Cohomology of line bundles on flag varieties

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1 Introduction

Fix a simply connected, simple algebraic group G over an algebraically closed field K of characteristic $p > 0$, with Lie algebra $\mathfrak{g}$. Then fix a maximal torus T and a Borel subgroup B containing it. We denote the group of characters (or integral weights) of T by $X = X(T)$ and the subset of dominant weights by X^+ (relative to B).

Following Kempf's vanishing theorem in 1976 (see Jantzen's book [26, II.4] and his extensive references), there has been substantial but sporadic progress in the study of the naturally occurring G-modules usually written as $H^i(\lambda)$: these are finite dimensional sheaf cohomology (or derived functor) modules for an arbitrary weight $\lambda \in X$, relative to induced line bundles on the flag variety G/B^- (see also the notes [21]). Here B^- is the opposite Borel subgroup, used for convenience to avoid sign problems. The Serre dual of H^i is H^{N-i} for suitably chosen weights; here $N = \dim G/B^- =$ number of positive roots.

Problem: Determine for any $p > 0$ *and any* $\lambda \in X$ *which* $H^i(\lambda)$ *are nonzero.*

This problem has a well-known solution in characteristic 0, obtained by adding to the results of Borel–Weil for dominant weights those of Bott for arbitrary weights. While it is far from being solved in prime characteristic, much has been accomplished since the mid-1970s. First Kempf showed for *dominant* weights $\lambda \in X^+$ that $H^0(\lambda)$ is the only nonvanishing cohomology (just as in characteristic 0); thus H^0 exhibits what may be called "standard vanishing behavior". This together with Serre duality settles the SL_2 case completely. Kempf's proof was geometric and somewhat complicated, but soon Andersen and Haboush found streamlined methods, explained in [26].

Then Mumford (starting with $p = 2$) and his student Larry Griffith (allowing odd p) worked out systematically the next case SL_3 (of type A_2). Here there is sometimes "nonstandard" vanishing of both H^1 and H^2 for weights near Weyl chamber walls. The 1977 MIT thesis of H.H. Andersen and his subsequent papers have identified many patterns in arbitrary rank but without settling matters in general.

There have been a number of surveys over the years, such as [10, 11, 20]. Here our aim is to provide some further background and updates, noting for example the more recent work by Andersen and Kaneda [12], while also filling in some of the historical details. Along the way we raise some natural questions, starting with the (meta)-question as to what a more detailed knowledge of the $H^i(\lambda)$ might contribute to the overall study of modular representations of G? There is for example still an open question as to how best to organize the internal structure of a Weyl module when the coefficients of λ relative to the fundamental weights are large compared to p.

For many exchanges over the years, I am grateful especially to Henning Haahr Andersen and Jens Carsten Jantzen as well as Masaharu Kaneda. More recently, correspondence with Will Hardesty has also been valuable. Besides listing some references especially relevant here, I have added other work such as [13, 14, 15].

2 Generic decomposition patterns

First we make a brief digression, whose relevance to the modules $H^i(\lambda)$ will soon become clear.

The notion of "generic decomposition" for a Weyl module $V(\lambda)$ with $\lambda \in X^+$ originates in Jantzen's 1977 paper [23]. While his treatment does not make explicit how large p has to be for such a decomposition to exist, the context makes it clear that *p should be at least h*, the Coxeter number of the root system Φ or Weyl group W. But in most cases p needs to be taken much larger in order for the generic pattern of composition factors of $V(\lambda)$ (with multiplicities counted) to fit into the lowest p^2-alcove of the dominant Weyl chamber. Even in type G_2 such a pattern requires 84 p-alcoves and a total of 119 composition factors.

Rank 1 and 2 examples have by now been well-studied, but it remains a difficult open question to find all composition factor multiplicities. And of course we have to assume that $p \geq h$. (For smaller p there is not yet a comprehensive conjecture about such multipicities, although computer calculations can be done up to a point.)

To make the problem manageable for large p-powers, the idea is to translate a generic pattern for a p-regular weight within the lowest p^r-alcove while preserving all the multiplicity data. For example, in rank 1 there are typically 2 composition factors (each with multiplicity 1) for a Weyl module in the lowest p^2-alcove and in general 2^m in the lowest p^{m+1}-alcove. Similarly, in rank 2 there are respectively 9, 20, and 119 composition factors in the lowest p^2-alcove for types A_2, B_2, G_2 (all multiplicities being 1 until type G_2, where some multiplicities are 2, 3, 4). In the lowest p^{m+1}-alcove, we typically find 9^m etc.

To simplify the discussion, we normally assume that $p \geq h$ and that all weights considered lie in the lowest p^2-alcove.

Remark: When λ fails to be p-regular (lies inside a p-alcove), the situation degenerates somewhat but remains under the control of Jantzen's translation functors if we start with some $p - regular$ weight and translate to a weight μ lying in a unique wall of the given alcove. The key fact here is that a composition factor of $H^i(\lambda)$ yields a composition factor of $H^i(\mu)$ if μ lies in the "upper closure" of the p-alcove in question but otherwise yields 0.

In a 1980 paper [25] and in his 1987 book (or its expanded second edition) [26], Jantzen observed that a streamlined proof of the existence of generic decompositions (for p suitably large but unspecified) can be drawn from his later formalism involving the mixed group schemes G_rT: here the generic induced modules can be compared systematically with Weyl modules. See Remark 1) following the proposition in [26, II.9.17], along with the earlier results.

3 Two general facts and some low rank examples

How many distinct generic patterns exist in the lowest p^2-alcove?

(1) *The number of distinct generic decomposition patterns within the lowest p^2-alcove is the same as the number of distinct p-alcove types, namely $|W|/f$ where f is the index of connection for the root system: the index of the root lattice in the weight lattice.* For example, this number is 1 for type A_1, respectively 2, 4, 12 for $A_2, B_2 = C_2, G_2$.

Somewhat more difficult is to compare the generic decomposition patterns arising from p-regular weights in different types of alcove. But here it turns out that all patterns involve the same number of composition factors:

(2) *All generic decomposition patterns in the lowest p^2-alcove involve linked weights in the same number of p-alcoves, with the same multiplicities.*

More precisely, all such patterns may be derived systematically from a single one. Keeping in mind that W is generated by simple reflections, one can start with a known generic pattern and transform it into any other by applying a sequence of affine reflections to the given highest weight. This is explained formally by Jantzen in [23], but is quite easy to carry out informally in rank 2. Start with the relevant alcoves and multiplicities marked by $X, 2X, \ldots$ in an alcove diagram, with X circled in the dominant alcove containing the unique highest weight. Then select a simple root α and cut the diagram into strips along affine hyperplanes parallel to the *alpha*-wall of the dominant Weyl chamber. Keep the strip with the circled X stationary, but move the strips on the left (resp. right) systematically to the right (resp. left) before reassembling the picture.

It is fairly easy to carry out this procedure in each of the irreducible rank 2 cases. The result is summarized in [23, p. 457]. For A_2, explicit diagrams are given in [16, Fig. 1,2]; for $B_2 = C_2$ (noting different conventions in the two cases about which simple root is long or short), see [23, p. 456]; for G_2 (where 84 p-alcoves and 119 composition factors are involved) see [19, Fig. 2] (but ignore for now the refined numbering of the Jantzen filtration, keeping track only of the number of composition factors attached to each alcove).

A symmetric "inverse" method is formalized by Lusztig [27, §12], to be explained further below.

Unfortunately, it is impossible to draw nice alcove diagrams in higher ranks (except possibly 3); so a more "analytic" approach would be needed, as in [2]. More seriously, one needs much information about simple modules in order to compute even a single generic decomposition pattern. For the moment G_2 seems to be the most intricate example one can picture, say for $p \geq 11$.

4 Conjectured connection with the modules $H^i(\lambda)$

In a Canadian lecture in the summer of 1977 [16], I tried to explain my initial insight into the nonstandard vanishing discovered by Mumford and Griffith. My hope then was to see how a generic decomposition pattern behaved when a dominant Weyl chamber wall was crossed. In type A_2, there is a clear "non-cancellation" possibility in just one of the two cases (the pattern with a highest weight inside the lowest type of alcove). But no mechanism governing such phenomena was yet apparent. Nor was the standard vanishing behavior of all H^0 (and their Serre duals H^N) explained.

Along with the appearance of the fundamental paper of Kazhdan–Lusztig in 1979, dealing with Hecke algebras and Kazhdan–Lusztig polynomials, several other developments took place. These included Jantzen's elaboration at the 1978 LMS Durham summer school of his ideas about filtrations and sum formulas for Weyl modules in characteristic $p > 0$ [24], Lusztig's 1979 lecture at the Santa Cruz summer institute in which he formulated a tentative conjecture on the characters of most simple G-modules along the lines of the Kazhdan–Lusztig Conjecture in characteristic 0 but involving an affine Weyl group relative to p, and Lusztig's 1980 "inverse" conjecture [27] (for large enough p) inspired by Jantzen's generic decomposition patterns.

5 Jantzen and Andersen filtrations

By now the results (including those of Andersen for H^1 in [2]), along with his general theorems and detailed computations in rank 2, have made it clear that at least some structural information about these modules should play an important role in solving the vanishing problem. On the other hand, one can't realistically expect to work out full information about submodules in general. Study of *filtrations* of Weyl modules and other sheaf cohomology modules is therefore a reasonable middle ground.

In the late 1970s Jantzen had already perfected many of the ideas in the somewhat parallel setting of Verma modules, including his striking Sum Formula. In the case of Weyl modules, similar ideas are written down less formally in the proceedings of the 1978 LMS Durham symposium [24] where he considers a typical Weyl module $V(\lambda)$ when $\lambda \in X^+$. He writes $V^0(\lambda) = V(\lambda)$ and $V^1(\lambda) =$ the unique maximal proper submodule. For his original method, based on the idea of a "contravariant form", he had to omit some small p. But then the Sum Formula expresses the sum $\sum_{i>0} \operatorname{ch} V^i(\lambda)$ as a computable sum of (known) Weyl characters. From this he was able in low ranks to obtain a lot of information about the simple modules involved in $V(\lambda)$, though naturally this soon leads to ambiguity about multiplicities of composition factors as the rank grows.

In characteristic 0, the Kazhdan–Lusztig Conjecture of 1979 and its proof soon afterwards was followed by the Beilinson–Bernstein proof of "Jantzen's conjecture", a natural formulation of an expected hereditary property of the Jantzen filtrations of embedded Verma modules. But the proof indicated that this was even stronger than the Kazhdan–Lusztig Conjecture on composition factor multiplicities of Verma modules.

Andersen [9] found an analogous implication for Weyl modules, having already removed in [6] Jantzen's restriction on p with a construction of the filtrations based on homomorphisms over $\mathbb{Z}$. But earlier he went further [6, 7, 8], inspired partly by [18] which had been accepted some years before publication and had circulated in preprint form. In my paper I had proposed filtrations similar to Jantzen's, keeping in mind that $V(\lambda) = H^N(\mu)$ for a suitable antidominant weight μ. Here the filtrations and sum formulas, for $H^i(\lambda)$ with $\lambda \in X$, relied on the (algorithmic) computation of inverse Kazhdan–Lusztig polynomials, as illustrated by the example of type $B_2 = C_2$.

Roughly speaking, the idea was to re-orient Lusztig's pictures in [27]. As shown in his rank 2 alcove diagrams in §12, there are "dual" *symmetric* patterns which determine Jantzen's generic decomposition patterns for weights in dominant p-alcoves far enough from walls of the dominant Weyl chamber. Here the distinguished alcove A is to be paired with any other fixed alcove B in his picture. The polynomial in q (or its inverse) placed in B is used to adjust the layer(s) on which the simple module with highest weight in the alcove A occurs in the Jantzen filtration of the Weyl module with highest weight in alcove B. The default layer depends just on the length difference of the two weights, while q is interpreted as moving the filtration level 2 units, and similarly for powers of q. (This parity condition is found throughout the theory, with q being thought of as the square of another indeterminate.)

Andersen was able to formalize my ideas and construct a rigorous proof of existence of such filrations in the modules $H^i(\lambda)$ (along with sum formulas) using his earlier construction of homomorphisms between sheaf cohomology modules. But this requires working first over $\mathbb{Z}$ and then reducing mod p, so in locations near Weyl chamber walls some torsion is encountered. For example, his work shows that "generically" the $H^i(\lambda)$ have a unique simple submodule and a unique highest composition factor. These occur for Weyl modules respectively on layers $N, 0$, with $N + 1$ the "generic" Loewy length of the filtration. (One caution is that the numbering of filtrations gets turned around, since a Weyl module is itself a cohomology module H^N.) Of course, the non-generic cases are more variable and often more interesting.

From my older conjectures about existence of reasonable filtrations came a specific "working hypothesis" about the precise condition for standard or nonstandard vanishing to occur. Here one starts with a weight λ inside a Weyl chamber (shifted by $-\rho$) where $H^i(\lambda) \neq 0$ in characteristic 0. Assume the generic filtration for λ crosses one or more chamber walls. For each alcove in the pattern outside the chamber, find an element $w \in W$ which takes the top vertex of its "box" (in Lusztig's sense) into the chamber; then consider the filtration layer(s) in each outside alcove and attach the sign of w to each. The net result of such signed cancellations is the formal character of $H^i(\lambda)$, which yields the Euler character given by Weyl's formula after combining (with alternating signs) all $H^j(\lambda)$. (See [19] for examples in G_2, which are worked out rigorously in [12].) The hypothesis (only checked so far in low rank examples) is:

There should be nonstandard vanishing for λ as above just when the filtration level of some linked weight in an alcove outside the given chamber fails to cancel the level of the corresponding alcove inside the chamber.

6 Weights on walls

While many questions remain open, the fit between the filtration proposals in [18] and the results written down so far (especially in low ranks) encourages the hope that my ideas and Andersen's methods will lead to more definitive results (probably after some refinements). The test case G_2 in [12] is encouraging, since the results achieved rigorously can be predicted formally from my viewpoint; already these predictions led to some corrections in Andersen's older treatment of G_2 in [5]. (Even so, the torsion obstruction he found later is cautionary, as are the results of Williamson and others on Lusztig's 1979 conjecture in spite of a rigorous proof of the latter by Andersen–Jantzen–Soergel in the 1990s when p is "large enough".)

It is instructive to look closely at what happens to weights lying in alcove walls (and possibly in Weyl chamber walls at the same time). Recall that the classical characteristic 0 theory developed by Bott leads to a uniform vanishing statement about weights lying in Weyl chamber walls: here all $H^i(\lambda)$ vanish. But the situation in characteristic $p > 0$ turns out to be much more delicate.

Under our standing assumption that $p \geq h$, there always exist p-regular weights $\lambda \in X$. In this situation the main tool for studying what happens on alcove walls is Jantzen's *translation principle* [26, II.7]. For $\lambda \in X^+$, translating λ to a weight $\mu \in X^+$ in the "upper closure" of its p-alcove takes $L(\lambda)$ to the simple module $L(\mu)$; otherwise the translate is 0.

What does this imply about a non-dominant but p-regular weight λ and the module $H^i(\lambda)$? Here one gets "generically" the formal character by passing to the dominant chamber using the dot-action of an element of the affine Weyl group relative to p. So the question is how a weight μ in an an adjacent alcove wall behaves under this transition to the dominant chamber.

The upshot is that some (though not all) weights on alcove walls yield nonzero sheaf cohomology, including even some for which the alcove walls lie in Weyl chamber walls. This is already seen in type A_2, where a weight such as for example $(p, -p-2)$ in standard coordinates lies in a wall separating an H^1 and an H^2 chamber. What Andersen's methods show is that both of these cohomology groups are nonzero; indeed, both are simple modules for G but cancel out in the Euler character. The adjacent p-alcoves yield nonzero H^1 and H^2, in fact, so one can appeal to the general principle in [12, Prop. 3.2].

7 Concluding remarks

While it's convenient to consider the case when $p \geq h$, the problem stated at the outset makes sense for all p (and all weights λ). So it remains a challenge to find a reasonably uniform solution.

Aside from this, there also remains the problem of accounting for the good behavior of dominant (or antidominant) weights in Kempf's vanishing theorem. If the theme of "cancellation" in canonical filtrations of cohomology modules is sensible, one must ask why the cancellation for Jantzen filtrations of Weyl modules is always well-behaved? Aside from trying to approach the proof of Kempf's theorem in a new way, this kind of question seems inevitable in its own right.

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