Two-sided cell attached to a minimal nilpotent orbit

April 2, 2012 (revised April 3, 2015)

In his series of papers on cells in affine Weyl groups (see [L2]), Lusztig established a bijection between the two-sided cells and the unipotent classes in a Langlands dual simple algebraic group G. In good characteristic, these classes are themselves in natural bijection with the nilpotent orbits of the corresponding Lie algebra $\mathfrak{g}$. (Lusztig's bijection has been revisited in the work of Bezrukavnikov, who also proved Lusztig's conjecture that the bijection is order-reversing for the two natural partial orderings involved.) Lusztig concluded part IV by formulating a detailed conjecture on the "asymptotic" Hecke algebra, in terms of a hypothetical finite set Y.

Under Lusztig's bijection, there is a (large) two-sided cell which corresponds to the (small) next-to-lowest unipotent class or nilpotent orbit. This orbit is the unique *minimal* nonzero orbit, the orbit of any long root vector in $\mathfrak{g}$. Its dimension is twice $h^\vee - 1$, where $h^\vee$ is the dual Coxeter number. Using combinatorial methods, Shi [S] proves that the number of left (or right) cells in the two-sided cell is always $\leq |W|/2$, where W is the finite Weyl group involved. (Shi denotes this finite group by W_0.) He conjectures that his bound is exact, which is known from calculations in some special cases.

We make two further observations:

(1) *Shi's conjecture would follow from the truth of Lusztig's older general conjecture on the number of left cells in an arbitrary two-sided cell* given in [L1], second paragraph of 3.6. Here the corresponding orbit of a nilpotent e has an associated Springer fiber $\mathcal{B}_e$ in the cotangent bundle of the flag variety, acted on by a finite component group $A(e)$ coming from the centralizer of e in G. A suitable version of cohomology for $\mathcal{B}_e$ then yields for the fixed point set of $A(e)$ an Euler characteristic (alternating sum of dimensions), predicted to count the desired number of left cells. Typically the odd cohomology all vanishes, so the conjecture involves just the fixed points of $A(e)$ on the total cohomology $H^*(\mathcal{B}_e)$.

The group $A(e)$ is trivial for e in the minimal nilpotent orbit: this follows from case-by-case study, as summarized in Carter's 1985 book for instance. On the other hand, for any nilpotent orbit (and a prime $p > h$, the Coxeter number), the dimension of the total cohomology of an associated Springer fiber is shown in [BMR], 5.4.3 and 7.1.1, to equal the number of simple modules in a "regular" block of the reduced enveloping algebra for $\mathfrak{g}$ associated to the nilpotent orbit in question. (When $A(e)$ is nontrivial, it permutes these simple modules in a natural way, with the number of orbits equal to the dimension of the fixed point set of $A(e)$ on $H^*(\mathcal{B}_e)$.)

In many cases the number of simple modules was calculated in [FP], Prop. 2.3 and Cor. 3.5), to be of the form $|W|/|W_I|$ with I a subset of simple reflections: for this e must be *regular* in a Levi subalgebra of the standard parabolic subalgebra of $\mathfrak{g}$ determined by I. (See [H1] for background.) When e lies in the minimal nilpotent orbit this "standard Levi type" condition is satisfied with I consisting of a single reflection, yielding the number $|W|/2$ which occurs in Shi's conjecture. Thus his conjecture would follow from the truth of Lusztig's conjecture (combined with known facts) in this special case.

(2) *In general, counting left cells in an arbitrary two-sided cell of an affine Weyl group might be approached using this indirect connection to counting the simple modules in a regular block of the reduced enveloping algebra for $\mathfrak{g}$ defined by the corresponding nilpotent orbit.* Indeed, the work of Bezrukavnikov et al. in [B] points strongly in this direction: Lusztig's conjectured finite set Y (of the correct cardinality arising from the dimension of the total cohomology of the Springer fiber) would be identified with the set of simple modules in a regular block. (This is outlined in some detail in [H2], based on conversations with Bezrukavnikov.)

[B] R. Bezrukavnikov et al., *On tensor categories attached to cells in affine Weyl groups*, I, II, III (2004, 2009); available on arXiv

[BMR] R. Bezrukavnikov, I. Mirkovic, D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, Ann. of Math. 167 (2008)

[FP] E.M. Friedlander, B.J. Parshall, *Deformations of Lie algebra representations*, Amer. J. Math. 112 (1990)

[H1] J.E. Humphreys, *Modular representations of simple Lie algebras*, Bull. Amer. Math. Soc. 35 (1998)

[H2] J.E. Humphreys, *Representations of reduced enveloping algebras and cells in the affine Weyl group*, Contemp. Math. 413, AMS (2006)

[L1] G. Lusztig, *Some examples of square-integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. 277 (1983)

[L2] G. Lusztig, *Cells in affine Weyl groups, IV*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (1989)

[S] Jian-yi Shi, *The second lowest two-sided cell in an affine Weyl group*, J. Algebra 333 (2011)