Partial orderings of an affine Weyl group

June 17, 2015 (revised December 23, 2015)

In this note we draw together some of the scattered literature dealing with several partial orderings of affine Weyl groups. Most of the theory was developed as a tool in the study of modular representations for groups of Lie type, but here we focus just on an affine Weyl group W_a in its elementary geometric setting while sometimes invoking also its structure as a Coxeter group. While notation and terminology vary considerably in the cited sources, the conventions here usually follow [6].

I'm grateful to Florian Herzig for calling my attention to problematic references needed for applications of modular representation theory to his current joint preprint posted at [5]. The help of a colleague in translating Jian-pan Wang's paper [13] into English is also appreciated.

1 Root systems and Weyl groups

The starting point for construction of an affine Weyl group is a finite Weyl group W together with an irreducible root system Φ. Here a Weyl group is a finite (irreducible) reflection group satisfying the crystallographic condition. Textbook references include [2, VI, §4] and [6, Chap. 4]. We may assume that W acts faithfully on a finite dimensional real euclidean space V with inner product $\langle \lambda, \mu \rangle$.

In terms of root systems, W arises as the group generated by the reflections s_{α_i} for some fixed choice of simple roots $\alpha_1, \ldots, \alpha_\ell$ of Φ; the set S of these simple reflections generates W as a Coxeter group. For any $\alpha \in \Phi$ and $\lambda \in V$, the reflection s_α is given by $s_\alpha \lambda := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha^\vee$, where $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ is the *coroot* of α. (Only the non-isomorphic root systems of types B_ℓ, C_ℓ of rank $\ell \geq 3$ give rise to isomorphic Weyl groups.)

Fixing a set of simple roots partitions Φ into sets of positive and negative roots. Since $s_\alpha = s_{-\alpha}$ for any root α, we usually work just with $\alpha > 0$. The orthogonal hyperplane through 0 is denoted H_α. Removing the finitely many hyperplanes H_α from V leaves an open set whose $|W|$ connected components are called *Weyl chambers*. The closure $\overline{C}$ of any such chamber C is then a fundamental domain for the natural action of W on V. Write C^+ for the *dominant* Weyl chamber: $\lambda \in C^+$ iff $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha > 0$ (or just simple roots, whose hyperplanes intersect the closure of C^+ in its *walls*).

2 Affine Weyl groups

Now the coroots $\alpha^\vee$ generate a lattice in V on which W acts naturally, and the resulting semidirect product is by definition the *affine Weyl group* W_a.

Sources (with varying notation and terminology) include the definitive treatise by Bourbaki [2, Chap. IV], the older paper by Iwahori and Matsumoto [7, § 1], and my more recent textbook [6, Chap. 4]. *Caution*: There is a Langlands dual version which instead uses the root lattice, as in Verma [12] and many other papers or in the book by Jantzen [10]; this version arises naturally in modular representation theory but won't concern us here. This duality interchanges the affine Weyl groups of types B_ℓ and C_ℓ, which are non-isomorphic for $\ell \geq 3$.

While the semidirect product description of W_a is adequate for some purposes, the viewpoint of groups generated by reflections is often more useful. Affine geometry ignores the origin, so we can get infinitely many translates of H_α by associating to each $n \in \mathbb{Z}$ a parallel hyperplane $H_{\alpha,n}$. The convention that $\alpha > 0$ associates to the hyperplane $H := H_{\alpha,n}$ upper and lower open half-spaces denoted H^+ and H^-; these are divided by H. For example, H^+ is obtained by translating the half-space for H_α which contains $\alpha > 0$. Denote by $\mathcal{H}$ the collection of all these affine hyperplanes.

Now associate to each $H \in \mathcal{H}$ an *affine reflection* s_H having H as fixed hyperplane. Since H is uniquely determined by a root $\alpha > 0$ along with $n \in \mathbb{Z}$, we may write $H = H_{\alpha,n}$ and $s_H = s_{\alpha,n}$. Explicitly,

$$s_{\alpha,n}(\lambda) := s_\alpha \lambda + n\alpha^\vee = \lambda - (\langle \lambda, \alpha^\vee \rangle - n)\alpha^\vee.$$

From the definitions we see that $ws_{\alpha,n} = s_{\beta,n}w$ whenever $w \in W$, where $\beta := w\alpha$ (which by our convention should then be replaced by $-\beta$ if $\beta < 0$). It turns out that W_a is generated by these affine reflections and is a Coxeter group relative to the finite generating set S_a consisting of the simple reflections in S along with one extra reflection $s_0 := s_{\tilde{\alpha},1}$ where $\tilde{\alpha}$ is the unique highest root.

The connected components of the open set obtained by removing all hyperplanes in $\mathcal{H}$ from V are called *alcoves*; write $\mathcal{A}$ for the set of all these. Each Weyl chamber is a union of alcoves. A basic fact is that the closure $\overline{A}$ of any fixed alcove A is a fundamental domain for the action of W_a on V. Nonempty intersections of hyperplanes with $\overline{A}$ are called the *walls* of A; there are $\ell + 1$ of these. Call an alcove *dominant* if it lies in C^+. There is a unique dominant alcove whose closure contains 0, which we denote by $A_\circ$. Its walls are determined by the $\ell + 1$ hyperplanes indexed by S_a. Another basic fact is that W_a permutes the alcoves in $\mathcal{A}$ simply transitively, so alcoves may be labelled unambiguously as $wA_\circ$ for $w \in W_a$.

3 The functions ℓ and d

In any Coxeter group, such as W_a, the choice of a canonical generating set of involutions (here S_a) determines a length function ℓ with $\ell(1) = 0$. As in the case of W, there is a natural geometric interpretation of ℓ for W_a (which was developed already by Iwahori–Matsumoto in [7]). First use the simple transitivity of W_a on $\mathcal{A}$ to assign to each alcove A a length $\ell(A) := \ell(w)$ when $A = wA_\circ$. Then: *For any alcove $A \in \mathcal{A}$, $\ell(A)$ is the number of affine hyperplanes separating A from $A_\circ$.* (For an exposition which takes advantage of Deodhar's treatment of "root systems" see [6, 4.5].)

A signed version of the length function has also been used extensively in the literature, for example in Jantzen's early paper [8, § 6] (see also [10, II.6.6]). For each $\alpha > 0$, an alcove $A \in \mathcal{A}$ lies between two hyperplanes H_{α,n_α} and $H_{\alpha,n_\alpha+1}$. Define $d(A) := \sum_{\alpha>0} n_\alpha \in \mathbb{Z}$. It follows from the geometric characterization of the length function that $-\ell(A) \leq d(A) \leq \ell(A)$ and that $d(A) \equiv \ell(A) \pmod 2$.

4 Partial orderings of W_a

The ideas special to W_a go back at least as far as Verma's paper [12, §1]. As before we denote typical alcoves in $\mathcal{A}$ by letters such as A, B, sometimes with added primes or subscripts. (Jantzen originally used the letter C, which suggests *chamber* or in German *Kammer*, but we reserve that term for the Weyl chambers in V relative to W.)

(1) Every Coxeter group has a natural partial ordering relative to the length function, called the *Bruhat ordering* (more accurately, the Chevalley–Bruhat ordering, since it first arose in Weyl groups relative to Chevalley's study of a geometric ordering for Schubert varieties in the flag variety). Any element of the Coxeter group conjugate to a canonical generator is called a *reflection*. In W or W_a this agrees with the respective geometric notion of orthogonal reflection relative to a root or affine reflection across a hyperplane $H \in \mathcal{H}$.

Initially one writes $x \leq w$ if $x = w$ or else $w = tx$ for some reflection t satisfying $\ell(x) < \ell(w)$. This generates a partial ordering. The definition agrees with that obtained by using right multiples by reflections and was explored in detail by Verma, Deodhar, and others (see for example [6, 5.9]). It turns out that every (upward) *path* in the partial ordering from x to w can be refined to one in which lengths increase by 1 at each step (though the reflections involved need not have length 1.)

In the special case of W_a, the bijection between group elements w and alcoves $wA_\circ$ allows one to view the Bruhat ordering as a partial ordering of $\mathcal{A}$. For this Verma [12] introduces the special notation $w' \overset{\mathrm{B}}{\leq} w$ if $B = w'A_\circ$ and $A = wA_\circ$, while Wang [13] writes $w' \leq_b w$.

(2) A more geometric ordering is natural for W_a, as Verma observed. He defines an *affine ordering*, starting with the relation $B = sA$ when $s = s_H$ is a reflection (with $H \in \mathcal{H}$) for which $B \subset H^-$ and $A \subset H^+$. This defines an *upward* reflection from B to A. (Elementwise, for $\lambda \in A$ the definition translates into the condition that $\langle \lambda, \alpha^\vee \rangle > n$ if $s = s_{\alpha,n}$.) In general he writes $B \overset{\mathrm{A}}{\leq} A$ if $B = A$ or else there is an upward path using affine reflections from B to A. (Wang also calls this the affine ordering but writes $B \leq_a A$, while Lusztig just uses the notation $B \leq A$ in [11, 1/5].)

In his 1977 paper Jantzen [8] (cf. [10, II.6.4–6.5]) introduced a similar partial ordering of weights (applicable to the alcoves for W_a) as a tool in modular representation theory, mimicking an earlier "strong linkage" ordering of W suggested by representation theory in characteristic 0 (Verma, BGG, Jantzen). His notation translates as $B \uparrow A$, which we also write. Outside C^+ this differs substantially from the Bruhat ordering on W_a (or $\mathcal{A}$) relative to the function ℓ: as Verma remarks, if $H = H_{\alpha,n}$ and $B = s_H A \uparrow A$, then $\ell(B) < \ell(A)$ if $n > 0$ but $\ell(B) > \ell(A)$ otherwise. We distinguish these two cases by referring to $H_{\alpha,n}$ as a *good* hyperplane if $n > 0$, *bad* otherwise. In other words, H is good if it intersects C^+ nontrivially, bad otherwise. In particular, the walls of this Weyl chamber are bad; then the corresponding reflections $s_\alpha = s_{\alpha,0}$ need to be handled with some care below.

The ordering $B \uparrow A$ is geometrically natural and behaves well with respect to the function d, as Jantzen proves in [8, Lemma 4] (cf. [10, II.6.6]):

Lemma. (Jantzen) *Given any two distinct alcoves $A, B \in \mathcal{A}$ with $B = sA$ for some reflection s, we have $B \uparrow A$ iff $d(B) < d(A)$.*

(3) There is a third useful partial ordering of $\mathcal{A}$, a more restrictive version of $\uparrow$, for which one requires that all reflections involved are relative to *good* hyperplanes H. For this relation we write $B \upuparrows A$, a notation apparently first used by Ye in [14]. However, Wang [13] calls $\upuparrows$ the *strong linkage ordering* and uses the notation $\uparrow$ for it, following the usage of Andersen [1] in the setting of modular representation theory. Later papers quoting Andersen such as [14, 4] contribute to the notational ambiguity. (We discuss Andersen's Proposition 1 in detail below.)

5 Comparison theorem

As Verma asserted (with some details omitted) in [12], the affine and Bruhat orderings turn out to coincide on pairs of *dominant* alcoves. In fact, the $\uparrow\uparrow$ ordering also agrees with both of these orderings on dominant alcoves. (Easy examples show that the three orderings can differ from each other on some pairs of alcoves.) Apparently the proof of these equivalences was first completed by Ye [15] (in a paper published only in Chinese), relying to some extent on case-by-case study of root systems. But a more unified and thorough treatment was soon given by Wang [13, Thm. 4.3] (also in Chinese).

Theorem 1. *Let $A, B \subset C^+$, where $A = wA_\circ$ and $B = w'A_\circ$ for $w, w' \in W_a$. Then the three relations $B \uparrow A$, $B \uparrow\uparrow A$, and $w' \overset{\mathrm{B}}{\leq} w$ are equivalent.*

The main issue we focus on here is the subtle implication: $B \uparrow A$ *implies* $B \uparrow\uparrow A$ *whenever A, B are dominant alcoves.* (Of course, the reverse implication follows immediately from the definition of the orderings.) One special case is easy to analyze directly. Say $B \uparrow A$, where both alcoves $A, B \subset C^+$ and moreover $B = s_H A$. Since C^+ (like any Weyl chamber) is a convex set in the real topology, a line segment joining a point in A to the reflected point in B must meet H in another point of C^+. In particular, H meets C^+ nontrivially, which by Verma's characterization ensures that H is good. Thus $B \uparrow\uparrow A$.

Example. If $A, B \subset C^+$ and a path involving more than one affine reflection leads from B up to A, there is no reason why the intermediate alcoves should be dominant (or the reflecting hyperplanes be good). It is easy to illustrate this when W_a arises from the Weyl group of type G_2. Take $\alpha = \alpha_1$ to be short and $\beta = \alpha_2$ to be long. Dominant alcoves other than $A_\circ$ can then be labelled by strings of integers $0, 1, 2$ to abbreviate products of the elements in S_a. Let the alcove B correspond to 10 and reflect it first upward across the hyperplane $H_{\alpha,1}$ to reach the non-dominant alcove labelled 12120 (which has ℓ-value 5 but d-value just 3). Next reflect upward across the bad hyperplane $H_{\beta,0}$ (which defines a wall of C^+) to reach the dominant alcove B labelled by 1210. Notice that the d-values increase by 1 at each step.

In this case we can find an alternate path of the same length within C^+, by reflecting A to the alcove labelled 210 and then to B. But in general the existence of such a dominant path is far from obvious.

6 Dominant paths

A second theorem, often discussed in tandem with Theorem 1, deals with the possible upward paths in one of these orderings between a pair of *dominant* alcoves $A, B \subset C^+$. The unavoidable problem (as we just saw) is that some such paths, even those of minimal length, might involve non-dominant alcoves. However, it turns out that this can always be avoided. Wang discusses versions of the following theorem, here stated for the affine ordering [13, Thm. 3.5]:

Theorem 2. *Let $A, B \subset C^+$, with $B \uparrow A$. Then there exist dominant alcoves A_i satisfying $B = A_0 \uparrow A_1 \uparrow \ldots \uparrow A_r = A$ along with $d(A) - d(A_i) = d(A) - i$ for all i.*

Recalling that $d = \ell$ for dominant alcoves, this provides a shortest possible dominant path in the $\uparrow$ ordering. The approaches to a proof in the literature, including [13], usually obtain it as a byproduct of the proof of Theorem 1. But it remains to be seen what the most efficient strategy is. At any rate, there are similar statements for the Bruhat ordering and (in [1]) for the more restrictive ordering studied by Andersen (using the same symbol $\uparrow$ as Jantzen). Below we revisit Andersen's variant of Theorem 2, after recalling a result of Jantzen which he invokes.

7 A key result of Jantzen

The following proposition due to Jantzen appears in his 1977 paper [8, Satz 9] (and is reproduced in his book [10, II.6.8]):

Proposition (Jantzen) *Let A be a dominant alcove. Assume that $B := s_{\alpha,n} A \uparrow A$ by a single good reflection. (Thus $B \upuparrows A$). Use the fact that the closure of C^+ is a fundamental domain for W to find $w \in W$ for which $wB \subset C^+$. Then there exist dominant alcoves A_i satisfying $wB = A_0 \uparrow A_1 \uparrow \ldots \uparrow A_r = A$ as well as $d(A) - d(A_i) = d(A) - i$ for all i.*

Note that Jantzen only assumes at first that $n \geq 0$, but if $n = 0$ he can simply choose $w = s_\alpha = s_{\alpha,0}$ and recover $wB = A$. So in this case nothing has to be proved. We remark further, based on our previous discussion, that the $\uparrow$ arrow between each pair of dominant alcoves in the statement can be replaced by $\upuparrows$.

8 Revisiting a result of Andersen

In his concise paper [1], Andersen refines (using his notion of "strong linkage") earlier results on the role of linkage of weights in modular representation theory, in the setting of sheaf cohomology of line bundles on a flag variety. As in Verma's formulation, Andersen uses the Langlands dual version of W_a with the root lattice expanded by a prime factor p.

An important step toward the main theorem of [1] is his Proposition 1, which relies on a partial ordering of weights (or alcoves) which he denotes by $\uparrow$ but with the qualification $n \geq 0$. This extra condition is needed in particular for his application of Jantzen's result quoted above. He is mainly concerned with dominant (or almost-dominant) weights, but his use of the symbol $\uparrow$ creates problems in further literature which quotes his result, notably papers by Ye [14] and by Doty–Sullivan [4]. Apart from this, the conclusion of Andersen's proof is too briefly written. So we rework the steps more carefully, in the framework of the usual affine Weyl group W_a (and with Jantzen's refinements using the d-function).

Proposition (Andersen) *Let $A, B \in \mathcal{A}$ be dominant alcoves. Assume that $B \uparrow A$ relative to a sequence of reflections $s_{\alpha,n}$ for which $n \geq 0$. Then there exist dominant alcoves A_i satisfying $B = A_0 \uparrow A_1 \uparrow \ldots \uparrow A_r = A$ along with $d(A) - d(A_i) = d(A) - i$ for all i.*

Recalling as before that $d = \ell$ for *dominant* alcoves, we see that the conclusion connects B with A by a sequence of upward affine reflections in C^+ across the permitted hyperplanes, with the length difference at each step being $+1$. As remarked earlier, the case $n = 0$ becomes trivial for Jantzen's method, so we usually ignore this possibility in what follows.

The proof of the proposition involves two basic tools: Jantzen's proposition above, along with the following observation: Start with a pair of alcoves $B' \uparrow A'$ (where $A' \subset C^+$) related by a single affine reflection $s = s_{\alpha,n}$ with the extra condition $n > 0$. Using the fact that the closure of C^+ is a fundamental domain for the action of W, find an element $w \in W$ for which $wA' \subset C^+$. The claim is that $(*)$ $wB' \uparrow wA'$ by a single good reflection.

The claim follows easily from Andersen's argument, using the assumption that $n > 0$: Since W normalizes the translation lattice in W_a, the definitions show that $ws = s_{\beta,n}w$ with $\beta = w\alpha$ (hence $\beta^\vee = w\alpha^\vee$) while n is unchanged (see for example [6, Prop. 4.1]). Using this notation, the W-invariance of the inner product on V then yields the following element-wise computation for any $\chi \in A'$:

$$\langle w\chi, \beta^\vee \rangle = \langle w\chi, w\alpha^\vee \rangle = \langle \chi, \alpha^\vee \rangle \geq n > 0.$$

Since $w\chi \in C^+$, this forces $\beta > 0$. Next we compute for $\lambda = s_{\alpha,n}\chi$:

$$w\lambda = w(s_\alpha \chi + n\alpha^\vee) = s_\beta w\chi + n\beta^\vee = s_{\beta,n}(w\chi),$$

which by the previous step implies $w\lambda \upuparrows w\chi$ (via the reflection $s_{\beta,n}$). Returning to the given alcoves, this translates into alcove language as $wB' \upuparrows wA'$.

Now we can prove the proposition. By assumption, there is a sequence $B = A'_0 \uparrow A'_1 \uparrow \ldots \uparrow A'_m = A$, using a good reflection (or a reflection across a Weyl chamber wall) at each step. But the intermediate alcoves A'_i are not necessarily dominant. While $d(A'_i) \leq d(A'_{i+1})$ at each step, by the lemma of Jantzen quoted earlier, the respective lengths are less predictable if some of the alcoves lie outside C^+. The idea is to work downward step-by-step (essentially an induction) to get a modified sequence of alcoves, first appealing to Jantzen's proposition (which trivializes whenever the translation parameter is 0) and then to Andersen's observation $(*)$. Denoting A'_{m-1} by B' and $A_m = A$ by A', Jantzen first provides a refined chain of dominant alcoves with d-difference ≤ 1 from wB' to A' for a suitable $w \in W$. (Naturally we take $w = 1$ in case A'_{m-1} already happens to be dominant.)

In turn, $(*)$ shows that $wA'_{m-2} \upuparrows wA'_{m-1}$ for some $w \in W$. So we can repeat the first step for this new pair of alcoves, accumulating a longer product $x \in W$. Eventually we arrive at a product in W, say y, for which $yA'_0 \upuparrows yA'_1$ with yA'_1 dominant. Along the way, the chain of further upward reflections has been refined so that all new alcoves involved are dominant and differ in length by ≤ 1. At the final step, we again apply Jantzen's proposition and finally reach a product $z \in W$ for which $zB = zA'_0$ is dominant and $zB \upuparrows yA'_1$ by a similar refined chain. But since B was initially assumed to be dominant, we can take $z = 1$ to conclude the proof.

9 Discussion of the comparison theorem

We continue to leave the comparison with Bruhat ordering in the background and consider at first just the two orderings $B \uparrow A$ and $B \upuparrows A$ for alcoves $A, B \subset C^+$. As we observed, the second of these implies the first. So we start with the assumption that $B \uparrow A$: there is a sequence of alcoves $A_i \in \mathcal{A}$ satisfying $B = A_0 \uparrow A_1 \uparrow \ldots \uparrow A_r = A$. Here each step involves a single affine reflection s_H. Since there may be many such paths, we cannot immediately rule out the occurrence of some bad hyperplanes H along the way. Instead, it has to be shown indirectly that another sequence involving only good hyperplanes must exist, thus proving that $B \upuparrows A$. For this Wang builds a complicated double induction, which at the moment offers the most uniform rigorous proof.

It is still worth asking whether a more transparent proof might exist. One approach suggested by Jantzen's work would be to *translate* the given configuration far enough from the walls of the dominant chamber to get a new sequence of alcoves $B' = B_0 \uparrow B_1 \uparrow \ldots \uparrow B_r = A'$, where now each pair of alcoves is related by a reflection across a *good* hyperplane. Clearly this is possible. In particular, we would get $B' \upuparrows A'$ with $B', A' \subset C^+$. Now apply Andersen's proposition above to this situation. This gives an alternative path involving only *dominant* alcoves with the $d = \ell$ distance increasing by 1 at each upward step.

Having replaced the translated path from B' to A' in this way, we now translate the new path back to the original setting and get a possibly new path from B up to A with the same number of steps (say t). The problem is of course (as our G_2 example in fact illustrates) that not all hyperplanes occurring here need be good. So a more careful choice would be needed. For this, one might use an elementary observation by Jantzen [8, p. 135] (which follows easily from the definitions involved): $d(A') - d(B') = d(A) - d(B)$. Keeping in mind that $d = \ell$ for these four dominant alcoves, we have $\ell(A') - \ell(B') = \ell(A) - \ell(B) = t$. But it remains a challenge to fine-tune the choice of paths from B to A so that only *good* hyperplanes are involved. If this could be done, it would yield an $\upuparrows$ relation at every step and show that $B \upuparrows A$ as desired. (In turn, Andersen's proposition would recover Theorem 2.)

Remark. Wang confirms that the two partial orderings agree on dominant alcoves, with upward increases by 1 in length. This in turn implies that $w' \overset{B}{\leq} w$ in the Bruhat ordering if $A = wA_\circ$ and $B = w'A_\circ$. (Verma [12, §1] states the equivalence of the $\uparrow$ and Bruhat orderings on C^+ but discusses briefly just the reverse implication.)

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