Notes on the minimal special nilpotent orbit

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Here we summarize some features of minimal and minimal special nilpotent orbits, especially their dimensions and component groups. An open question is how much can be proved without case-by-case methods.

1 Notation and background

Let G be a simple algebraic group, say of adjoint type, over an algebraically closed field of *good characteristic*: either 0 or a prime p not dividing any coefficient of the highest root. While the adjoint type restriction sometimes affects the structure of the Lie algebra $\mathfrak{g}$ of G, this is not very serious in good characteristic: see [5, 0.12–0.13] and references there. (The "bad" prime characteristics are 2 in all types except A_ℓ, while 3 is bad for the five exceptional types and 5 is bad only for E_8, cf. [5, 3.9].)

The main advantage of the good characteristic assumption is that we can invoke a G-equivariant Springer isomorphism to pass back and forth between the variety $\mathcal{U}$ of unipotent elements in G and the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$ (cf. [5, 6.20]). Either variety has dimension equal to the number $2N$ of roots.

This variety involves just finitely many unipotent classes or nilpotent orbits (all having even dimension), parametrized in the usual way by Dynkin diagrams or Bala–Carter method (or for classical types $A_\ell - D_\ell$ by partitions). In good prime characteristic, all of this agrees with the classical Dynkin–Kostant theory in characteristic 0 as developed in [2, 3]. We use throughout the language of nilpotent orbits. The orbits are partially ordered in a natural way by the closure ordering, which for partitions in type A_ℓ is the usual partial ordering for partitions of $\ell + 1$.

Just below the *regular* orbit (which is dense in $\mathcal{N}$) there lies a unique *subregular* orbit having dimension $2(N-1)$. At the opposite extreme is the zero orbit, above which lies a unique *minimal* nonzero orbit: this consists of all long root vectors in $\mathfrak{g}$ relative to various choices of Cartan subalgebra (see §2 below). In simply-laced cases, all roots are regarded by convention as long. When there are two root lengths, the short root vectors form another orbit; this orbit turns out to be the minimal nonzero "special" one except in type G_2 (see §5).

2 Minimal nilpotent orbit

Proposition. *There exists a nonzero nilpotent orbit $\mathcal{O}_m$ of minimal dimension, which is contained in the closure of every nonzero nilpotent orbit. In particular, $\mathcal{O}_m$ is the unique nilpotent orbit of its dimension.*

This result (see [3, Thm. 4.3.3]) is proved in characteristic 0, but like other such facts it carries over to any good prime characteristic. The idea of the proof is to show that a nonzero orbit of minimum dimension contains elements arbitarily close to a root vector for the highest root relative to some fixed Cartan subalgebra and simple system of roots. (Standard conjugacy theorems show that the latter choices do not matter.) As a consequence, $\mathcal{O}_m$ is seen to consist of all possible root vectors for long roots.

3 Dimension of the minimal nilpotent orbit

To express the dimension of the minimal or minimal special nilpotent orbit in $\mathfrak{g}$ most efficiently, one starts with the *Coxeter number* h of the Weyl group. This originated in the study of finite reflection groups by Coxeter, where it is the order of the product of any generating set of simple reflections (taken in any order). Thus h agrees for Lie types B_ℓ, C_ℓ.

As Kostant explained in a classic 1959 paper, it is helpful in the study of $\mathfrak{g}$ to interpret h as 1 plus the sum of coeficients of the highest root when it is written as a $\mathbb{Z}^+$-linear combination of simple roots. In particular, h is easily computed case-by-case for the simple Lie types: for types $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$, the value of h is (respectively) equal to $\ell+1, 2\ell, 2\ell, 2(\ell-1), 12, 18, 30, 12, 6$.

One also needs the *dual Coxeter number* $h^\vee$ introduced by Kac in the study of affine Lie algebras. This can be defined in the usual setting of finite root systems to be 1 plus the sum of coefficients of the highest short root of the dual root system (involving interchange of types B_ℓ, C_ℓ) when written as a $\mathbb{Z}^+$-linear combination of the simple roots. (The original definition by Kac involves affine Lie algebras and extended Dynkin diagrams.) Unlike h, this number depends on the root system, not just on the Weyl group. Explicitly, $h = h^\vee$ for the simply-laced root systems (types A_ℓ, D_ℓ, E_ℓ), while $h^\vee = 2\ell - 1, \ell + 1, 9, 4$ for the respective root systems B_ℓ, C_ℓ, F_4, G_2.

In a short note [12, Thm. 1] Weiqiang Wang gives a case-free proof of the following general dimension formula:

Proposition. *The dimension of the minimal nilpotent orbit in* $\mathfrak{g}$ *is* $2h^\vee - 2$.

For the proof, he starts with the standard fact that the orbit dimension is the codimension of the isotropy of a typical element e in the orbit (such as a highest root vector): the centralizer of e in G or the fixed points of ad e in

$\mathfrak{g}$. Here the dimension is characterized as 1 plus the number of positive roots not orthogonal to the highest root [3, 4.3.5]. Wang then transforms this into the more readily computable number here.

4 Special nilpotent orbits and LS duality

In his study of Weyl group representations in the late 1970s, Lusztig discovered a type of unipotent class in G which he came to call "special". These classes or corresponding nilpotent orbits have arisen also in other sorts of representation theory, for example in the work of Barbasch and Vogan on unitary representations of complex semisimple Lie groups. There is no easy way to identify the special nilpotent orbits intrinsically within $\mathcal{N}$, but the work of Lusztig and Spaltenstein revealed a new type of duality on the set of special classes or orbits (we call it for short LS duality): see [10, Chap. III]. For example, this duality interchanges the zero orbit with the (dense) orbit of regular elements in $\mathcal{N}$.

The easiest classical case is type A_ℓ; its nilpotent orbits are parametrized by Jordan forms or equivalently by partitions of $\ell+1$. In this case all orbits are special. Moreover, the standard transpose operation on partitions coincides with LS duality. For example, the subregular orbit with partition $[\ell, 1]$ corresponds to the minimal orbit with partition $[2, 1^{\ell-1}]$.

But in all other simple Lie types, some nilpotent orbits fail to be special. In classical type B_ℓ, the orbits are parametrized by partitions of $2\ell+1$ in which all even parts $(2, 4, \dots)$ occur an even number $(0, 2, 4, \dots)$ of times. Similarly, in type C_ℓ one gets the partitions of 2ℓ in which all odd parts occur an even number of times. In either case, an orbit is special if and only if its transpose permutation meets the stated requirement. (See [3, 5.1, 6.3].) (But simply-laced type D_ℓ requires more delicate adjustments.)

The exceptional types no longer have obvious natural representations and require case-by-case study. See [2, 13.1, 13.4] for convenient tables of data along with graphs showing the partial ordering of $\mathcal{N}$ and of its subset of special orbits. Much of this data comes from the work of Spaltenstein, whose partial ordering diagrams [10, Chap. IV] for E_7, E_8 are correct even though Carter's lack several bonds.

In general, it turns out that any Richardson orbit in $\mathfrak{g}$ (such as the subregular orbit) is special. Moreover, case-by-case study shows that the minimal nilpotent orbit is special in all simply-laced types A_ℓ, D_ℓ, E_ℓ. Although this fails when there are two root lengths, but LS dual of the (special) subregular orbit is always minimal among the nonzero special orbits.

5 Dimension of the minimal special nilpotent orbit

It turns out that there is a similar dimension formula for the minimal special nilpotent orbit, valid in all types:

Proposition. *The minimal special nilpotent orbit has dimension* $2h-2$.

For simply-laced types the formula is consistent with the proposition of §3 in view of our earlier remarks: here $h = h^\vee$, while "minimal" is the same as "minimal special".

Though the formula is easy to check case-by-case using the data in [2], it so far lacks a uniform conceptual proof. This was posed as a question at `http://mathoverflow.net/questions/67143/`. While there is no complete answer yet, P. Levy commented that one could get some insight from the paper [1]. For example, the minimal special orbit in (say) F_4 is obtained indirectly from the minimal orbit in E_6 via the "folding" of Dynkin diagrams (using an involutive graph automorphism). The two root systems share the same h and the orbits in question have the same dimension. In G_2 the minimal nonzero special orbit is subregular, not the smaller orbit involving short root vectors; here the folding from D_4 is relevant.

Among the classical types, only B_ℓ and C_ℓ need discussion; here $h = 2\ell$ in both cases. For these types, LS duality transposes the partition for each special orbit. In type B_ℓ, [3, 5.4] gives the partition of $2\ell+1$ for the subregular orbit: $[2\ell-1, 1^2]$. From this one gets the partition for the minimal special orbit: $[3, 1^{2\ell-2}]$. Then an easy calculation using the dimension formula for classical types in [3, 6.1] yields $4\ell - 2 = 2h - 2$.

Similarly, in type C_ℓ the subregular orbit corresponds to the partition of 2ℓ given by $[2\ell-2, 2]$. So the minimal special orbit has the transpose partition $[2^2, 1^{2\ell-4}]$. In turn, the orbit again has dimension $2h-2$.

Remark: In his early work on the role of special unipotent classes in the character theory of finite groups of Lie type, Lusztig [8, §3,Rem.!(b)] actually noticed that G has a unique unipotent class of dimension $2h-2$ and that this is the minimal unipotent class precisely in the simply-laced case. He also developed some geometric properties along with the number of rational points in the closures of these classes (minus the identity element) over finite fields, extending unpublished partial results of Kostant.

6 Component groups

The *component group* $A(e) := C_G(e)/C_G^\circ(e)$ is an orbit invariant and has been computed case-by-case in all types. For classical groups, the study of centralizers by Springer and Steinberg [11, IV, §2] leads to an explicit recipe (cf. [3, 6.1]), while for exceptional groups Spaltenstein's work is reflected in Carter's tables [2, 13.1].

Proposition. *Let $e \in \mathfrak{g}$ be nilpotent.*

(a) *If e lies in the minimal orbit of $\mathfrak{g}$, then $A(e) = 1$ in all cases.*

(b) *If e lies in the minimal special nilpotent orbit of $\mathfrak{g}$, then $A(e)$ is trivial if $\mathfrak{g}$ is simply-laced, but of order 2 in types B_ℓ, C_ℓ, F_4 and isomorphic to the symmetric group S_3 in type G_2.*

Remark: After defining "special pieces" in the unipotent variety, Lusztig [9] considered the case when C is a special unipotent class whose special piece involves at least one smaller non-special class. His Thm. 0.4 implies that the component group (which he denotes A_C) of such a special class is nontrivial. (Case-by-case study of the exceptional types, using the tables cited earlier, pins down more precisely the structure of each nontrivial component group: see [9, Prop. 0.7].)

7 Kazhdan–Lusztig graphs

It is worth mentioning also a related problem which was posed online some years ago: `http://mathoverflow.net/questions/60627/`. The notion of *Kazhdan–Lusztig graph* occurs at the end of their influential 1979 paper [7]. In a follow-up paper, Dolgachev and Goldstein [4] considered the case of the minimal nilpotent orbit. So it seems natural to ask also about the graph for a minimal special nilpotent orbit. All of this has a strongly geometric flavor, arising from Springer's resolution of singularities of $\mathcal{N}$ and the resulting Springer fibers in the flag variety. These are not necessarily irreducible varieties, so the behavior of their irreducible components is well worth studying.

References

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