Tensor-closed objects of the BGG category $\mathcal{O}$

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This note responds to a question raised by Zhaoting Wei on MathOverflow in July 2015 (see http:mathoverflow.net/questions/211535, where I outlined a very elementary proof in rank 1):

If an object M in the BGG category $\mathcal{O}$ is "tensor-closed" (meaning that $M \otimes N$ is in $\mathcal{O}$ whenever N is), must $\dim M < \infty$*?*

The answer is yes. This is part of the folklore of the subject but apparently not written down explicitly. Of course it then shifts attention to tensoring in $\mathcal{O}$ just with finite dimensional M, which has been a standard emphasis in the literature. Here we provide a proof, relying only on the most basic facts about $\mathcal{O}$ (see for example the 1976 BGG paper [2] or the early chapters of my textbook [4], whose notational conventions are used here).

Fix a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteritic 0, as well as a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$ be the resulting Cartan decomposition. Write $U(\mathfrak{g})$ for the universal enveloping algebra. Recall that the category $\mathcal{O}$ of $U(\mathfrak{g})$-modules consists of finitely generated modules M which are direct sums of their weight spaces relative to $\mathfrak{h}$, while the action of $\mathfrak{n}$ is locally finite dimensional. It follows from the axioms that each $M \in \mathcal{O}$ is finitely generated. Moreover, $\mathcal{O}$ is closed under taking submodules and taking quotients. All weight spaces M_ν (with $\nu \in \mathfrak{h}^*$) of $M \in \mathcal{O}$ are finite dimensional, even though M itself is usually infinite dimensional.

Objects in $\mathcal{O}$ include the universal highest weight modules (Verma modules) $M(\lambda)$ and their unique simple quotients $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$; the latter exhaust the simple objects in $\mathcal{O}$. Moreover, each $M \in \mathcal{O}$ has a finite Jordan–Hölder series with simple subquotients.

Proposition. Suppose $M \in \mathcal{O}$ satisfies the property: $M \otimes N \in \mathcal{O}$ for all $N \in \mathcal{O}$. Then $\dim M < \infty$.

Proof. Since M has finite length, and $\mathcal{O}$ is closed under taking subquotients, it is clearly enough to assume that $M = L(\lambda)$ for some $\lambda \in \Lambda$. Also, we may take $N = M(\mu)$ to be a Verma module. Set $T := M \otimes N$. Assuming that M is infinite dimensional, we aim to derive a contradiction. Of course, it is the finite generation axiom that T violates, but this is hard to prove directly. Instead, the idea is to show T fails to have finite length, essentially because its weight space dimensions grow "too fast" in this situation.

Since T is assumed to lie in $\mathcal{O}$, it has a formal character [4, 1.15]. So we can extend the reasoning in the standard BGG argument from [1, §4, Lemma 5] outlined as an exercise in [4, 3.6]. The only change is that the weights of $L(\lambda)$ form an infinite list, compatible with the usual partial ordering of weights; we write them as $\lambda_1, \lambda_2, \ldots$. Then the formal character of T involves all characters of Verma modules having highest weights $\lambda_i + \mu$, which contradicts the finite length of T as an object in $\mathcal{O}$.

References

1. J. Bernstein, I.M. Gelfand, S.I. Gelfand, *Structure of representations generated by vectors of highest weight*, Funktional. Anal. i Prilozhen. **5**, no. 1 (1971), 1–9; English transl., Funct. Anal. Appl. **5** (1971), 1–8 (reprinted in [3], 556–563).

2. ———, *A category of $\mathfrak{g}$-modules*, Funktional. Anal. i Prilozhen. **10**, no. 2 (1976), 1-8; English trans., Funct. Anal. Appl. **10** (1976), 87–92 (reprinted in [3], 596–601).

3. I.M. Gelfand, *Collected Papers*, vol. II, Springer, 1988.

4. J.E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Amer. Math. Soc., 2008.