

Composition factors of arbitrary Weyl modules?

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In this note we ask an apparently straightforward question: *What are the composition factor multiplicities of a Weyl module $V(\lambda)$ with arbitrary highest weight λ , for a simple algebraic group G over an algebraically closed field K of characteristic $p > 0$?*

This would be a natural follow-up to the determination of all dimensions of weight spaces in simple modules $L(\lambda)$ for G . The latter problem has of course been solved only for “sufficiently large” p , but thanks to Steinberg’s tensor product theorem it reduces in principle (for each fixed p) to the study of the finitely many p -restricted highest weights λ : those with coordinates between 0 and $p - 1$ relative to fundamental dominant weights. Even for large p , the solution to the problem is expressed only in a recursive fashion in terms of values of Kazhdan–Lusztig polynomials at 1 (for a suitable affine Weyl group).

One further remark: Although its relationship with indecomposable tilting modules for G remains speculative, the question we have asked about composition factors of a Weyl module would arise naturally if the multiplicities of Weyl modules in a filtration of such a tilting module were governed by a reciprocity law analogous to that for Verma modules: see 11.7(1) in [H08]. (However, a substitute for the longest element in W in such reciprocity might be rather subtle in prime characteristic, involving for example symmetry around a suitable special point relative to W_p for a tilting module of dominant highest weight λ .)

1 Weyl modules and simple modules

To get some perspective it helps to look back at the history involved, which was sketched with references in [H14]. For a systematic development in prime characteristic, see [J03]. Initially a Weyl module $V(\lambda)$ was thought of non-intrinsically as a certain reduction modulo p of the classical finite dimensional simple module with dominant integral highest weight λ . Here the formal character and dimension of $V(\lambda)$ are given by Weyl’s formulas. A key fact is that $V(\lambda)$ has a unique simple quotient $L(\lambda)$, leading to a version of Chevalley’s classification of (rational) simple G -modules.

Following the proof of Kempf’s Vanishing Theorem in the 1970s it became clear that $V(\lambda)$ also has an intrinsic description as the universal highest weight module for λ in the category of (rational) finite dimensional G -modules. It is realized as the top cohomology of a line bundle \mathcal{L} related to λ on the flag variety G/B for a suitable Borel subgroup B of G . Thus $V(\lambda)$ is the Serre dual of the space $H^0(G/B, \mathcal{L})$ of global sections, which Chevalley had in effect used to get his classification. Over the years a substantial

amount of work has been done by Jantzen, Andersen, Donkin, Parshall–Scott, and others in the direction of understanding the internal structure (e.g., filtrations) of $V(\lambda)$ and its homomorphisms to other Weyl modules.

Even in characteristic 0 it is highly nontrivial to work out weight space dimensions in simple modules for a given highest weight, though the computational problem is reasonably well-structured by Freudenthal’s recursive formula. In prime characteristic there is less to go on beyond brute force. Given a dominant weight λ and assuming we know everything about weight multiplicities for the relevant simple modules $L(\mu)$ with $\mu \leq \lambda$, it is possible in principle to peel off the composition factors recursively starting with $L(\lambda)$ and relying on the partial ordering of weights. But this quickly becomes a daunting task for a high-speed computer, even if one takes advantage of general facts including the strong linkage principle, and it gives no insight into the resulting numbers.

2 Characteristic 0 analogue

Underlying most of the theoretical work done so far is a partial analogy with the study of Verma modules and their simple quotients in the characteristic 0 BGG category \mathcal{O} . Here all integral weights λ appear, each occurring as the highest weight of an infinite dimensional Verma module $M(\lambda)$ and of its unique simple quotient $L(\lambda)$ (which is finite dimensional just in the classical case when λ is dominant). In this setting, the 1979 Kazhdan–Lusztig Conjecture (soon proved using some recently developed geometry) predicted that in a suitable Grothendieck group, each $L(\lambda)$ can be written formally as a \mathbb{Z} -linear combination of $M(\lambda)$ along with various $M(\mu)$ for lower linked weights. (See Chapter 8 in [H08] for a detailed exposition.)

In this formula the multiplicities are given explicitly as a signed sum of values at 1 of certain Kazhdan–Lusztig polynomials $P_{x,w}$ with coefficients in \mathbb{Z}^+ . Here $x \leq w$ in the Bruhat ordering of the finite Weyl group W and signs depend on length differences in W . Elements of W parametrize highest weights of Verma modules in a regular block, and the coefficients of the polynomials are recursively computable. Jantzen’s translation functors then allow one to read off the composition factors of $M(\lambda)$ when λ is arbitrary and possibly irregular. Equivalently, the composition factor multiplicities $[M(\lambda) : L(\mu)]$ can be expressed as values at 1 of correlated *inverse* Kazhdan–Lusztig polynomials. Moreover, these polynomials are just the original ones with a small twist in the two Weyl group parameters attached.

To make this recursive procedure precise, one starts with a fixed regular

integral weight λ such as 0 inside the shift of the dominant Weyl chamber having lowest vertex $-\rho$ (where ρ is the sum of fundamental dominant weights). Then the (shifted) Weyl group action $w \cdot \lambda := w(\lambda + \rho) - \rho$ on weights yields all linked weights. Simple modules in the resulting block can be written as $L_x := L(x \cdot \lambda)$ and similarly for Verma modules. In the corresponding Grothendieck group, where brackets denote isomorphism classes of modules, the formula becomes: $[L_w] = \sum_{x \leq w} (-1)^{\ell(w) - \ell(x)} P_{x,w}(1) [M_x]$. (In order to make the labeling of weights compatible with the Bruhat ordering of W , one usually starts with $\lambda = -2\rho$ rather than 0.)

3 Lusztig's conjectures in characteristic p

In characteristic $p > 0$ the situation is roughly parallel, provided one requires p to be “not too small” and λ to be “not too large”. Here one substitutes for W the corresponding (Langlands dual!) *affine Weyl group* W_p , using the lattice of translations given by p times the root lattice. The dot-action on weights is again relative to a shifted origin $-\rho$, but reflections now involve arbitrary p -translations of the root hyperplanes defining Weyl chambers in the underlying euclidean space; there are infinitely many such hyperplanes, the complement of whose union comprises infinitely many p -alcoves. An inherent complication here is that each power p^r of p defines a subgroup W_{p^r} of W_p : all of these are abstractly isomorphic as Coxeter groups, and all might arise in the study of arbitrary dominant weights.

How does the category \mathcal{O} theory help to organize the characteristic p theory? In the latter case only the *dominant* integral weights parametrize the relevant highest weight modules, though arbitrary weights enter into the associated sheaf cohomology formalism for line bundles on the flag variety. Often p is required to exceed the Coxeter number h of W , in order that p -alcoves for W_p should contain integral weights. On the other hand, Steinberg's twisted tensor product theorem reduces matters in principle to the case of finitely many restricted dominant weights. (But later work by Williamson has shown difficulties with “intermediate” p as well, and the only uniform results known so far require that p be very large.)

As in characteristic 0 one relies on Jantzen's translation functors to fill in the formulas for arbitrary weights once the formulas are pinned down for a single W_p -orbit of p -regular weights (always assuming such weights exist, which requires $p \geq h$). Again the weight 0 is a reasonable starting point, since it lies inside the lowest dominant p -alcove. Here we know in general that $\dim V(0) = \dim L(0) = 1$. If we write L_w, V_x for the modules

$L(w \cdot 0), V(x \cdot 0)$ when $x \leq w$ in the Bruhat ordering of W_p (with the relevant weights dominant and lying well within the lowest p^2 -alcove), it makes sense to ask whether the Kazhdan–Lusztig formalism will still go through:

$$[L_w] = \sum_{x \leq w} (-1)^{\ell(w) - \ell(x)} P_{x,w}(1) [V_x]?$$

In this direction Lusztig formulated two distinct conjectures (for large enough p and small enough λ in the lowest p^2 -alcove): both appeared in 1980, the first as a brief conference report [L79] after the 1979 AMS Summer Institute at Santa Cruz incorporating the displayed equation (but using a more subtle parametrization of weights), the second as a detailed journal paper [L80].

In the latter paper, he built on Jantzen’s *generic decomposition patterns* [J77] for Weyl modules and assumed only that p is sufficiently large. Here there is already a visible need for p to be fairly large in order to get such patterns to fit inside the permitted region of the lowest p^2 -alcove. Lusztig’s conjecture is stated in a more symmetric form which takes advantage of a sort of reciprocity Jantzen worked out in §5 of [J80]: the composition factor multiplicity $[M_w : L_x]$ is the same as the multiplicity with which M_w occurs in a G -module Q_x lifted from the injective hull of L_x as modules for a subscheme involving the first Frobenius kernel of G . (At this point the existence of such a lifting requires $p \geq 2h - 2$.) For this conjecture, Lusztig uses indirectly the values at 1 of appropriate *inverse* Kazhdan–Lusztig polynomials. (Later work by Shin-ichi Kato showed that Lusztig’s two conjectures become equivalent for “large enough” but unspecified p .)

A serious complication is that the inverse polynomials for W_p are typically not just obtained by re-parametrizing the original polynomials as they were for W . Though these polynomials have been studied more closely by Andersen and Kaneda, there has been little direct progress toward proving this conjecture of Lusztig beyond the few small rank cases handled by Jantzen.

4 Possible further steps

While there are not yet any promising conjectures for primes $p < h$ or for arbitrarily large weights λ , some of the internal structure of Weyl modules has been studied far enough to make observations about patterns here. In particular, there is some hope of filtering $V(\lambda)$ by suitable p^r -translates of sheaf cohomology modules, each endowed with a filtration of Jantzen–Andersen type: see [H84]. This is partly suggested by unpublished results

of Cline in rank 1 [C79]. Related filtrations of cohomology modules have been constructed generically by Andersen and should be determined by the inverse Kazhdan–Lusztig polynomials in a way suggested by the successful study of Jantzen filtrations in characteristic 0.

But when λ is too close to a Weyl chamber or alcove wall for $V(\lambda)$ to admit generic filtration behavior, there is likely to be a problem with the composition factor formalism that would result. This involves in part the breakdown of Jantzen’s generic decomposition patterns near Weyl chamber walls, which are implicitly treated in his paper, but would also involve similar phenomena for composition factor patterns of cohomology modules when there is “nonstandard” vanishing in more than one degree. All of this would inevitably complicate any attempt to formulate a uniform conjecture on composition factor multiplicities in $V(\lambda)$ for “large” λ . Still, there may well be some pattern here which depends ultimately just on affine Weyl groups and Kazhdan–Lusztig theory. In particular, the dependence on p or λ may turn out to be minimal.

5 Theory vs. Computation

It’s worth emphasizing that computational techniques have been in use for some decades to determine for each pair of dominant weights $\mu \leq \lambda$ the dimension of the weight space for μ in $L(\lambda)$ when p is fixed. Such computations rely on the use of a suitable \mathbb{Z} -form in the characteristic 0 simple module of highest weight λ which yields $V(\lambda)$ after reduction mod p . The basic ideas were formulated by Burgoyne and Wong, then refined for computer implementation by Gilkey–Seitz and others; some of the most recent tables have been produced by Frank Lübeck. All such results are recursive in nature, based on the usual partial ordering of weights, and of course they are limited by the sizes of p and the coordinates of λ . Moreover, the resulting dimensions tell one very little conceptually.

Even in characteristic 0, computations of weight multiplicities (typically based on Freudenthal’s formula) are by themselves not very informative and are difficult to carry out in large ranks or for weights with large coordinates. Weyl’s character formula, on the other hand, is conceptually elegant and leads to a simple explicit formula for overall dimensions of simple modules.

While no direct analogue of Weyl’s formula is expected in characteristic p , the basic problem of finding weight space multiplicities in $L(\lambda)$ does reduce to finitely many weights thanks to Steinberg’s tensor product theorem. Unfortunately, there is still no definitive result for p -restricted highest

weights. And again one would prefer a conceptual proof.

As we observed at the outset, if one knew all about the simple modules here, one could in principle deduce the composition factor multiplicities of arbitrary Weyl modules. However, this would require that we first work out all weight space dimensions in characteristic 0, which as remarked above is a challenging computational problem. Even if we obtained arbitrarily large tables of multiplicities (working recursively), we would not gain any real insight into what is going on. So the search for conceptual methods is bound to continue.

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