Counting Coxeter elements

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After a quick review of Coxeter elements in a finite Coxeter group W (such as a Weyl group), we focus in this short note on the question of counting them. This mainly involves pinning down the order of the centralizer $C_W(c)$ of a typical Coxeter element c.

Consider a finite (irreducible) Coxeter system (W, S) as in [1, Chap. 4], [5], etc. Here W is a finite group generated by reflections, acting irreducibly on an n-dimensional real euclidean space V, while $S = \{s_1, s_2, \ldots, s_n\}$ is a set of n generating reflections satisfying the usual Coxeter relations for W. The set S is often viewed as the set of orthogonal reflections relative to some simple system of "roots", taken to be unit vectors orthogonal to suitable reflecting hyperplanes in V. The number n is called the *rank* of W (often written ℓ as in Bourbaki, or perhaps r). Although S is often fixed, any other choice is W-conjugate to it and therefore geometrically indistinguishable.

Textbook treatments vary a lot in style but are similar in content for finite Coxeter groups: see for example [1, 3, 5]. In particular, the classification of the finite irreducible groups (by *Coxeter graphs*, which are trees in this case) leads to the Weyl groups of simple Lie algebras (say over $\mathbb{C}$) together with the remaining non-crystallographic finite reflection groups: most of the dihedral groups $I_2(m)$ (of rank 2) and two exceptional groups H_3 of rank 3 and order 120, and H_4 of rank 4 and order $14,400$.

By definition, a *Coxeter element* of W (relative to any choice of $S = \{s_1, \ldots, s_n\}$) is the product of reflections $s_1, s_2, \ldots, s_n$ taken in any order. Since all sets S are W-conjugate, it is usually enough to work with a fixed choice; this allows one for example to define the length $\ell(w)$ for each $w \in W$. A basic fact is that all Coxeter elements lie in a single conjugacy class of W and thus have the same order h.

This and other striking facts were discovered by Coxeter in his geometric and combinatorial explorations of finite (real) reflection groups: see for example [4]. In particular, his discovery of the natural relations on elements of S which define W led Bourbaki to their more general notion of "Coxeter group" (usually infinite, but having a Coxeter-type presentation with n involutions as generators).

Textbook accounts are given for example in [1, V,§6], [3, 10.3–10.6], and [5, 3.16–3.20]. The key ingredient, discovered by Coxeter, is a special plane P in V on which a Coxeter element c acts by a rotation of order precisely $h=$ the order of c in W, nowadays called the *Coxeter number* of W. The cyclic group $\langle c \rangle$ is in fact normal in a dihedral subgroup of W which acts naturally on the Coxeter plane as a reflection group. (Since the case $n = 1$ is easily handled directly, it is convenient to leave it aside in the following discussion.)

Proposition. *Let W be a finite irreducible Coxeter group and $c \in W$ be an arbitrary Coxeter element. Then the centralizer $C_w(c)$ is the cyclic group $\langle c \rangle$.*

Corollary. *The single conjugacy class consisting of all Coxeter elements of W has order $|W|/h$.*

We remark that h is the unique largest *degree* of W, while $|W|$ is the product of the *degrees* $2 = d_1 < d_2 \leq d_3 \leq \cdots \leq d_{n-1} < d_n = h$. Thus the number of Coxeter elements in W is the product of the first $n - 1$ degrees. (Here the d_i are the degrees in increasing order of any minimal set of homogeneous polynomials generating the W-invariants in the ring of all polynomial functions on V.)

To conclude this note we comment briefly on the proofs in the literature of the above proposition. Although it does not seem to be stated explicitly in the papers of Coxeter or in the textbooks cited earlier, the most direct proof is based mainly on taking a closer look at the action of c on the *Coxeter plane* $P \subset V$.

A straightforward argument is given by Carter [2, Prop. 30] (on p. 35). While his paper primarily deals with classes and centralizers in a finite Weyl group (including a lot of case-by-case work), the general theory applies equally well to any finite Coxeter group. [But his early discussion of "length" in W deals in fact just with what is usually called "reflection length".]

The idea of the proof is roughly as follows. It is convenient for Coxeter's construction of a plane $P \subset V$ to assume that our Coxeter element c is the product of two involutions y, z (always assuming $n > 1$ without loss of generality). Such a choice of c is possible since the Coxeter graph of W is a tree (an easy first step in the classification). All Coxeter elements being conjugate, this choice is harmless.

The eigenspace for c in V relative to the (nonreal!) eigenvalue $\zeta := e^{(2\pi i)/h}$ has dimension 1, following the work of Coxeter explained in the cited books. Moreover, an eigenvector for ζ is not orthogonal to any root. The construction of P shows that the reflecting hyperplanes of y and z intersect P in two lines, while c itself acts as a rotation of order h. In particular, ζ and its complex conjugate both occur as eigenvalues of c on P, so the complexification of P is spanned by their eigenvectors. Studying the Weyl chambers of P relative to the dihedral group generated by y and z, one sees that any $w \in W$ commuting with c must lie in this dihedral group and therefore must agree with some power of c. [For background on P, Carter cites [7] but not the papers of Coxeter. In his own paper, Steinberg refines Coxeter's earlier work in various directions.]

In a slightly later paper [6] Springer investigates "regular" elements in a finite (real or complex) reflection group: for example, Coxeter elements in a real reflection group W. He studies the eigenspaces and centralizers of regular elements, obtaining in [6, Cor. 4.4] a proof of the above proposition in this more general setting along with many other details, e.g., that all regular elements are conjugate. (He refers to Carter's paper for the more direct proof sketched above.)

A final remark: If one considers only those Coxeter elements resulting from a fixed choice of S, it is a more standard combinatorial problem to work out the cardinality of the resulting set. This number seems to depend only on the rank.

References

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV–VI, Hermann, 1968 [English translation published by Springer, 2002].

2. R.W. Carter, *Conjugacy classes of the Weyl groups*, Compos. Math. **25** (1972), 1–59.

3. ——-, *Simple Groups of Lie Type*, Wiley, 1972.

4. H.S.M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Math. J. **18** (1951), 765–782.

5. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, 1990.

6. T.A. Springer *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–198.

7. R. Steinberg, *Finite reflection groups*, Trans. Amer. Math. Soc. **91** (1959), 493–504.