What is a billiard system?

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Overview of the talk

- What are billiards?
- Let's play a game!
- Why billiards are important?
- Applications

Billiards appear as natural models in many problems of mathematics and physics. The mechanic of billiard system is very similar to that of the cue sport of billiards.

- The billiard particle moves along straight lines inside the table;
- When it reaches the wall, the angle of incidence is equal to the angle of reflection (elastic collision).

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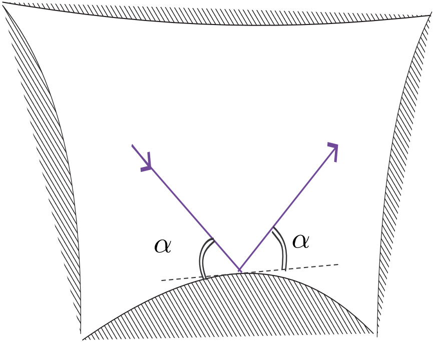

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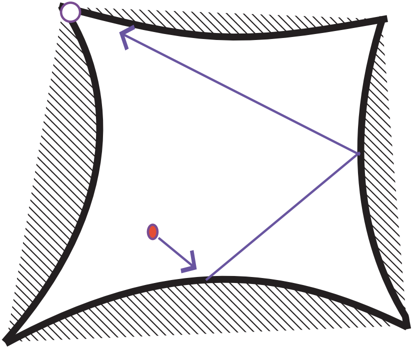
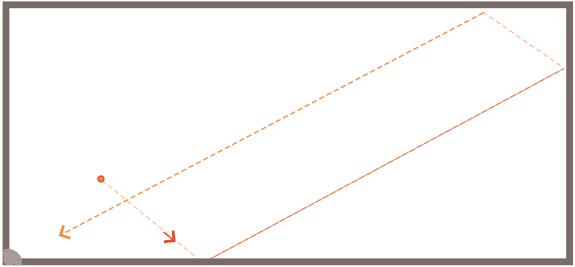
A game of billiard

- Assume the billiard table is smooth – no friction, but does not have to be rectangular;
- Assume the particle is tiny
- There is a hole somewhere on the table
- Goal: Hit the particle and make it collide with the wall at least once before running into the hole. Can we win the game?



Rectangular billiards



If the initial error is small, we can win the game easily.

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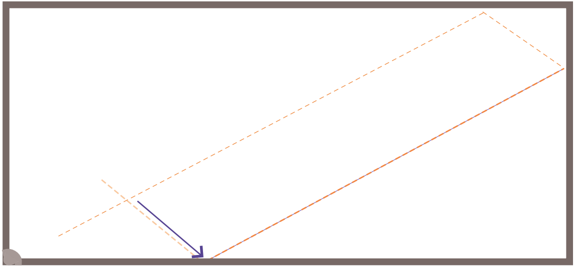

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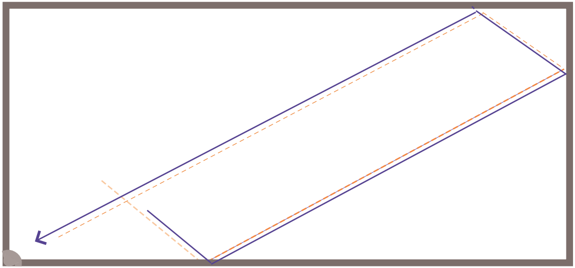

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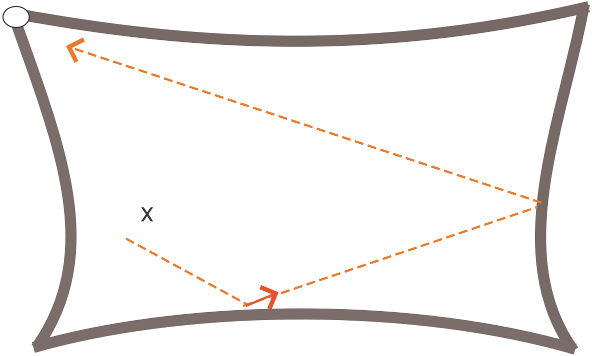
Dispersing billiard



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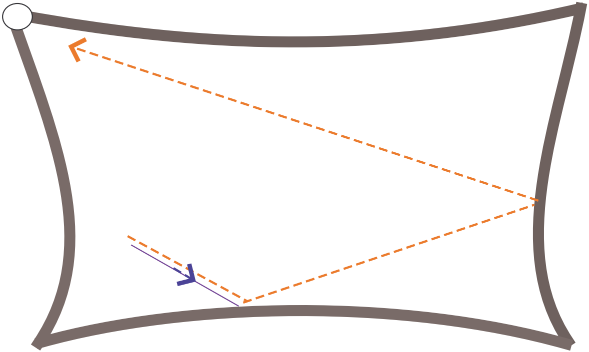

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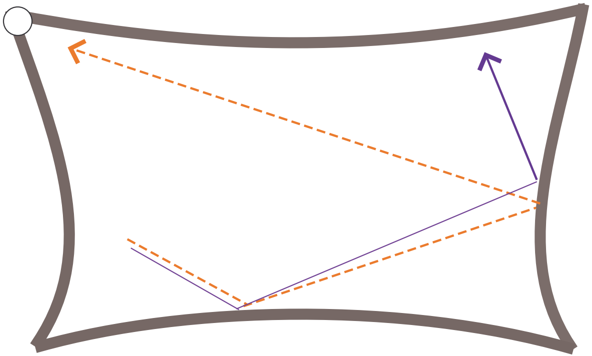

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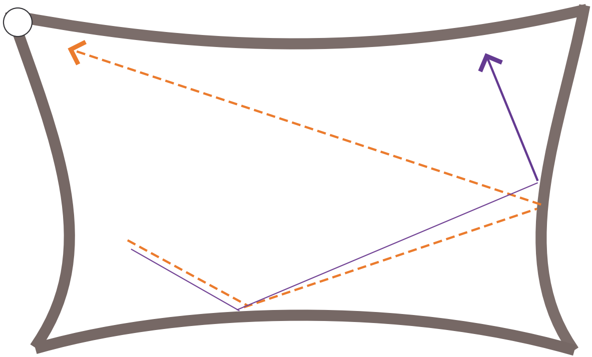
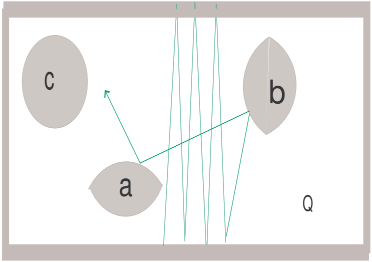


The error grows very fast, it is hopeless to win mostly even if the initial error is small. We call this billiard chaotic, since we can not predict a trajectory by knowing its adjacent ones.

Another chaotic billiard - Sinai billiard



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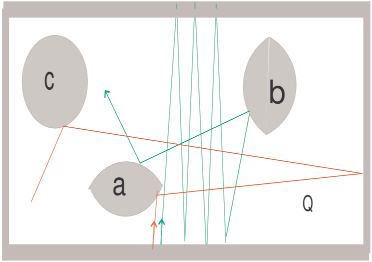

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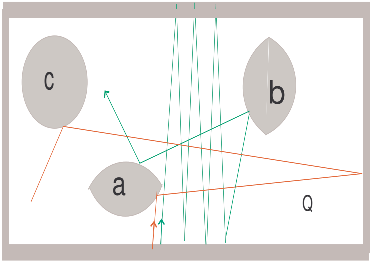
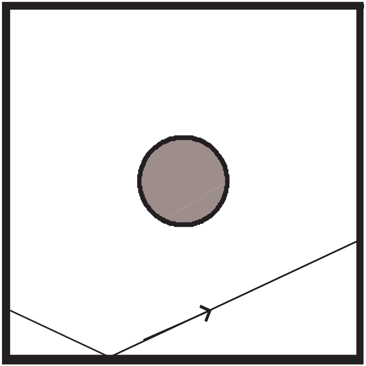
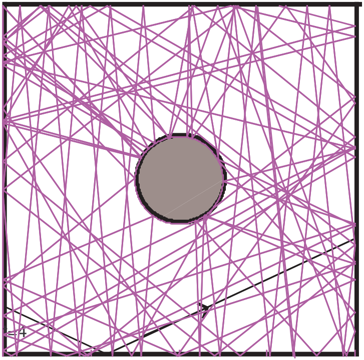


It seems there is NO way that we can win the game since it is a chaotic billiard!



- Wait: did we assume that the table is smooth? .....
- Can we see any hope in the long run?
- Yes, we can win the game certainly even for the chaotic Sinai billiard!



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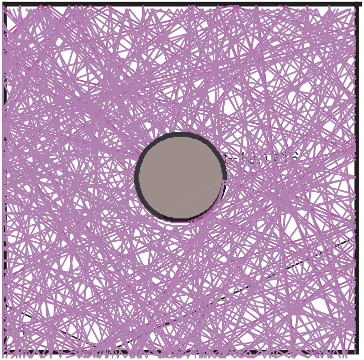
The game of billiards



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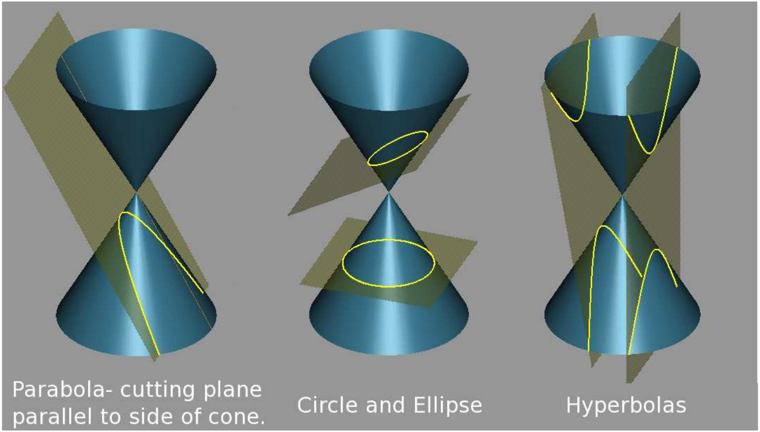
regular billiards

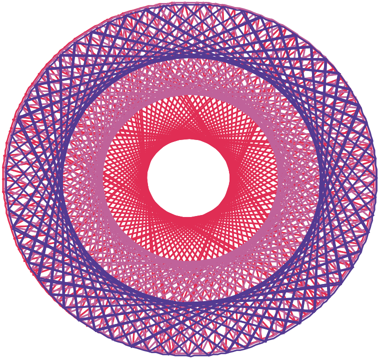
A regular billiard has properties "opposite" to chaotic one, i.e. we can predict a path very well according to its neighbors. For example, billiards on a circle or ellipse are regular.

Circle, ellipse, hyperbola and parabola can be obtained by cutting a cone using a plane. So these curves are also called conic sections.



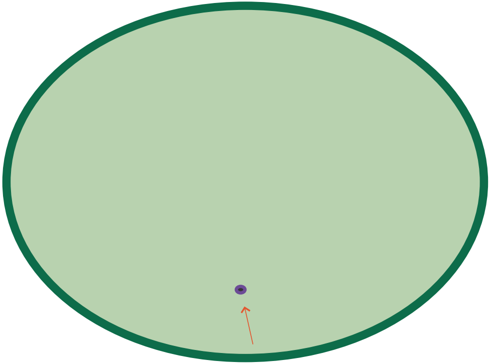
circular billiard

Regular system –circular billiards



- Any billiard path has a circular caustic; reversely, any concentric circle inside the table is the caustic of a path to a billiard path. We call a system regular if it has the above property.
- If two initial conditions are very close, then the two path remain close and have similar caustics.

Another regular billiard – elliptic billiards



Billiards in an ellipse has two families of caustics (confocal ellipses and confocal hyperbolas).

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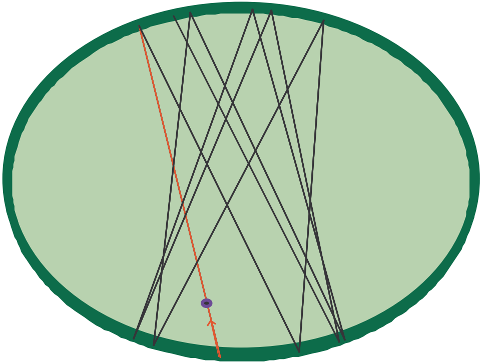

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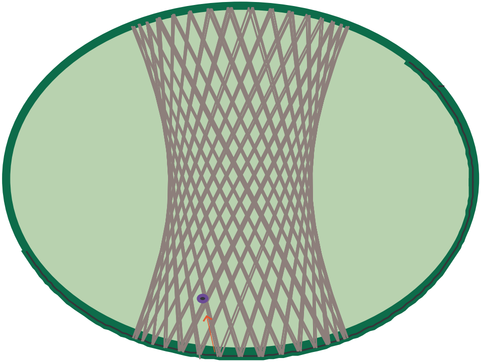

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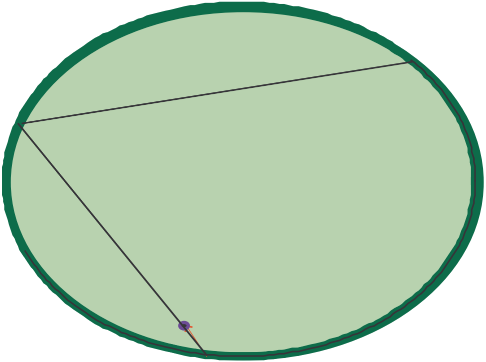

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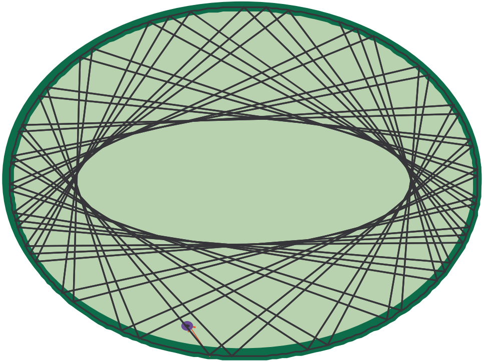

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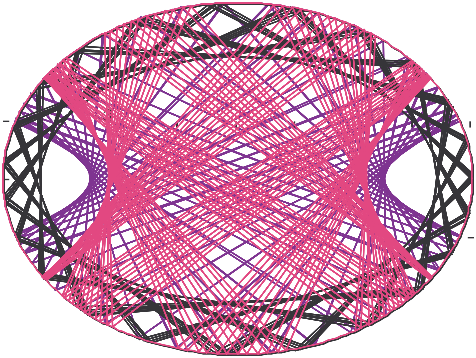

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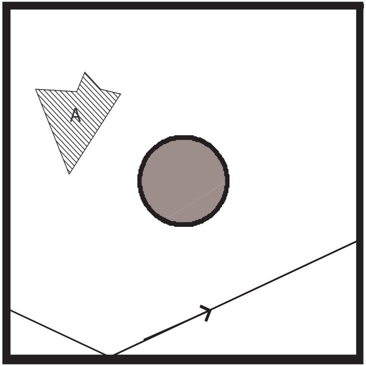
Birkhoff conjecture

It was conjectured by Birkhoff in 1927 that the only regular billiards are elliptic (including circular) billiards.
This conjecture is still unsolved – so anyone who is interested in math is encouraged to think about it.

Another question

Pick a region A in the Sinai billiard table, what is the frequency that a typical billiard trajectory visits A in the long run?



- Since eventually the images of x will be almost uniformly distributed in the phase space,
- Our conjecture:

$$\text{frequency of visiting } A \approx \text{area } A$$

- Any system has this property is called an ergodic system.

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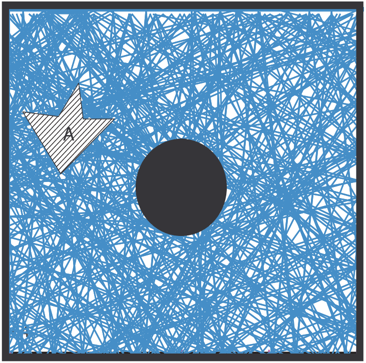

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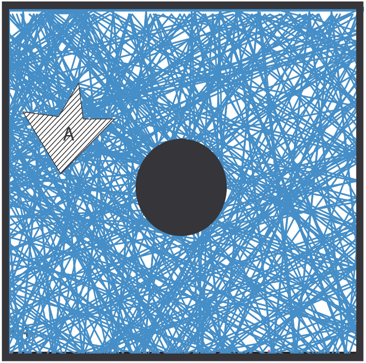

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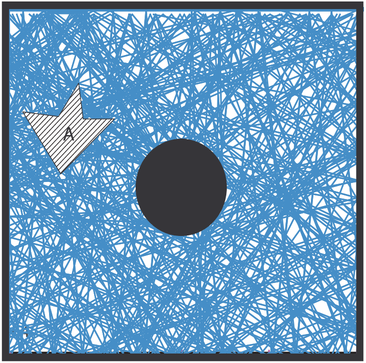

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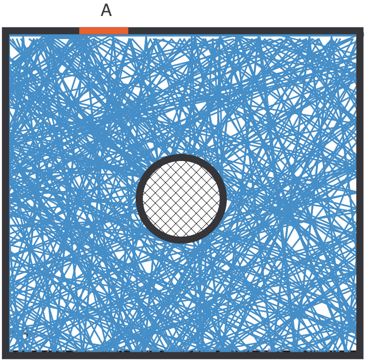

- The Sinai billiard is ergodic

Boltzmann's Hypothesis - a Conjecture for centuries?

Boltzmann's Ergodic Hypothesis, 1870's

For any closed system of large number of interacting particles in equilibrium, time averages are close to the space average. (Physical model)

Sinai Ergodic Hypothesis, 1961

The $n-$ dimensional Sinai billiard is ergodic for any $n \geq 2$. (Mathematical model)

Ya. G. Sinai

- Professor in Princeton.
- Proved the ergodicity of 2-d Sinai billiard in 1970's.

Discrete billiard models and applications

Billiards appear as natural models for many problems in physics, chemical engineering, and other fields

- Fermi acceleration
- Brownian motion
- Used to prove the Ohm's law