examples of infinite dimensional rational G-modules are the injective modules, which play an essential role in [10]: here $\mathcal{C}$ has enough injectives. (On the other hand, Donkin observed that $\mathcal{C}$ has no projective objects.)

In category $\mathcal{C}$ one can single out modules V with a *good filtration*: by definition $0 = V_0 \subset V_1 \subset V_2 \subset \dots$ with $V = \bigcup_i V_i$, where the V_i are finite dimensional G-modules and each quotient V_i/V_{i-1} is isomorphic to some $H^0(\lambda_i)$. Dually one considers modules V with a *Weyl filtration*: here there is again an ascending filtration but with quotients isomorphic to various Weyl modules $V(\lambda_i)$. (See [10, II.4.16].) It is clear, from rank 1 examples where V is either a Weyl module or an induced module and is indecomposable with two composition factors $L(\lambda) \neq V(\lambda)$ and $L(\mu) = V(\mu)$, that the existence of one type of filtration does not imply the existence of the other. A basic fact is that any injective G-module has a good filtration.

3.2

By definition, a *tilting module* in $\mathcal{C}$ is a G-module which has both a good filtration and a Weyl filtration. Obvious examples are those $L(\lambda)$ for which $V(\lambda) = L(\lambda) = H^0(\lambda)$; here λ is minimal in its linkage class in X^+. More generally, it can be shown that for every $\lambda \in X^+$ there exists a finite dimensional indecomposable tilting module $T(\lambda)$ having $L(\lambda)$ as its head and also as its socle. It has λ as unique highest weight, with multiplicity 1, but is usually not a highest weight module. In Chapter E added in the second edition of his book [10], Jantzen discusses all of this with references to the expanding literature. The starting point is a much-cited paper of Donkin [6], which was motivated especially by Ringel's formulation of tilting in a way which could be applied to algebraic groups.

3.3

The term "tilting" came to be applied here in a devious way, which Jantzen explains briefly with reference to the representation theory of finite dimensional associative algebras and Schur–Weyl duality for general linear and symmetric groups. (Strictly speaking, one should refer to "partial tilting modules" in the algebraic group setting.) In any case, the terminology neatly fills a gap in our situation even if it has no intuitive meaning there. By now the "tilting" notion has in fact been used very broadly, as seen for example in [1, 2, 11].

3.4

Two basic facts have emerged from the study of good and Weyl filtrations along with tilting modules: (1) the tensor product of modules having a good or Weyl filtration has the same property; (2) restricting a module with a good or Weyl filtration to a Levi subgroup of a parabolic subgroup of G yields a module with the same property. As recalled in an appendix to [11], knowing both (1) and (2) for all modules with a good filtration (resp. a Weyl filtration) is equivalent to knowing that (1) and (2) hold for all tilting modules.

In the case of (1), the initial question which I raised in the late 1970s was whether the tensor product of two Weyl modules must have a Weyl filtration. The question was only weakly motivated at the time, partly by the roughly analogous infinite dimensional situation when one tensors a Verma module with a finite dimensional irreducible module in characteristic 0. In characteristic $p > 0$ my question was at least compatible with the classical fact that the product of any two Weyl characters is known to be a sum of various Weyl characters.

The question was shown to have a positive answer in many cases by Jian-pan Wang [14], a participant in my 1980 lectures in Shanghai. Then Donkin [5] extended this to almost all cases and treated restrictions to Levi subgroups in the spirit of (2). A general (and case-free) proof of (1) and (2) using Frobenius splitting was then given by Mathieu [12].

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