Parabolic subgroups of affine Weyl groups: cells and Kazhdan–Lusztig polynomials

July 11, 2012 (expanded September 22, 2016)

1 Some recollections

As background we first recall some of the standard relationships between a general Coxeter group W (with canonical generating set S) and its proper parabolic subgroups W_I for $I \subset S$. These are drawn mainly from the original paper by Kazhdan–Lusztig [KL], as exposed in Chapters 5 and 7 of the textbook [H90]:

(1) The length function ℓ on W relative to S agrees on W_I with its length function relative to I [H, Thm. 5.5].

(2) The (Chevalley–)Bruhat ordering on W_I is the restriction of the corresponding ordering on W [H, Cor. 5.10].

(3) The Iwahori–Hecke algebra $\mathcal{H}_I$ of the Coxeter group W_I embeds naturally into the corresponding algebra $\mathcal{H}$, in such a way that the standard basis $\{T_w\}$ of $\mathcal{H}_I$ lies in the corresponding subset of $\mathcal{H}$. (This and the following observations result immediately from the constructions in [KL], cf. [H, Chap. 7]. taking into account (1) and (2) above; cf. the proof of Lemma 6.3(e) in [L85].)

(4) The inverse of T_w in $\mathcal{H}_I$ for $w \in W_I$ agrees with the inverse of $T_w \in \mathcal{H}$.

(5) The Kazhdan–Lusztig involution in $\mathcal{H}_I$ is the restriction of the corresponding involution in $\mathcal{H}$.

(6) The Kazhdan–Lusztig basis $\{C_w\}$ of $\mathcal{H}_I$ (with $w \in W_I$) embeds naturally in the corresponding basis of $\mathcal{H}$. In particular, the Kazhdan-Lusztig polynomials $P_{x,w}$ for $x \leq w$ in W_I coincide with the corresponding polynomials relative to $\mathcal{H}$.

2 Affine Weyl groups

From now on, fix an irreducible affine Weyl group (W, S). Its proper parabolic subgroups W_I are finite and are in natural bijection with subsets $I \subsetneq S$.

From Lusztig's series of papers (see [L83], [L85], [L89] and their references), one gets a lot of specific information about the finitely many 2-sided cells Ω in W and the finitely many left cells Γ contained in each of them. (However, his conjecture in [L83, 3.6] on the number of left cells in a given 2-sided cell remains open in general.) Here we summarize a few connections between cells in W and cells in the finite Coxeter groups W_I.

(A) W is a disjoint union of finitely many 2-sided cells 'Ω, each with an a-invariant taking constant value $a(w)$ on its elements. Here $a(w)$ varies between 0 (when Ω has the identity element $1 \in W$ as sole element) and N, the number of positive roots in the ambient root system. In general, there is an inequality

$$(*) \quad a(w) \leq \ell(w) - 2\delta(w),$$

where $\delta(w) = \deg P_{1,w}(q)$.

(B) By Thm. 4.8(d) in [L89], each 2-sided cell Ω in W intersects some finite parabolic subgroup W_I nontrivially. This intersection is then a 2-sided cell Ω_I in W_I having the same a-invariant as Ω.

(C) Each 2-sided cell Ω in W is a disjoint finite union of left cells Γ (whose inverses form a corresponding right cell). Each left cell contains a unique *distinguished involution* (DI) w characterized by equality in $(*)$ above. When Ω_I is defined and Γ intersects W_I nontrivially, the intersection Γ_I is a left cell of W_I and contains the DI of Γ.

3 Inverse Kazhdan–Lusztig polynomials

Here we discuss briefly the important special case where the parabolic subgroup of W is the underlying Weyl group W_0. Its standard Coxeter generators comprise a subset $I := S_0$ of S consisting of simple reflections relative to a fixed basis of the underlying root system. [When one starts with a finite Weyl group, it is more common to denote it by W and to denote the resulting affine Weyl group by W_a. Note too that in the Bourbaki set-up, W_a is a semidirect product of the coroot translation lattice and the group W; but in modular representation theory, it is more natural to work with the Langlands dual, which interchanges types B_ℓ and C_ℓ.]

In any Coxeter group one has a unipotent matrix with $P_{x,w}$ as the (x, w)-entry whenever $x \leq w$ in the Bruhat ordering of W, or 0 elsewhere. Here the rows and columns are labelled relative to some total ordering of W compatible with the length function and beginning with $w = 1$. Such a matrix is infinite when W is. Even so, it is upper triangular unipotent with all but finitely many elements in each column equal to 0. Thus it has in principle a recursively defined "inverse" matrix. To keep the signs under control, we write $\varepsilon_w := (-1)^{\ell(w)}$ and then seek polynomials $Q_{z,w}$ for all $z \leq w$ satisfying:

$$\sum_{x \leq z \leq w} \varepsilon_w \varepsilon_z P_{x,z} Q_{z,w} = \delta_{x,w}.$$

Here $Q_{w,w} = 1$; other *inverse Kazhdan–Lusztig polynomials* are more elusive.

Only for finite Coxeter groups such as W_0 can the inverse polynomials $Q_{z,w}$ be conveniently described: here $Q_{z,w} = P_{w_\circ w, w_\circ z}$, where $w_\circ$ is the unique longest element of W_0. (The basic ideas originate in [KL], §3; cf. [H90], 7.13, [H08], 8.4.)

Algorithmic approaches to the inverse polynomials for an affine Weyl group have been developed especially by Andersen [A] and Kaneda [K], following the work of Lusztig in [L80]. (Note that Andersen's paper was actually written and accepted several years before it appeared.)

4 Comparing characteristic 0 and characteristic p representations

Much of the Coxeter group machinery sketched above has been developed in response to questions in representation theory. It is sometimes useful to compare the well-developed characteristic 0 theory (as in the Kazhdan–Lusztig conjecture [KL], now a theorem) with the much less complete modular theory in characteristic p (sometimes required to be "sufficiently large").

Roughly speaking, many of the key results in characteristic 0 involve infinite dimesnional representations such as Verma modules, having arbitrary integral weights, but turn out to depend just on data for the Weyl group W_0. In the modular theory the key representations tend to be finite dimensional but involve only dominant highest weights (relative to a fixed choice of S_0); here the results and conjectures emphasize an associated affine Weyl group W. One adapts the general theory by working with p times the translation lattice of roots used to define W_0, whereas in the abstract Bourbaki theory treated in [H90] the coroot lattice is more natural. These modifications along with the ρ-shift have essentially no effect on the Coxeter group theory apart from sometimes interchanging types B_ℓ and C_ℓ.

The emphasis on dominant rather than arbitary weights often makes the origin $-\rho$ for W_0 less useful than a *special* point inside the dominant Weyl chamber (one lying on affine hyperplanes for all positive roots), e.g., the first Steinberg weight $(p-1)\rho$. In this context it is still possible to carry over some information from characteristic 0 to characteristic p, as illustrated by the appendix to Jantzen's Habilitationsschrift (see [J], Anhang). (His work preceded the statement of the Kazhdan–Lusztig conjecture in [KL].) Such comparisons are limited but still valuable in some situations.

References

[A] H.H. Andersen, *An inversion formula for the Kazhdan–Lusztig polynomials for affine Weyl groups*, Adv. Math. **60** (1986), no. 2, 125–153.

[H90] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, 1990.

[H08] ——–, *Representations of Semisimple Lie Algebras in the BGG Cagtegoy $\mathcal{O}$*, Amer. Math. Soc., 2008.

[J] J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lect. Notes. in Math. **750**, Springer, 1979.

[K] M. Kaneda, *On the inverse Kazhdan–Lusztig polynomials for affine Weyl groups*, J. Reine Angew. Math. **381** (1987), 116–135.

[KL] D. Kazhdan and G. Lusztig, *Representations of Hecke algebras and Coxeter groups*, Invent. Math. **53** (1979), no. 2, 165–184.

[L80] G. Lusztig, *Hecke algebras and Jantzen's generic decomposition patterns*, Adv. in Math. **37** (1980), no. 2, 121–164.

[L83] ——–, *Some examples of square-integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 623–653.

[L85] ——–, *Cells in affine Weyl groups*, pp. 255–287 in: *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, Adv. Studies in Pure Math. **6** (1985).

[L89] ——–, *Cells in affine Weyl groups, IV*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), no. 2, 297–328.