Notes on nilpotent elements in modular Lie algebras

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These notes should be viewed as background for the immediately preceding unpublished notes (and later notes on support varieties), which involve more open-ended questions. Our main purpose here is to sketch briefly some of the basic ideas about how to define "unipotent" or "nilpotent" (as well as "semisimple") elements in arbitrary linear algebraic groups and their Lie algebras. Special emphasis is placed on prime characteristic.

1 Jordan–Chevalley decomposition: G and $\mathfrak{g}$

The key idea behind this decomposition is that any (always meaning affine) algebraic group and its Lie algebra have intrinsically defined *semisimple* and *unipotent* (or *nilpotent*) elements which propagate naturally under morphisms.

We begin with an algebraic group G over an algebraically closed field K of characteristic $p \geq 0$, together with its Lie algebra $\mathfrak{g}$. But the ideas mostly adapt to smaller fields of definition and also to the setting of group schemes over a commutative ring. The main complications here arise when $p > 0$. [In practice G may usually be assumed to be *connected*, which does not affect $\mathfrak{g}$. For example, when G is just a finite group, one has the usual type of unique decomposition of $x \in G$ as a commuting product of an element of order prime to p with an element of order a power of p when $p > 0$; both factors are powers of x. Here of course $\mathfrak{g} = 0$.]

In his 1951 book, Chevalley found the first way to approach such a decomposition: in fact the semisimple and unipotent (or nilpotent) parts are obtained as polynomials in the given element x. Thus in a given matrix realization of G (hence of $\mathfrak{g}$) these parts commute with every matrix commuting with x. Moreover, they are uniquely defined by x. For a slightly later account in the more modern setting he adopted after the 1951 treatment, see [6, 4.4]. (Much more recently, a careful exploration has been made by D. Couty et al., with emphasis on history and pedagogy: see [7, 8].)

At first such intrinsic notions appear unexpected and hard to explain directly just by the fact that the groups are defined by polynomial conditions. Indeed, the semisimple and unipotent parts of an arbitrary square matrix over K certainly need not lie in a given matrix group containing it. This special property of algebraic groups becomes more transparent when viewed indirectly in terms of how such a group (or its Lie algebra) acts on the *infinite* *dimensional* algebra $K[G]$ of regular functions on G. The point is that the left (or right) translation action of G and the associated convolution action of $\mathfrak{g}$ are *locally finite dimensional.* For a systematic treatment along this line, see [2, §4], [9, 15.3], [17, 2.4.8, 4.4].

Remark: By way of contrast, in general there is no such unique decomposition in a Lie group or its Lie algebra. For example, the additive group and the multiplicative group have the same 1-dimensional Lie algebra, which over a field such as $\mathbb{C}$ fails to distinguish the two groups. In the algebraic group setting, the first group consists of unipotent elements (so its Lie algebra consists of nilpotent elements), whereas the other group or its Lie algebra consists of semisimple elements. (However, for a *semisimple* Lie group over $\mathbb{C}$, the algebraic theory does turn out to apply thanks to Chevalley's classification. This is seen even more directly in the Lie algebra using non-degeneracy of the Killing form.)

2 Algebraic group definitions of "semisimple", "unipotent", "nilpotent"

The cited treatments of Jordan–Chevalley decomposition leave the intrinsic meaning of terms such as "semisimple" and "unipotent" somewhat implicit. A two-part paper by Borel and Springer [3, 4] makes this much more explicit in the context of an algebraic group G and its Lie algebra $\mathfrak{g}$. (See also [16, §5].) Without loss of generality, G can always be assumed to be connected. Their main objective was a better understanding of what happens over an arbitrary field of definition, which we leave aside here.

By definition, an element $x \in G$ is called *semisimple* if it lies in a maximal torus of G or *unipotent* if it lies in a connected unipotent subgroup of G. Similarly, an element $x \in \mathfrak{g}$ is called *semisimple* (resp. *nilpotent*) if it lies in the Lie algebra of such a closed subgroup of G. (The treatment is similar in the textbooks [2, 9, 17]; for example, in Borel's book see 4.8–4.9 and 14.26.)

These notions are then seen to agree with those occurring in the Jordan–Chevalley decomposition, by working in a fixed linear realization and using the relevant uniqueness property. The direct definitions are applied along with the Borel–Chevalley structure theory of G as well as the important early observation of Grothendieck that $\mathfrak{g}$ is always the union of its "Borel subalgebras" (Lie algebras of Borel subgroups): proofs can be found in [2, 14.25], Exp. 14 of SGA3, or the 1965 IHES Borel–Tits paper on reductive groups.

It is well known that semisimple elements of G behave similarly in all characteristics; moreover, their centralizers and conjugacy classes have been worked out rather uniformly. However, unipotent elements other than 1 differ considerably in characteristics 0 and $p > 0$: they have infinite order in G in the former case, but are characterized in the latter case as the elements of G having order equal to some power of p.

We remark that another definition of nilpotent elememts in $\mathfrak{g}$ is used in the introduction of lectures by Jantzen [13]: here an element is called *nilpotent* if the operator corresponding to it in the derived Lie algebra version of any faithful finite dimensional (algebraic group) representation of G is nilpotent in the usual sense. As he points out, such an element is ad-nilpotent (but not always conversely). We leave it to the reader to compare his definition with that of Borel–Springer recalled above.

3 Unipotent and nilpotent varieties

The collection of all unipotent elements in G may be denoted $\mathcal{U}$. Similarly, the collection of all nilpotent elements in $\mathfrak{g}$ may be denoted $\mathcal{N}$. Both of these are closed irreducible subsets in the respective Zariski topologies, having the same dimension (which is the total number of roots if G is reductive). Moreover, both turn out (by difficult arguments) to be unions of finitely many conjugacy classes or $\operatorname{Ad} G$-orbits, though the numbers can differ when the characteristic of K is a "bad" prime. But neither variety is non-singular, which leads to subtle problems involving singularities, normality, and the like. These varieties were studied initially by Springer and Steinberg; many details are given for example in [19, 5, 10, 13]. Further important contributions have been made by Richardson, Slodowy, Lusztig, Spaltenstein, and others.

In contrast, the collection of all semisimple elements in G (or $\mathfrak{g}$) is seldom closed, for example when G is semisimple: in that case the semisimple elements instead form a Zariski-dense proper subset.

4 Scheme-theoretic approach

What we have so far discussed is the traditional setting adopted by Borel (and Tits), in which an algebraic group G over K can usually be identified with its group of rational points $G(K)$. When a smaller field of definition is involved, or even a commutative ring such as $\mathbb{Z}$, the language of schemes normally works much better. Here G is viewed (as in part I of the book [12]) as a special type of functor from groups to commutative rings.

The notions involving $\mathcal{U}$ and $\mathcal{N}$ are often studied instead in the context of an *adjoint quotient* map, as in Steinberg's Tata lecture notes [19] and Slodowy's monograph [15, 3.9, 3.14]. For example, there is a canonical map $\mathfrak{g} \to \mathfrak{h}/W$ when G is reductive, sending an arbitrary element of $\mathfrak{g}$ to the W-orbit of its semisimple part. The fiber over 0 is essentially $\mathcal{N}$ (now often referred to as the *nullcone*). (One advantage of this viewpoint is that it often generalizes to other spaces on which G acts.)

5 Restricted Lie algebras

Finally, we point out a rather different approach taken (in characteristic $p > 0$ only) to a larger class of finite dimensional Lie algebras which leads to decompositions of Jordan–Chevalley type.

In his 1967 book Seligman [14, §7] emphasizes *restricted Lie algebras* (also known as p-restricted Lie algebras or Lie p-algebras). Here one works over a field of characteristic $p > 0$ and imposes on an abstract finite dimensional Lie algebra $\mathfrak{g}$ an extra $[p]$-operation satisfying certain axioms which are satisfied when $\mathfrak{g}$ is a matrix Lie algebra with $[p]$-opeation given by the ordinary pth power. The notion was introduced by Jacobson, who had observed for example that in a restricted Lie algebra one has the nice identity: $\operatorname{ad} x^{[p]} = (\operatorname{ad} x)^p$ for all $x \in \mathfrak{g}$. Obviously the $[p]$-operation can be iterated, for which we write $x^{[p]^2}$, etc. There are many accounts, sometimes using variants of the definition (as in Jantzen's lectures [11, §2]).

An easy standard fact is that each $\operatorname{Lie} G$ is restricted. But in the other direction, it is known that not every restricted Lie algebra (even a simple one) is of this type. One can still introduce intrinsic notions of "semisimple" and "nilpotent" element in any restricted $\mathfrak{g}$. Call $x \in \mathfrak{g}$ *semisimple* if it lies in the restricted subalgebra of $\mathfrak{g}$ generated by $x^{[p]}$, or *nilpotent* if some power of the $[p]$-operation applied to x yields 0. Using these definitions, Seligman

[14, V.7.2] derives a general version of the Jordan–Chevalley decomposition.

When $\mathfrak{g}$ is of the form $\operatorname{Lie} G$, it is easy to see that these notions agree with those discussed earlier: take a faithful embedding of G as a closed subgroup of some general linear group, and use the fact that $\mathfrak{g}$ has p-operation given by the restriction of the usual associative pth power in the general linear algebra.

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