Notes on C_3

July 20, 2009

Consider the simple Lie algebra $\mathfrak{g}$ of type C_3 over an algebraically closed field K of characteristic $p > h = 6$ (the Coxeter number). Here $\mathfrak{g}$ is the Lie algebra of $\mathrm{Sp}_3(K)$. We have $\dim \mathfrak{g} = 21$, with $N = 9$ positive roots, while the Weyl group W has order 48.

As in the notes for type D_4 [9], we assemble some details about three aspects of $\mathfrak{g}$ which are known or conjectured to be closely related: nilpotent orbits; cells in a corresponding affine Weyl group; representations (not necessary restricted) of $\mathfrak{g}$ over K attached to nilpotent orbits. Some of the general background in [9] is repeated here, for convenience.

First we summarize concisely in a table known data in the case of C_3, followed by remarks and references to sources. Here are some abbreviations used in the table and elaborated in more detail below.

$\mathcal{O}_e$	Orbit of given $e \in \mathcal{N}$ (= nilpotent variety of $\mathfrak{g}$)
d	Half-dimension of $\mathcal{O}_e$
$A(e)$	Component group $A(e) := C_G(e)/C_G(e)^\circ$ if nontrivial
$\mathcal{B}_e$	Fiber over e in Springer's desingularization of $\mathcal{N}$, identified with the set of Borel subalgebras containing e
a	$\dim \mathcal{B}_e$ (= Lusztig's a-invariant of cell Ω_e associated to $\mathcal{O}_e$)
IC	Number of irreducible components of $\mathcal{B}_e$
R	Is $\mathcal{O}_e$ a Richardson orbit?
S	Is $\mathcal{O}_e$ special?
LT	Is $\mathcal{O}_e$ of standard Levi type (e is regular in some Levi subalgebra)?
M	Number of simple modules in a regular block of the reduced enveloping algebra attached to e
LC	Number of left cells in the two-sided cell Ω_e of the Coxeter group of affine type B_3 (dual to C_3)

orbit	d	$A(e)$	a	IC	R	S	LT	M	LC
$[6]$	9		0	1	Y	Y	Y	1	1
$[4,2]$	8	$\mathbb{Z}_2$	1	4	Y	Y	N	5	4
$[4,1^2]$	7		2	2	N	N	Y	6	6
$[3^2]$	7		2	3	Y	Y	Y	8	8
$[2^3]$	6		3	3	Y	Y	Y	12	12
$[2^2,1^2]$	5	$\mathbb{Z}_2$	4	5	Y	Y	Y	24	18
$[2,1^4]$	3		6	1	N	N	Y	24	24
$[1^6]$	0		9	1	Y	Y	Y	48	48

Table 1: Data for type C_3

Sources and remarks

(1) The nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$ comprises 8 nilpotent orbits relative to the adjoint group G. Labels for orbits involve the partitions of 6 whose odd parts have even multiplicity. The partial ordering of orbits by inclusion of one orbit in the closure of another is linear except for the two noncomparable orbits with $d = 7$. The table gives data about the C_3 orbits along with data on representations attached to an orbit and on the left cells of the associated 2-sided cell of the affine Weyl group of (dual) type B_3, relative to Lusztig's bijection recalled below.

(2) The nilpotent orbits of C_3 as well as component groups $A(e)$ have been well-studied: see for example [3, 4, 15]. In [3] see pp. 400, 435. In [4] see pp. 82, 96, 103.

(3) The Weyl group W of type C_3 has also been well-studied, together with its characters (in the general setting of type C_n). Here W is a semidirect product of S_3 with an elementary abelian group of order 8, the latter being normal. Thus $|W| = 48$. The set $\widehat{W}$ of its characters has 10 elements, of degrees $1, 1, 1, 1, 2, 2, 3, 3, 3, 3$. Of these, 8 are Springer characters having degrees $1, 1, 1, 2, 3, 3, 3, 3$; in turn, two of these having degrees $1, 2$ are nonspecial in Lusztig's sense. The characters of W are realized by Springer theory [25, App. III]. In case the component group $A(e)$ is trivial, the top cohomology of the Springer fiber $\mathcal{B}_e$ affords an irreducible character of W having degree equal to the number IC of irreducible components of $\mathcal{B}_e$. Values of IC for type C_3 are found in [26, p. 234].

(4) Lusztig conjectured and later proved (by using deep geometric methods) that there is a bijection between nilpotent orbits $\mathcal{O}_e$ of $\mathfrak{g}$ (or rather unipotent classes of G) and 2-sided cells Ω_e in the (dual) affine Weyl group: see [18] and the references there. In his bijection, the a-invariant of Ω_e agrees with the dimension of the Springer fiber $\mathcal{B}_e$ for a typical e in the corresponding orbit. The values range from $a = 0$ for the regular orbit to $a = N$ for the zero orbit. Here $d = N - a$ is half the dimension of the orbit, as seen in the table.

(5) In a 1983 paper, Lusztig [17, 3.6] conjectured that the number of left cells in the two-sided cell Ω_e is given in good characteristic by

$$\sum_{i\geq 0}(-1)^i \dim H^i(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell)^{A(e)}.$$

This has not yet been proved in general. He formulated the conjecture for unipotent elements and arbitrary p, but it carries over to nilpotents in good characteristic. In that case all odd degree cohomology is known to vanish, so the sum gives the dimension of the fixed point space of $A(e)$ on the total cohomology of the Springer fiber $\mathcal{B}_e$. (The dimension of this total cohomology is computable in most cases using Lusztig's induction theorem [23] for Springer representations.)

(6) Column LC in the table specifies the number of left cells in each correlated 2-sided cell of the affine Weyl group of dual type B_3: see Du [5]. The results here agree with Lusztig's conjecture just quoted. The 8 canonical left cells have been drawn in various colors by Gunnells [6]. (Note that in Du's figure on p. 1407 of [5], the labels E and F must be reversed, so that E corresponds to the special orbit $[3^2]$ of C_3 and F to the nonspecial orbit $[4, 1^2]$. These are colored respectively blue and green by Gunnells.)

(7) The zero orbit corresponds to restricted representations of $\mathfrak{g}$, coming from representations of a simply connected group of the same type: see [13]. For $p \geq h$, Lusztig's 1980 conjecture should provide recursively the dimensions and formal characters of simple modules in this case. This is not yet proved in full generality, but in any case the number of simple modules in each regular block for $\mathfrak{g}$ is given uniformly by $|W|$.

(8) For background on the non-restricted representations of $\mathfrak{g}$, see [7]; many details are worked out by Jantzen [10, 11, 12, 14, 16]. Simple modules attached by Kac–Weisfeiler to nilpotent orbits are the crucial ones to understand. As they conjectured and Premet proved (under mild restrictions), all $\mathfrak{g}$-modules for a given orbit of dimension $2d$ have dimensions divisible by p^d.

(9) A nilpotent orbit $\mathcal{O}_e$ has "standard Levi form" if e is regular in some Levi subalgebra of $\mathfrak{g}$, say determined by a subset I of simple reflections in W. In this case the number M of simple modules in a regular block is always $|W|/|W_I|$, where W_I is the subgroup of W generated by I (Friedlander–Parshall). More detailed information predicted by Lusztig [20] is verified by Jantzen in special cases (some unpublished).

(10) For the regular orbit (here $d = 9$), a regular block has only one simple module and its dimension is p^9. The subregular case ($d = 8$) is worked out for C_3 and most other cases by Jantzen [11]. In unpublished notes [16] (recently expanded) on type C_n he also gives details about both orbits with $d = 7$ (including explicit dimension formulas, which sometimes verify Lusztig's conjecture in [20] but sometimes assume it).

(11) For C_3 the number M can be computed by using the algebraic results just quoted. In general the work of Bezrukavnikov, Mirković, and Rumynin [1, 2] has shown for $p > h$ that the number M is given by the dimension of the total cohomology of the associated Springer fiber $\mathcal{B}_e$.

(12) Lusztig's proposed formalism [18, §10] for the asymptotic Hecke algebra associated with a 2-sided cell Ω_e of an affine Weyl group is expected to be modeled by the set of simple modules in a regular block of the corresponding reduced enveloping algebra for a simple Lie algebra (of dual type): see [22, 8] and forthcoming joint work of Bezrukavnikov and Mirković. Here each $A(e)$-orbit in the set of M simple modules in a regular block should be assigned uniquely to a left cell. (The numbers here make sense in view of Lusztig's approach [17] to counting left cells, combined with the result of [1] just quoted and the equivariance of their category equivalences relative to $A(e)$. This insures that $A(e)$ acts on the total cohomology of $\mathcal{B}_e$ by a permutation representation, forcing the number of orbits in the set of simple modules to agree with the dimension of the fixed point space.) Example: for C_3, the nilpotent

orbit of type $[2^2, 1^2]$ has component group $\mathbb{Z}_2$ and should act on the set of 24 simple modules with 18 orbits in all: 12 singletons and 6 pairs, in a natural bijection with the 18 left cells.

(13) In general it is reasonable to ask when a higher power of p than p^d can divide one or more dimensions of simple modules in a regular block attached to a nilpotent orbit of dimension $2d$. The relatively few examples known so far from Jantzen's work (in rank ≤ 4 or involving "small" blocks) behave consistently: In each instance there is a *special piece* of $\mathcal{N}$, involving a special orbit $\mathcal{O}_e$ together with one or more smaller nonspecial orbits in its closure; then $A(e) \neq 1$ according to Lusztig [19, Thm. 0.4]. Two or more simple modules attached to $\mathcal{O}_e$ form an $A(e)$-orbit, with a common dimension of the form $p^d m$ (p not dividing m). These should "deform" (or "degenerate") to a single module of the same dimension attached to a nonspecial orbit; such a pattern might be repeated (as occurs for G_2) in passing to a smaller nonspecial orbit, leading again to a higher than expected p-power in some dimension there. So far it is precisely for nonspecial orbits that examples are known where an unexpected p-power occurs; is this a general fact?

(14) In the case C_3 there are two special pieces: one involving the subregular orbit $[4, 2]$ with $d = 8$ and $A(e) = \mathbb{Z}_2$ along with the nonspecial orbit $[4, 1^2]$ with $d = 7$ and the other involving the special orbit $[2^2, 1^2]$ with $d = 5$ and $A(e) = \mathbb{Z}_2$ along with the nonspecial (minimal) orbit $[2, 1^4]$ with $d = 3$. Jantzen [16] worked out dimensions in the case when $d = 7$, confirming the expectation that one simple module has dimension divisible by p^8. It would be especially interesting to compute the p-powers dividing dimensions when $d = 3$. From the cell data one expects to find 6 simple modules in a regular block having dimension divisible by p^5 and involved in the deformations of 6 pairs of simple modules of equal dimension for the orbit with $d = 5$. A feature of this special piece not encountered in smaller examples is that d drops by 2 from the larger to the smaller orbit.

(15) The known dimension formulas all support the idea that there should be a uniquely defined "deformation" (or "degeneration") of simple modules attached to one orbit into modules of the same dimension typically having two or more composition factors attached to a smaller orbit in the closure of the first one. In almost all cases studied so far (with

an interesting exception already for G_2 [24]), the dimension formulas themselves allow for only one possible deformation.

(16) It is interesting to ask which features of the representation theory of $\mathfrak{g}$ (including the expected deformations) depend just on the relevant Springer fibers. For example, in the subregular case the Dynkin curves for types G_2, C_3, and D_4 are all isomorphic: a configuration of four projective lines with three parallel and the fourth intersecting each in a point. The component group $A(e)$ differs in each case, but we leave this aspect aside. Here the total cohomology of the Springer fiber in each case has dimen sion $5 = 1 + 4$, so there are 5 simple modules for $\mathfrak{g}$ in a regular block for the given nilpotent orbit.

Consider types G_2 and C_3. In each case the subregular orbit has a nonspecial orbit with $a = 2$ in its closure, and in each case a regular block relative to that orbit involves 6 simple modules while the total cohomology of each Springer fiber has dimension $6 = 1 + 3 + 2$ (by comparing numbers of irreducible components). It is natural to suspect that the two surfaces here are isomorphic, though even for dimension 2 the geometry of a Springer fiber is complicated to pin down. Moreover, the expected deformation pattern from the subregular case to this one yields in each case the same 5×6 matrix (with entries 0 and 1). Note that the Dynkin diagram of G_2 can be obtained from that of C_3 by "folding".

Similarly, consider types C_3 and D_4 (where the diagram of C_3 can be obtained from that of D_4 by "folding"). In each case the subregular orbit has a special orbit with $a = 2$ in its closure, and in each case a regular block relative to that orbit involves 8 simple modules while the total cohomology of each Springer fiber has dimension $8 = 1 + 4 + 3$. Again it is natural to suspect that the two surfaces here are isomorphic. And again the expected deformation pattern in each case is the same 5×8 matrix (with entries 0 and 1).

These speculations are strongly reinforced by the observed fact that the dimension formula for a simple module attached to the larger rank case in each situation can be "folded" naturally to produce the dimension formula for the smaller rank case.

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