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Stat 516 Handout - Properties of MLE

Suppose $Y_1, \ldots, Y_n$ be iid rvs from $f_\theta(y)$. Let $\hat{\theta}$ be the MLE for θ. Then we have

(i) $\hat{\theta}$ is consitent, i.e., $\hat{\theta} \to \theta$ as $n \to \infty$.

(ii) when n is large, $\hat{\theta}$ approximately has a normal distribution with mean θ and variance $I(\theta)$ (with θ replaced by $\hat{\theta}$), where $I(\theta)$ is the Cramer-Rao lower bound based on n observation defined by

$$I(\theta) = \frac{1}{n\ (-1)E\left\{\frac{\partial^2}{\partial^2\theta} \ln\ f_\theta(Y)\right\}}$$

(iii) An approximate $(1-\alpha)$ confidence interval for θ is given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{I(\hat{\theta})}$$

Asymptotic distribution of functions of MLE

Let $g(\theta)$ be a function of θ, say, $g(\theta) = \theta^2$. Let $\hat{\theta}$ be an MLE for θ. Then we have the following results:

(i) **Invariance property of MLE**: $g(\hat{\theta})$ is an MLE for $g(\theta)$.

(ii) When n is large, $g(\hat{\theta})$ approximately has a normal distribution with mean $g(\theta)$ and variance $[g'(\theta)]^2 I(\theta)$, where $g'(\theta) = d\, g(\theta)/d\, \theta$ (the derivative of $g(\theta)$ wrt θ), and $I(\theta)$ is the Cramer-Rao lower bound defined above.

(iii) An approximate $(1-\alpha)$ confidence interval for $g(\theta)$ is given by

$$g(\hat{\theta}) \pm z_{\alpha/2}\sqrt{[g'(\hat{\theta})]^2 I(\hat{\theta})}$$

Example: Suppose $Y \sim \text{Bino}(n,p)$. The mle for p is $\hat{p} = Y/n$. From problem 9.99 (verify it),

$$I(p) = p(1-p)/n$$

Consider $g(p) = p(1-p)$, we want to get an approximate C.I. for $p(1-p)$. From the invariance property of mle, the mle for $p(1-p)$ is $g(\hat{p}) = \hat{p}(1-\hat{p})$. Now $g'(p) = 1-2p$, the asympotic variance for $g(\hat{p})$ is

$$\begin{aligned} \text{AV}[g(\hat{p})] &= (1-2p)^2 p(1-p)/n \\ &= p(1-p)(1-2p)^2/n \end{aligned}$$

Consequently, An approximate $(1-\alpha)$ confidence interval for $p(1-p)$ is given by

$$\hat{p}(1-\hat{p}) \pm z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})(1-2\hat{p})^2/n}.$$