Contact Dynamics in 3 Dimensions: Some Explorations of Reeb-Beltrami Fields

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Abstract

We give a brief overview of the setting and philosophy of contact dynamics, and then focus on the 3-dimensional case, where the astounding results of Etnyre and Ghrist [4] give a correspondence between Reeb vector fields and Beltrami fields. We then turn our attention to some specific examples of both while in select cases the correspondence is explicitly made. Using simple numerics, we try to understand and visualize the qualitative behavior of the chosen examples.

1 What is a Contact Structure?

1.1 Defining Contact Structures

We begin with a mild amusement. A friend proposes a game: first, this friend specifies a smooth non-vanishing vector field $F \in \mathfrak{X}(\mathbb{R}^3) := \Gamma(T\mathbb{R}^3)$ (i.e. smoothly assigns a tangent vector at every point of $\mathbb{R}^3$), and then you specify a smooth non-vanishing vector field $G \in \mathfrak{X}(\mathbb{R}^3)$ which is nowhere parallel to F. Thus, together you have specified a smoothly varying distribution of tangent planes. Next, you choose a point $p \in \mathbb{R}^3$, and your friend chooses another point $q \in \mathbb{R}^3$. You are then seated in a flying saucer hovering at p, designed so that it can thrust forward and rotate without needing to adjust the pitch of the saucer plane. However, the flying saucer can only travel along paths such that the saucer plane (which we will assume contains your velocity vector) remains tangent to the specified plane distribution. The mission is to transport a tangle of topologists in this UFO from p to q. Your friend wins the game if the point q is chosen so that you are unable to arrive at your destination, and thus are subjected to spending an indeterminate amount of time flying around with the topologists. Otherwise, you win by successfully delivering the topologists to their destination. Under what conditions can you accomplish your mission and win the game?

Before accepting a potentially futile mission (which involves spending time with a knot of topologists, all the worse[1]!) you seek conditions on the distribution of planes that would ensure you can complete the flight. Feeling too shy to ask the giddy gang for assistance, you decide to review the theory of vector field distributions on manifolds yourself. Let $\mathfrak{H} \subset T\mathbb{R}^3$ be the distribution specified by your friend, identified as a sub-bundle of the tangent bundle of $\mathbb{R}^3$ (which is naturally diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}^3$— a fact convenient for visualization). By the well known theorem of Frobenius, if this distribution is completely integrable, then there exists a foliation of $\mathbb{R}^3$ by surfaces tangent to $\mathfrak{H}$, and if the system is locally integrable around a point $x \in \mathbb{R}^3$ then there exists a local foliation by integrable subsurfaces. To say you can always accomplish your mission is to say that for any pair of points $p, q \in \mathbb{R}^3$ there's a path connecting them, such that the tangent bundle of the path is a subbundle of the distribution $\mathfrak{H}$.

You realize quickly that your mission cannot be a success unless there are no neighborhoods where the distribution is integrable, for, within any neighborhood possessing integral subsurfaces, to ensure that you cannot travel to q from your chosen point p your friend merely has to specify q on a leaf of

[1] I consider this joke at the expense of topologists acceptable, since my primary research interests at present belong to the area of low-dimensional differential topology; I'm also known to occasionally travel in gangs of ≤ 4 low-dimensional topologists.

the foliation disjoint from any leaf containing p (if the leaves extend to a neighborhood containing p; otherwise your friend could simply choose q in a neighborhood whose leaves cannot be extended to a neighborhood of p, which would imply an obstruction, e.g. a leaf consisting of a closed surface enclosing some neighborhood).

Thus, to ensure that you can win no matter which point your friend specifies, in the very least you must choose your vector field so that the distribution is *maximally non-integrable*: writing the distribution $\mathfrak{H}$ as the span of the two smooth pointwise linearly-independent vector fields F and G, it must be the case that the Lie bracket $[F, G]$ is a vector field everywhere transverse to $\mathfrak{H}$, else there is a neighborhood where Frobenius tells us we can find integral submanifolds. We will henceforth refer to a maximally non-integrable distribution of (hyper-)planes as a contact distribution. The remarkable fact is that, if the distribution defined by F and G is a contact distribution, then you can always find a path tangent to the distribution from p to q, no matter choice of p and q in $\mathbb{R}^3$. A curve whose tangent space is contained in a contact distribution is called a *Legendre curve* for that contact distribution.

We switch perspectives: consider $\mathfrak{H} = \text{span}\{F, G\}$ as the kernel of some smooth one-form, $\alpha \in \Omega^1(\mathbb{R}^3)$. Then since $\alpha(F) = 0 = \alpha(G)$, and $[F, G] \notin \ker\alpha$ we note that

$$0 \neq \alpha([F, G]) = \alpha([F, G]) - F(\alpha(G)) + G(\alpha(F)) = -\mathrm{d}\alpha(F, G)\,,$$

whence $\mathrm{d}\alpha$ is a non-degenerate two form when restricted to the contact distribution. It follows that $\alpha \wedge \mathrm{d}\alpha$ is a non-vanishing top degree form, called a *contact volume form.*

We are now prepared to define contact structures on manifolds in greater generality. For simplicity we consider the case of an orientable manifold $\mathcal{M}$ with hyperplane bundle $\mathfrak{H}$ such that the quotient bundle $T\mathcal{M}/\mathfrak{H}$ is orientable (as a bundle).

Definition 1.1. Given $\mathcal{M}$ as above endowed with a maximally non-integrable hyperplane distribution $\mathfrak{H} \subset T\mathcal{M}$, the pair $(\mathcal{M}, \mathfrak{H})$ is referred to as a *contact manifold*, and the specification of $\mathfrak{H}$ on a given $\mathcal{M}$ is called a *contact structure*. Because $T\mathcal{M}/\mathfrak{H}$ is orientable one can find a globally defined one form $\alpha \in \Omega^1(\mathcal{M})$ such that $\mathfrak{H} = \ker\alpha$; α is then called a *contact form* for the contact structure $\mathfrak{H}$.

Contact structures can exist without global contact forms [2, p. 70] , but we will restrict our attention to the case where we can define global contact forms, since in the sequel our considerations of contact dynamics evolve from the choice of a contact form. There are also restrictions on the existence of contact structures, the most notable being dimension. Even if there is no global contact form, there are always local contact forms, and as in the preceding discussion, for a contact form α the two form $\mathrm{d}\alpha$ is nondegenerate when restricted to $\mathfrak{H}$. Since the rank of the subbundle $\mathfrak{H}$ is one less than the dimension of $\mathcal{M}$ and $\mathrm{d}\alpha$ is a non-degenerate two form on $\mathfrak{H}$, we conclude that the dimension of $\mathcal{M}$ is odd since the rank of $\mathfrak{H}$ is necessarily even.

A quick criterion for determining if a one form $\alpha \in \Omega^1(\mathcal{M}^{2n+1})$ is a contact form for some contact structure is to check if it can be used to build a contact volume form: $\alpha \wedge (\mathrm{d}\alpha)^n$ must be a volume form if α is a contact form, and conversely, if $\alpha \wedge (\mathrm{d}\alpha)^n \in \Omega^{2n+1}(\mathcal{M}^{2n+1})$ is a non-vanishing $2n+1$ form, then it is a contact volume form and α defines a contact distribution $\mathfrak{H} := \ker\alpha$. Note that there are many contact forms for a given contact structure: if $\mathfrak{H} = \ker\alpha$, then $\mathfrak{H} = \ker f\alpha$ for any nowhere vanishing $f \in \mathcal{C}^\infty(\mathcal{M})$. Thus there is not a uniquely specified choice of contact form or contact volume for a given contact structure. It is often convenient however to fix a contact form to specify a contact structure. An example, to be explored in greater depth later, is the *standard form* for $\mathbb{R}^3$: $\alpha_{std} := \mathrm{d}z + x\mathrm{d}y$. More generally, on $\mathbb{R}^{2n+1}$ with coordinates $(z, x^1, y^1, \ldots x^n, y^n)$, the standard form is $\alpha_{std} := \mathrm{d}z + \sum_{i=1}^n x^i \mathrm{d}y^i$. It earns its name due to the following theorem:

Theorem (Darboux)**.** *Given a contact manifold $(\mathcal{M}, \mathfrak{H})$ and a point $p \in \mathcal{M}$, there exists a coordinate chart $\big(\mathcal{U}, (z, x^1, y^1, \ldots x^n, y^n)\big)$ centered at p such that*

$$\mathfrak{H}|_{\mathcal{U}} = \ker\left(\mathrm{d}z + \sum_{i=1}^n x^i \mathrm{d}y^i\right)\,.$$

Returning to $\mathbb{R}^3$ with the standard contact structure, as determined by the form $\alpha_{std} := \mathrm{d}z + x\mathrm{d}y$, we can exploit the diffeomorphism $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ to create a vivid image of the plane distribution. To each point of $\mathbb{R}^3$ we may attach a plane spanned by the vectors $\mathbf{i}$ and $\mathbf{j} - x\mathbf{k}$ which correspond to the tangent vector fields ∂_x and $\partial_y - x\partial_z$ which span $\ker \alpha_{std}$.

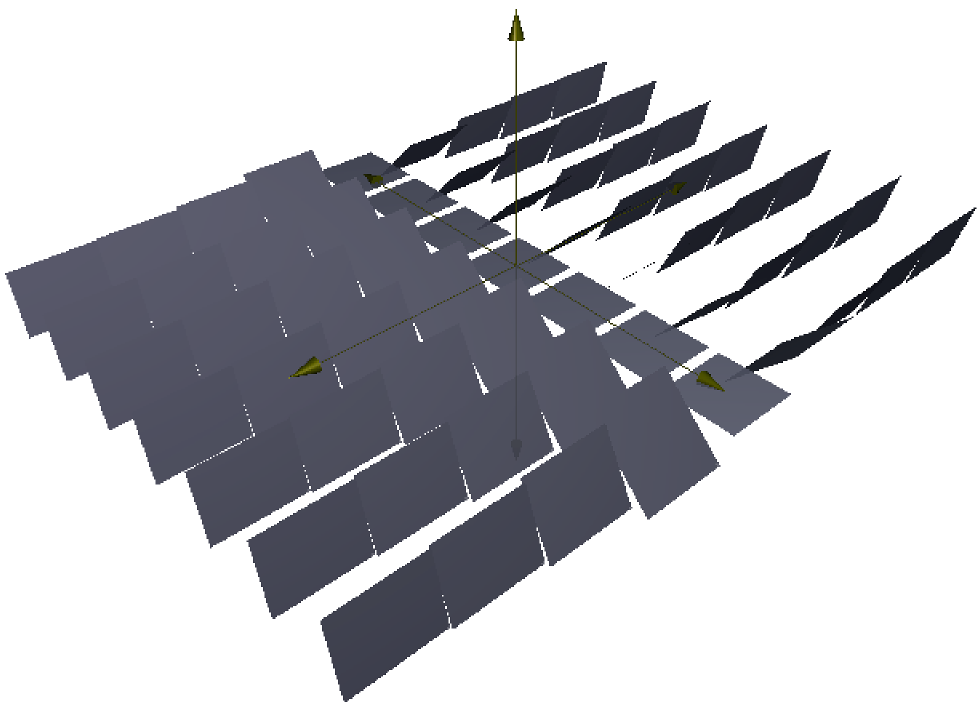

Figure 1: A visualization of some planes of the plane field $\mathfrak{H}_{std}$ on $\mathbb{R}^3$. Note the symmetry induced by translations $(x, y, z) \mapsto (x, y + t, z)$. There is another, unillustrated symmetry: vertically translating this swatch of planes produces another swatch of contact planes in $\mathfrak{H}_{std}$; this corresponds to the *contactomorpishms* $(x, y, z) \mapsto (x, y, z+t)$ generated by the flow of the *Reeb vector field* $R_{\alpha_{std}} = \partial_z$ of the form α_{std}. See the next section for definitions.

Note that the contact volume associated to α_{std} is the usual metric volume on $\mathbb{R}^3$ with respect to the Euclidean metric. We can construct a normal form for a contact structure on $\mathbb{R}^3$ in cylindrical coordinates as well: the form $\alpha_{rstd} := \mathrm{d}z + r^2\,\mathrm{d}\theta$ defines a contact structure on $\mathbb{R}^3$ with plane distribution

$$\mathfrak{H}_{rstd} = \operatorname{span}\left\{\partial_r, r^2\partial_z - \partial_\theta\right\} .$$

1.2 Characteristic Foliations and Tight versus Overtwisted Contact Structures

Contact structures on 3 manifolds come in two very topologically different flavors: *tight* and *overtwisted* [1, p. 33]. To define these for a contact manifold $(\mathcal{M}^3, \mathfrak{H})$ we first consider embeddings of disks $\mathcal{D} \hookrightarrow \mathcal{M}$, though the following construction applies to any embedded surface $\mathcal{S} \hookrightarrow \mathcal{M}$. Let $T\mathcal{D} \subset T\mathcal{M}$ be the tangent bundle of the $\mathcal{D}$ viewed as a subbundle of $T\mathcal{M}$. The key observation is that $\mathfrak{H} \cap T\mathcal{D}$ defines a line field with (generically finitely many) singular points $p \in \mathcal{D}$ where the disk is tangent to the plane distribution $\mathfrak{H}$). This line field can then be integrated to yield a foliation of the disk, with singularities occurring precisely at points of tangency [8, p. 78]. We call this foliation the *characteristic foliation* $\mathcal{F}_{\mathcal{D},\mathfrak{H}}$ of the disk $\mathcal{D}$ (or more generally of the embedded surface $\mathcal{S}$). For example, the characteristic foliation of a flat disk embedded with center on the z-axis and parallel to the coordinate plane $\{z = 0\}$ in $(\mathbb{R}^3, \mathfrak{H}_{rstd})$ consists of radial segments emanating from the center, which is the only singularity. If one bubbles this disk into a hemisphere while leaving the boundary fixed, the radial lines become "spirals", twisting up into a single singularity at the north pole.

One might suspect a "less tame" contact structure might support more interesting foliations on embedded disks. Let us provide an example: consider the one form defined in cylindrical coordinates on $\mathbb{R}^3$ by $\alpha_{o\tau} = r\cos r\,\mathrm{d}z + r\sin r\,\mathrm{d}\theta$. Note that this indeed is a contact form, as

$$\alpha_{o\tau} \wedge \mathrm{d}\alpha_{o\tau} = \left(1 + \frac{\sin 2r}{2r}\right) r\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z \neq 0 \text{ for } r > 0\,,$$

and so $\alpha_{o\tau} \wedge \mathrm{d}\alpha_{o\tau}$ defines a volume form where the coordinates are well defined. Unlike $\mathfrak{H}_{rstd}$ whose planes never rotate past the vertical as one travels along a ray $\theta = \theta_0$ away from the origin, for the contact structure $\mathfrak{H}_{o\tau} := \ker \alpha_{o\tau}$ as one travels along rays $\theta = \theta_0$ the planes complete *infinitely many complete rotations*. Consider now an embedding of a disk $\mathcal{D}$ with radius π in $(\mathbb{R}^3, \mathfrak{H}_{o\tau})$ centered on the z-axis, tangent to the coordinate plane $\{z = 0\}$ on the boundary, but raised slightly in the center. Note that for $r = \pi$, $\alpha_{o\tau}(\pi) = -\pi\,\mathrm{d}z$. Thus the boundary is a *limit cycle* for the leaves of the characteristic foliation, which spiral from the center of the disk out toward the boundary. We say such a disk is *overtwisted.*

Definition 1.2. A contact 3-manifold $(\mathcal{M}^3, \mathfrak{H})$ is said to be *overtwisted* if there exists a disk $\mathcal{D} \hookrightarrow \mathcal{M}$ such that the characteristic foliation $\mathcal{F}_{\mathcal{D},\mathfrak{H}}$ arising from the singular line field $\mathfrak{H} \cap T\mathcal{D}$ possess a limit cycle, i.e. a closed leaf which is an accumulation orbit for other leaves of the foliation. If no overtwisted disks exist in $(\mathcal{M}^3, \mathfrak{H})$ we say that the contact structure $\mathfrak{H}$ is *tight.*

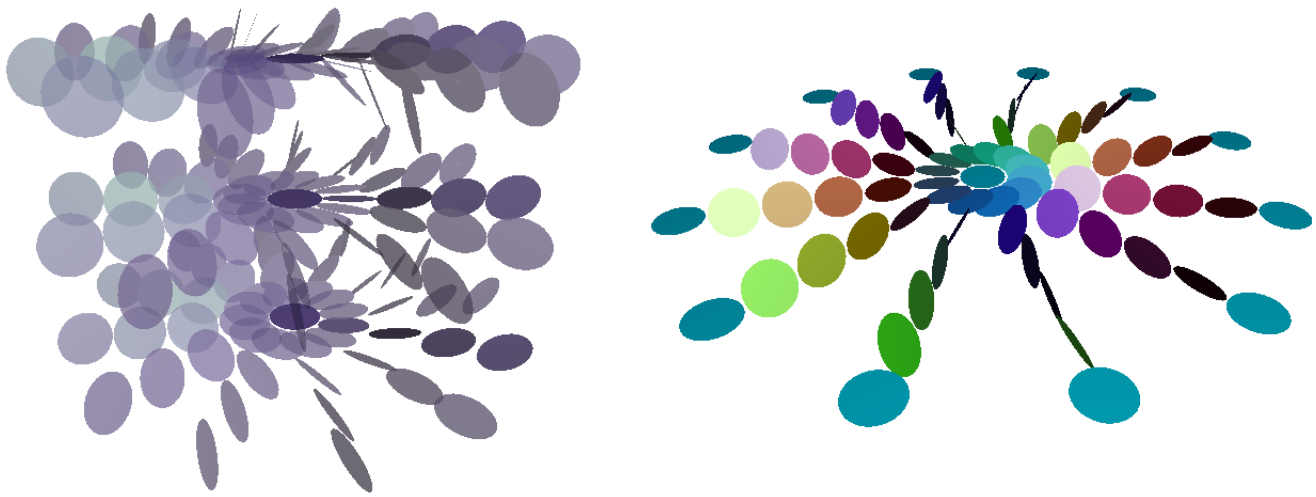

Figure 2: On the right is a visualization of some planes of $\mathfrak{H}_{rstd}$, and on the left is a visualization of the overtwisted structure $\mathfrak{H}_{o\tau}$ on $\mathbb{R}^3$.

In particular, the standard structure $\mathfrak{H}_{std}$ on $\mathbb{R}^3$ and its radially-symmetric counter part $\mathfrak{H}_{rstd}$ are tight, while $\mathfrak{H}_{o\tau}$ is overtwisted.

For the remainder of the paper, we would like to investigate the connections between certain dynamical constructions arising from a contact structure on a three manifold, and steady-state fluid mechanics on three manifolds. Let us fix some notation going forward: $(\mathcal{M}, \mathfrak{H})$ will always denote an oriented $2n+1$ dimensional smooth contact manifold, and $(\mathcal{M}, \mathfrak{H}, \alpha)$ will denote a smooth $2n+1$ dimensional contact manifold with $\alpha \in \Omega^1(\mathcal{M})$ a fixed global contact form for $\mathfrak{H}$. However, whenever possible, examples will be taken to be three dimensional.

2 Contact Dynamics - An Overview

2.1 Reeb and Contact Fields

In 1948 Seifert posed the following question: Does every non-vanishing vector field on $\mathbb{S}^3$ possess closed orbits? Counterexamples of class $\mathcal{C}^1$ and even $\mathcal{C}^\infty$ are known [2], so we look to a slightly more rigid generalization. Suppose instead that we are interested in nonsingular flows preserving a distinguished volume form $\mu \in \Omega^3(\mathcal{M})$ on $\mathcal{M} = \mathbb{S}^3$ or $\mathbb{R}^3$. Thus we investigate vector fields $F \in \mathfrak{X}(\mathcal{M})$ such that

$$0 = \mathcal{L}_F\mu = \mathrm{d}(\iota_F\mu)\,.$$

Since $H^2_{\mathrm{dR}}(\mathcal{M}) = 0$, we conclude that $\iota_F\mu \in \Omega^2(\mathcal{M})$ is exact, so there is some $\alpha \in \Omega^1(\mathcal{M})$ such that $\iota_F\mu = \mathrm{d}\alpha$. Then $\iota_F\mathrm{d}\alpha = \iota_F\iota_F\mu = 0$. If $\iota_F\alpha > 0$ throughout $\mathcal{M}$, we call F *positive* for α. If a volume preserving flow on $\mathcal{M}$ comes from a field F which is positive for α, then $\alpha \wedge \mathrm{d}\alpha$ is also a volume form. In this case we may normalize F by setting $R = F/\iota_F\alpha$ so that $\iota_R\alpha = 1$, and observe that R preserves the new volume form $\alpha \wedge \mathrm{d}\alpha$:

$$\mathcal{L}_R(\alpha \wedge \mathrm{d}\alpha) = (\mathcal{L}_R\alpha) \wedge \mathrm{d}\alpha - \alpha \wedge (\mathcal{L}_R\mathrm{d}\alpha) = 0 - \alpha \wedge \mathrm{d}(\mathcal{L}_R\alpha) = 0\,.$$

This motivates the following philosophy: to study volume preserving flows on $\mathbb{R}^3$, $\mathbb{S}^3$, or any other 3-manifold with vanishing second de Rham cohomology, one may instead choose to investigate vector fields R satisfying

$$\begin{cases} \iota_R\mathrm{d}\alpha = 0 \\ \iota_R\alpha = 1 \\ \alpha \wedge \mathrm{d}\alpha \text{ is a volume form.} \end{cases}$$

Since $\alpha \wedge \mathrm{d}\alpha$ is in fact a contact volume form, we implicitly arrived at a contact structure on our manifold of interest, $\mathcal{M}$. Vector fields satisfying these conditions with respect to a contact structure are themselves objects worthy of study, even without the additional structure of a preserved volume, or the assumption that there is no second cohomology.

Definition 2.1. Given a contact manifold $(\mathcal{M}, \mathfrak{H})$ and a global contact form α for $\mathfrak{H}$, there is a unique vector field R_α satisfying $\iota_R\mathrm{d}\alpha = 0$ and $\iota_R\alpha = 1$, called a *Reeb vector field* for the contact form α on $(\mathcal{M}, \mathfrak{H})$. A vector field $F \in \ker \mathrm{d}\alpha$ which is positive with respect to α, i.e. $\iota_F\alpha > 0$ throughout $\mathcal{M}$, is called a *Reeb-like* vector field, and its flow is said to be a *Reeb-like flow.*

A first crucial observation is that the flow $\Psi_{R_\alpha} : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ of the Reeb vector field R_α associated to a contact form α actually preserves α and thus also preserves the contact structure: for all $t \in \mathbb{R}$, $(\Psi^t_{R_\alpha})^*\alpha = \alpha$, and for any contact plane $\Xi_p \in \mathfrak{H}$, $p \in \mathcal{M}$, $(\Psi^t_{R_\alpha})_*\Xi_p = \Xi_{\Psi^t_{R_\alpha}(p)}$. A diffeomorphism with the property of preserving the contact structure is called a *contactomorphism.* Note that preserving the contact form is a stronger condition than preserving the contact structure. To distinguish these, we consider, in the language adopted by Geiges [6, p. 32] (in the spirit of Sophus Lie) the notions of *infinitesimal automorphisms* and *strict infinitesimal automorphisms* of the contact manifold.

Definition 2.2. Let $(\mathcal{M}, \mathfrak{H})$ be a contact manifold. For a vector field $X \in \mathfrak{X}(\mathcal{M})$, let Ψ_t be the (local) flow associated to X. We call X an *infinitesimal automorphism of the contact structure* $\mathfrak{H}$ if $\Psi^t_*(\mathfrak{H}) = \mathfrak{H}$ (wherever Ψ^t_* is well defined); we may also refer to X as a *contact vector field.* A contact vector field on $(\mathcal{M}, \mathfrak{H}, \alpha)$ which further satisfies $\Psi^*_t\alpha = \alpha$ is known as a *strict infinitesimal automorphism for the contact form* α, or occasionally as a *strict contact vector field.*

Example 2.1. Let us show that the Reeb vector field of $(\mathbb{R}^3, \mathfrak{H}_{std}, \alpha_{std})$ is in fact $R = \partial_z$. Let $R_{std} = R_x\partial_x + R_y\partial_x + R_z\partial_x$ be the desired Reeb vector field. Then

$$\begin{aligned} 0 &= \iota_{R_{std}}\mathrm{d}\alpha_{std} = \mathrm{d}x \wedge \mathrm{d}y(R_x\partial_x + R_y\partial_x + R_z\partial_x, \cdot) = R_x\,\mathrm{d}x - R_y\,\mathrm{d}y \\ &\implies R_x = 0 = R_y \\ 1 &= \alpha_{std}(R_{std}) = R_z + xR_y = R_z \\ &\implies R_z = 1\,, \end{aligned}$$

and thus $R_{std} = \partial_z$ as claimed. One might now be concerned that Reeb fields aren't very interesting, but on the contrary, we will later see just how chaotic they can be, even for the simple contact structures described so far.

The richness of the dynamics of contact and Reeb vector fields comes in part from their variety: a given contact structure supports unfathomably diverse dynamics via uncountably many vector fields of contact and Reeb type:

Theorem 2.1. *Given a contact manifold* $(\mathcal{M}, \mathfrak{H}, \alpha)$*, there is a bijective correspondence between smooth functions* $H \in \mathcal{C}^\infty(\mathcal{M})$ *functions on* $\mathcal{M}$ *and infinitesimal automorphisms of* $\mathfrak{H}$ *(dependent on* α*). Moreover, given our initial contact form* α*, there is a bijection between non-vanishing smooth functions on* $\mathcal{M}$ *and pairs* $(\tilde{\alpha}, R_{\tilde{\alpha}})$ *of contact forms and their Reeb fields on* $(\mathcal{M}, \mathfrak{H})$*.*

Proof. The second assertion follows readily from the fact that the space of global annihilating one forms $\operatorname{Ann}(\mathfrak{H})$ for $\mathfrak{H}$ is a line bundle, whence given a non-vanishing section of $T^*\mathcal{M}$ which is in $\operatorname{Ann}(\mathfrak{H})$, one can obtain all other forms $\tilde{\alpha} \in \operatorname{Ann}(\mathfrak{H})$ via scaling by non-vanishing smooth functions $f \in \mathcal{C}^\infty(\mathcal{M})$. Since the set of vector fields annihilating $\mathrm{d}\tilde{\alpha}$ is also a line bundle ($\mathrm{d}\tilde{\alpha}$ is nondegenerate on a codimension 1 subbundle of $T\mathcal{M}$), it follows that the normalization $\tilde{\alpha}(R_{\tilde{\alpha}}) = 1$ determines a unique Reeb vector field for $\tilde{\alpha}$, which establishes the uniqueness asserted in definition 2.2.

For the second assertion, the correspondence is explicitly defined as follows: given $H \in \mathcal{C}^\infty(\mathcal{M})$, we produce an infinitesimal automorphism X_H satisfying the relations

$$\begin{cases} \alpha(X_H) = H & \text{and} \\ \iota_{X_H}\mathrm{d}\alpha = \iota_{R_\alpha}\mathrm{d}(H\alpha)\,, \end{cases}$$

where R_α is the Reeb vector field for α. It is immediate that R_α is annihilated by the one form $\iota_{R_\alpha}\mathrm{d}(H\alpha) = \mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H$. By non-degeneracy of $\mathrm{d}\alpha$ on $\mathfrak{H}$, inclusion of contact vector fields into $\mathrm{d}\alpha$ induces an isomorphism from $\operatorname{Ann}(R_\alpha)$ to $T\mathcal{M}/\mathfrak{H} \cong \mathcal{C}^\infty(\mathcal{M})$. Thus given H, we can find $X_H = HR_\alpha + Y$ where $Y \in \mathfrak{H}$ which satisfies $\iota_{X_H}\mathrm{d}\alpha = \mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H$, and conversely given a contact vector field X there is a unique vector field Y such that $X = \alpha(X)R_\alpha + Y$ is determined by the above conditions, where $H = \alpha(X)$. We just need to check that the vector field X_H corresponding to H is indeed a contact vector field. It suffices to show that $\mathcal{L}_{X_H}\alpha = \mu\alpha$ for some $\mu\colon \mathcal{M} \to \mathbb{R}$. Indeed, if $\mathcal{L}_{X_H}\alpha = \mu\alpha$, then one obtains the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi^t_{X_H}\right)^*\alpha = \left(\Psi^t_{X_H}\right)^*\mathcal{L}_{X_H}\alpha = \left(\Psi^t_{X_H}\right)^*(\mu\alpha) = \left(\mu \circ \Psi^t_{X_H}\right)\left(\Psi^t_{X_H}\right)^*\alpha\,,$$

whose solution is

$$\left(\Psi^t_{X_H}\right)^*\alpha = \left(\exp\int_0^t \mu \circ \Psi^\tau_{X_H}\,\mathrm{d}\tau\right)\alpha\,.$$

Thus, if $\mathcal{L}_{X_H}\alpha = \mu\alpha$ then the flow $\Psi^t_{X_H}$ preserves the line bundle $\ker\alpha = \mathfrak{H}$, i.e. X_H is an infinitesimal automorphism. Now, if X_H is the vector field produced from $H \in \mathcal{C}^\infty(\mathcal{M})$ by the procedure above, then by Cartan's magic formula

$$\begin{aligned} \mathcal{L}_{X_H}\alpha &= \iota_{X_H}\mathrm{d}\alpha + \mathrm{d}(\iota_{X_H}\alpha) \\ &= \mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H + \mathrm{d}\big(\alpha(X_H)\big) \\ &= \mathrm{d}H(R_\alpha)\alpha\,. \end{aligned}$$

□

This correspondence is perhaps not a mere coincidence of linear algebra and differential equations, but points to something deeper about contact structures. The role of the "potential" H associated to an infinitesimal automorphism of $(\mathcal{M}, \mathfrak{H})$ is strongly analogous to the case of Hamiltonian functions on a symplectic manifold and their Hamiltonian vector fields (or *symplectic gradients*; the correspondence depends on the symplectic form just as our contact automorphism correspondence depends on the contact form). One thus arrives at the following language which emphasizes this parallel:

Definition 2.3. For a contact manifold $(\mathcal{M}, \mathfrak{H}, \alpha)$ and a given $H \in \mathcal{C}^\infty(\mathcal{M})$, we say X_H is a *contact vector field for the contact Hamiltonian* H*.*

Just as with the symplectic case, one can extend this language to *time dependent contact Hamiltonians* by showing that there is a parallel correspondence between smooth families of functions $\{H_t\}_{t\in\mathbb{R}}$, $H \in \mathcal{C}^\infty(\mathcal{M}\times\mathbb{R})$ and smooth one parameter families $X_{H_t} \in \mathfrak{X}(\mathcal{M}\times\mathbb{R})$ of contact vector fields, which yield *contact isotopies*. The proof is completely analogous, except that one appeals to the existence and uniqueness of solutions to non-autonomous ordinary differential equations, in this case arising from smooth time dependent vector fields, or smooth families of contact forms. The machinery of time-dependent contact Hamiltonian dynamics allow for many differential topological constructions sensitive to the contact structure of a given contact manifold but these would take us far afield.

2.2 The Weinstein Conjecture

Weinstein like Seifert chose to consider the question of existence of periodic orbits, but confined his attention to the class of Reeb flows on contact manifolds. He proposed the following conjecture in 1979 [10]:

Conjecture 2.1. *Let $(\mathcal{M}, \mathfrak{H}, \alpha)$ be a closed contact manifold with contact form α, and let R_α be the associated Reeb vector field. Then the flow $\Psi^t_{R_\alpha}$ of the Reeb field admits a periodic orbit.*

Much work had been done to understand the state of conjecture For example, Hofer and Viterbo proved that it holds in special cases such as $\mathbb{S}^3$, contact manifolds with vanishing second homotopy, and for overtwisted contact structures [2, p. 78]. For a closed three manifold, the conjecture is now resolved in general by the work of Clifford Taubes, who applied gauge-theoretic invariants (Seiberg-Witten-Floer theory) to prove the result [9].

2.3 Classifying Reeb fields for a given Contact Distribution

We now explore the variety of Reeb fields for a given contact structure. Suppose a contact manifold $(\mathcal{M}, \mathfrak{H})$ admits a global contact form $\alpha \in \Omega^1(\mathcal{M})$. Then $\mathfrak{H} = \ker f\alpha$ for any $f \in \mathcal{C}^\infty(\mathcal{M})$ such that f is nowhere vanishing. A natural question is "how does rescaling α to $f\alpha$ alter the Reeb field?"

If $R_{f\alpha}$ is the Reeb field for the contact form $f\alpha$, then we know that $R_{f\alpha} \in \ker \mathrm{d}(f\alpha)$ and $f\alpha(R_{f\alpha}) = 1$. Thus the new conditions can be written as

$$\begin{cases} \iota_{R_{f\alpha}}\mathrm{d}(f\alpha) = \mathrm{d}f(R_{f\alpha})\alpha - \alpha(R_{f\alpha})\mathrm{d}f + f\iota_{R_{f\alpha}}\mathrm{d}\alpha = 0\,, \\ \iota_{R_{f\alpha}}\alpha = \frac{1}{f}\,. \end{cases}$$

Little can be said about the specific dynamics which can result from such a rescaling of the contact form without working with some explicit contact form in coordinates. A naïve first question might be "can every vector field transverse to the contact field be realized as a Reeb field for some choice of rescaled contact form?" The answer is resoundingly "no" even in the simplest cases, as we shall presently see. The above conditions are linear algebraic conditions on $R_{f\alpha}$ once $f\alpha$ is known, but if given a non-vanishing $R \in \mathfrak{X}(\mathcal{M})$ and a contact form α, if one wishes to determine if $R = R_{f\alpha}$ for some f, it is necessary that $f = 1/\alpha(R)$ while simultaneously f must satisfy the *partial differential equation*

$$\mathrm{d}f(R)\alpha - \alpha(R)\mathrm{d}f + f\iota_R\mathrm{d}\alpha = 0\,.$$

One can reduce this to a PDE for the coefficients of R and hope to classify all solutions of the PDE, but alas, the resulting equation is highly nonlinear even for the simplest choices of contact form α, such as the standard structure on $\mathbb{R}^{2n+1}$. But upon inspection it is apparent that the PDE presents a condition strictly stronger than transversality.

A better approach is to fix a contact form α for $\mathfrak{H}$ and let the set of smooth, non-vanishing functions on $\mathcal{M}$ parametrize the set of Reeb fields $R_{f\alpha}$ which can be associated to $\mathfrak{H} = \ker f\alpha$. Let us see what we can say concretely for the standard form $\alpha_{std} = \mathrm{d}z + x\,\mathrm{d}y$ for $\mathbb{R}^3$ with the standard tight structure.

Let $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ be a non-vanishing function. Writing $R = R^x\partial_x + R^y\partial_y + R^z\partial_z$ for our Reeb field and using the notation $f_x := \partial_x f$, $f_y := \partial_y f$, and $f_z := \partial_z f$ for the coordinate derivatives of f, we consider solutions of

$$\iota_R \mathrm{d}(f\alpha_{std}) = (R^x f_x + R^y f_y + R^z f_z)(\mathrm{d}z + x\mathrm{d}y) - (R^z + xR^y)(f_x\mathrm{d}x + f_y\mathrm{d}y + f_z\mathrm{d}z) + f(R^x\mathrm{d}y - R^y\mathrm{d}x) = 0\,,$$

subject to the normalization $f(R^z + xR^y) = 1$. This is equivalent to seeking a normalized element of the kernel of the matrix

$$\mathrm{A}_f := \begin{bmatrix} 0 & -f - xf_x & -f_x \\ f + xf_x & 0 & xf_z - f_y \\ f_x & f_y - xf_z & 0 \end{bmatrix}.$$

Since this is a 3×3 skew-symmetric matrix, it corresponds to a cross product operator, e.g. it can be realized as the matrix of

$$-\Big((f_y - xf_z)\partial_x - f_x\partial_y + (f + xf_x)\partial_z\Big)\times : T\mathbb{R}^3 \to T\mathbb{R}^3\,,$$

in the coordinate basis $\{\partial_x, \partial_y, \partial_z\}$ of $T\mathbb{R}^3$. Thus an element of the kernel of A_f is parallel to the vector field $(f_y - xf_z)\partial_x - f_x\partial_y + (f + xf_x)\partial_z$. Using the normalization, one finds that the desired Reeb vector field of the contact form $f\,\alpha_{std} = f(\mathrm{d}z + x\,\mathrm{d}y)$ is thus

$$R = \frac{1}{f^2}\Big((f_y - xf_z)\,\partial_x - f_x\,\partial_y + (f + xf_x)\,\partial_z\Big).$$

Note that choosing $f \equiv 1$ yields $R_{std} = \partial_z$, as expected.

A similar computation shows that on $(\mathbb{R}^3, \mathfrak{H}_{rstd})$ for $f\alpha_{rstd}$, where $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ is nonzero for $r \neq 0$, the Reeb vector fields have the form

$$R = \frac{1}{2rf^2}\Big((f_\theta - r^2 f_z)\,\partial_r - f_r\,\partial_\theta + (2rf + r^2 f_r)\,\partial_z\Big)$$

Finally, we also look at the overtwisted contact structure on $\mathbb{R}^3$ determined by $\alpha_{o\tau} = r\cos r\,\mathrm{d}z + r\sin r\,\mathrm{d}\theta$. Here one finds that

$$R = \frac{1}{(r + \sin r\cos r)f^2}\Big((f_\theta\cos r - rf_z\sin r)\,\partial_r + (f\sin r - f_r\cos r)\partial_\theta + (rf_r\sin r + f\sin r + rf\cos r)\partial_z)\Big).$$

The case where $f \equiv 1$ yields the vector field

$$R_{o\tau} = \frac{1}{(r + \sin r\cos r)}\Big(\sin r\,\partial_\theta + (\sin r + r\cos r)\,\partial_z\Big)$$

We will later examine a few cases of choices of f and visualize the integral curves of the preceding Reeb fields for the chosen f using simple numerics. First we introduce the astounding connection between contact dynamics and hydrodynamics.

3 The Etnyre-Ghrist Correspondence

3.1 General Euler Equations and Beltrami Fields

Let $\mathbb{R}^3$ be considered as a Riemannian manifold with the usual flat Riemannian metric given by the standard inner product structure on $(T\mathbb{R}^3, \langle\cdot,\cdot\rangle) \cong (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, let $\mu := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \in \Omega^3(\mathbb{R}^3)$ be the metric volume form, and for vector fields $X, Y \in T\mathbb{R}^3$ let $\nabla_X Y$ denote covariant differentiation of Y

along X (∇ denotes the Levi-civita connection). Then the classical Euler equations for an inviscid, incompressible fluid in $\mathbb{R}^3$ take the form

$$\begin{aligned}\partial_t X + \nabla_X X &= -\operatorname{grad} p \\ \operatorname{div} X &= 0\,,\end{aligned}$$

where p is a generally time-dependent function modeling pressure. We wish to consider Euler equations in a more general setting of a Riemannian 3-manifold $(\mathcal{M}, g)$ with a distinguished volume form (which may or may not be the metric volume form). In this generality, the first Euler equation may be written down identically where it is understood that $\operatorname{grad} p$ is defined using the metric g, and ∇ is the Levi-Civita connection associated to g. On $\mathbb{R}^3$ the second equation is equivalent to stipulating that the flow of X preserves the metric volume μ. For a general Riemannian 3-manifold $(\mathcal{M}, g)$ we would equivalently require $\mathcal{L}_X \mathrm{Vol}_g = 0$. Thus if we would like our Euler fluid to preserve a distinguished volume form μ (not necessarily the metric volume), then we replace the second condition by the equation

$$\mathcal{L}_X \mu = 0\,.$$

As Etnyre and Ghrist show [4], the Euler equations on a Riemannian three manifold preserving the volume form μ can be described by an equivalent differential system

$$\begin{aligned}\partial_t(\iota_X g) + \mathcal{L}_X(\iota_X g) &= -\mathrm{d}\left(p - \tfrac{1}{2}\iota_X \iota_X g\right) \\ \mathcal{L}_X \mu &= 0\,.\end{aligned}$$

To interpret the term $\mathcal{L}_X(\iota_X g)$, one can define a general curl operator on a Riemannian three manifold, dependent on the distinguished volume μ as well as the Riemannian structure:

Definition 3.1. Given 3-manifold $(\mathcal{M}, g)$ with a distinguished volume form $\mu \in \Omega^3(\mathcal{M})$, and a vector field $X \in \mathfrak{X}(\mathcal{M})$, define the (g, μ)-curl of X to be the unique vector field $\operatorname{curl}_\mu(X)$ which satisfies

$$\iota_{\operatorname{curl}_\mu(X)} \mu = \mathrm{d}(\iota_X g)\,.$$

When there is no ambiguity about the chosen volume form or the metric, one simply speaks of curl and writes $\operatorname{curl}_\mu(X) = \nabla \times X$.

Note that by the definition of the (g, μ)-curl and from Cartan's magic formula, one can write

$$\mathcal{L}_X(\iota_X g) = \mathrm{d}(\iota_X(\iota_X g)) + \iota_X \mathrm{d}(\iota_X g) = \mathrm{d}(\iota_X \iota_X g) + \iota_X \iota_{\nabla \times X} \mu\,,$$

whence our differential system for the Euler equations becomes

$$\begin{aligned}\partial_t(\iota_X g) + \iota_X \iota_{\nabla \times X} \mu &= -\mathrm{d}\left(p + \tfrac{1}{2}\iota_X \iota_X g\right) \\ \mathcal{L}_X \mu &= 0\,.\end{aligned}$$

A special class of interesting and well understood solutions to the Euler equations are *steady state* solutions, which exhibit no time dependence. Steady state solutions are well understood in part because they arise as fixed points of the time-evolution operator of the system, and steady state flows generically have quite rigid topology. The exceptional case to this is the class of *Beltrami flows*, which are flows associated to vector fields B that satisfy the equation

$$\nabla \times B = \lambda B\,,$$

where $\lambda \in \mathcal{C}^\infty(\mathcal{M})$. We will generally assume is non-vanishing so that the flow is non-singular. Among Beltrami fields are the eigenfields of the curl operator (i.e. $\nabla \times B = \lambda B$ when $\lambda \in \mathbb{R}^3$ is a constant), which include the ABC flows investigated in the next section. Nonvanishing Beltrami fields are sometimes called *rotational Beltrami fields.* Note that a vector field $B \in \mathfrak{X}(\mathcal{M})$ is a Beltrami field if and only if given a volume form $\mu \in \Omega^3(\mathcal{M})$, B satisfies $\iota_B \iota_{\nabla \times B} \mu \equiv 0$.

A natural question in the context of steady state solutions to the Euler equations is the "hyrdodynamical Seifert conjecture": *all analytic steady state solutions to the Euler equations on $\mathbb{S}^3$ admit periodic orbits, independent of the Riemannian structure and preserved volume on $\mathbb{S}^3$.* For non-Beltrami flows, this can be answered in the positive using topological techniques (see [4]) and we will see that in fact this conjecture is resolved positively for the Beltrami case via a correspondence connecting Beltrami flows to contact dynamics.

3.2 The Theorem and Its Consequences

The conditions of being a Beltrami field, like the conditions to be a Reeb field, are constrained by partial differential equations which capture geometric structure; in the case of Reeb fields it was the contact structure, and in the case of a Beltrami field it is the pairing of a volume form and a Riemannian metric. What makes Etnyre and Ghrist's work so impactful is that they are able to show that these two classes are closely related, so that one can use information about one geometric structure to inform one about dynamics arising in the other.

Theorem 3.1 (Etnyre-Ghrist Correspondence)**.** *On any Riemannian 3-manifold $(\mathcal{M}, g)$, every smooth, nonsingular rotational Beltrami field is Reeb-like (and thus a re-parametrization of a Reeb vector field) for some contact structure, and conversely, given any Reeb field R for some contact structure on $\mathcal{M}$, for any smooth nonzero rescaling of R there is some metric g and a volume form with respect to which the rescaled field is a Beltrami field.*

We sketch a proof. Given $X \in \mathfrak{X}(\mathcal{M})$ a nonsingular rotational Beltrami field on $(\mathcal{M}, g)$, let $\alpha := \iota_X g$. It is straightforward to check that the Beltrami condition $\nabla \times X = \lambda X$ for $\lambda \in \mathcal{C}^\infty(\mathcal{M})$ nonzero implies that $\alpha \wedge \mathrm{d}\alpha$ is a volume form, and hence α is a contact form on $\mathcal{M}$. The contact distribution consists of planes orthogonal to the Beltrami field X as determined by the metric g. To show that X is a Reeb-like vector field, one notes that $\iota_X \mathrm{d}\alpha = \lambda \iota_X \iota_X \mu = 0$, and since $\alpha(X) = \iota_X \iota_X g > 0$, $X/\|X\|$ is a Reeb field, so X is Reeb-like.

For the converse, let $(\mathcal{M}, \mathfrak{H}, \alpha)$ be a contact 3-manifold and R_α its Reeb vector field, and let fR_α be a nonzero rescaling. Working in any chart $\mathcal{U}$, we may choose parallelizations $\{e_1, e_2, e_3\}$ where $e_1 = R_\alpha$, and $\{e_2, e_3\}$ are a symplectic basis of $\mathfrak{H}$, whence $\mathrm{d}\alpha(e_2, e_3) = 1$, and define a local Riemannian structure by

$$g = \frac{1}{f} \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \varepsilon_3 \otimes \varepsilon_3 \,,$$

where $\varepsilon_i \in T^*\mathcal{U}$ are the dual covectors to the e_i in the chart U, i.e. $\varepsilon_i(e_j) = \delta_{ij}$. One then shows that this local metric can be glued together to a Riemannian metric since $e_1 = X$ is globally defined, and $\{e_2, e_3\}$ is a symplectic basis of $\mathfrak{H}$, and thus their transitions are unitary. Finally, one has to check that the flow of Y preserves the volume form $\mu := \frac{1}{f} \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$, and is parallel to its own curl.

This correspondence motivates the following terminology:

Definition 3.2. Given a Riemannian 3-manifold $(\mathcal{M}, g)$ with distinguished volume μ and with a contact structure given by a contact form α we say a vector field $X \in \mathfrak{X}(\mathcal{M})$ is a *Reeb-Beltrami field for* $(\mathcal{M}, g, \mu, \alpha)$ if it is a Beltrami field with respect to g preserving μ and Reeb-like with respect to α. If a Reeb-Beltrami field X is itself a Reeb vector field for its contact form, and if the preserved volume is the metric volume, we say X is a *strict metric Reeb-Beltrami field.* If a Reeb-Beltrami field X is itself a Reeb vector field for its contact form, and if the preserved volume is the contact volume, then we say X is a *strict contact Reeb-Beltrami field.*

Remark 3.1. The existence of a strict metric or contact Reeb-Beltrami field for $(\mathcal{M}, g, \mu, \alpha)$ gives compatibility conditions between g and α. For example it is not clear that one can find a strict contact or metric Reeb-Beltrami field if one preselects a Beltrami field for a given metric structure and attempts to construct a contact form for which it is a Reeb field. However, given a contact structure with contact form α, it is always possible to construct a metric as in the proof so that R_α is a strict contact and metric Reeb-Beltrami field, by taking the unit scaling of R_α and thus setting $g := \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_2 \otimes \varepsilon_2 + \varepsilon_3 \otimes \varepsilon_3$, and $\mu := \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$.

An immediate corollary of the correspondence is that *every-Reeb-like vector field gives rise to a steady state solution for a perfect incompressible fluid satisfying Euler equations with respect to some suitable Riemannian metric.* Of further interest is the use of contact geometry to prove results about existence of closed orbits in Beltrami fields. From Taube's proof of the Weinstein conjecture, this correspondence yields that *any nonsingular rotational Beltrami flow on a closed 3-manifold has a periodic orbit.*

A once open matter of investigation, in the spirit of Seifert's conjecture and the general thrust of dynamical system theory, was to understand *the topology* of possible closed orbits for a Beltrami field. The Etnyre-Ghrist correspondence translates the condition of being Beltrami, which comes from a partial differential equation, to the condition of being Reeb-like, which for a given contact form is a linear algebraic condition (annihilating the exterior derivative of the contact form) coupled with an analytic condition (non-vanishing of contraction with the contact form). Hence one converts the problem of understanding the possible topologies of Beltrami flows into understanding the topologies of solutions to the nonlinear ordinary differential equations for Reeb Fields arising in contact geometry. But, as we've seen, Reeb fields come parametrized by nonzero smooth functions, which gives incredible freedom to construct complex dynamics. Moreover, one can apply surgery techniques to piece together contact structures on different pieces of a manifold, in order to produce closed orbits of a Reeb field with any selected knotting and linking type. Thus, the class of Beltrami fields on a given 3-manifold also must possess the ability to support closed orbits of arbitrary knotting and linking type.

More surprising however, is the following result of Etnyre and Ghrist [5]:

Theorem 3.2. *For some Riemannian structure on $\mathbb{S}^3$, there exists a steady, nonsingular analytic Beltrami field $X \in \mathfrak{X}^\omega(\mathbb{S}^3)$ whose flow simultaneously possesses periodic orbits of all knot and link types.*

Of course, to isolate a particular such flow *for a given metric* and witness the infinitude of knotted and linked orbits on $\mathbb{S}^3$ is perhaps untenable: the various cut and paste constructions needed to produce the knotted flowlines involve altering the Riemannian structure. Etnyre and Ghrist pose as an open question the exhibition of such a flow on the standard round $\mathbb{S}^3$, and also on Euclidean $\mathbb{R}^3$. Ghrist and Holmes showed that there are simple ODE systems on $\mathbb{R}^3$ possessing all knot and link types as closed orbits [7], but to date the it is unknown whether there are steady state solutions to Euler's equations for Euclidean $\mathbb{R}^3$ with this knotted flowline property.

4 Examples of Beltrami and Reeb flows

4.1 The ABC fields

Let us turn to what are perhaps the most well known examples of Beltrami fields: the ABC fields, named for Arnold, Beltrami, and Childress [3]. In their honor, the parameters are labeled A, B, C. Given $A, B, C \in \mathbb{R}$, consider the vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ defined in standard coordinates (x, y, z) by

$$X(x,y,z) := \big(A\sin z + C\cos y\big)\partial_x + (B\sin x + A\cos z)\partial_y + (C\sin y + B\cos x)\partial_z\,.$$

It is a simple computation to show that in the standard Euclidean metric on $\mathbb{R}^3$, $\nabla \times X = X$, so this is a curl eigenfield with eigenvalue 1, and thus also a rotational Beltrami field. The corresponding system of ordinary differential equations for the integral curves is

$$\begin{aligned}\dot{x} &= A\sin z + C\cos y\,,\\ \dot{y} &= B\sin x + A\cos z\,,\\ \dot{z} &= C\sin y + B\cos x\,.\end{aligned}$$

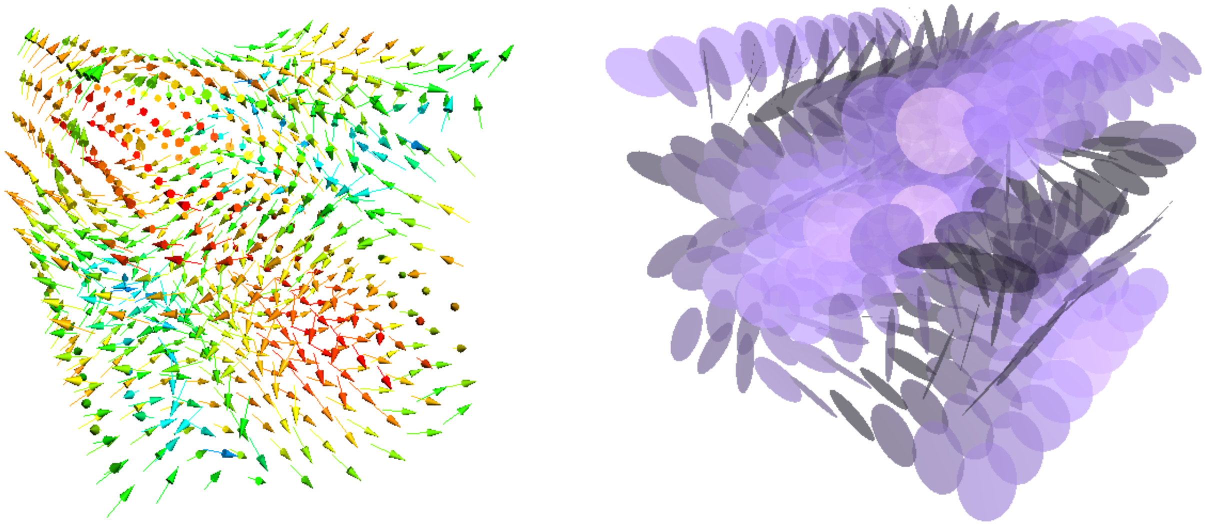

Figure 3: On the left is a visualization of the ABC field X as a vector field on $\mathbb{R}^3$ for parameters $A = 1$, $B = 9\sqrt{3}/20$, $C = 7/20$, and on the right is a visualization of the corresponding plane field $\mathfrak{H}_{ABC}$ on $\mathbb{R}^3$.

Note that X is not just smooth–X is $\mathcal{C}^\omega$. The flow will be nonsingular if and only if the vector field is non-vanishing, which puts conditions on the admissible parameters. Up to rescaling, a suitable condition to ensure that the flow is nonsingular is to take $A = 1$ and $B, C \in \mathbb{R}_+$ with $0 < B^2 + C^2 < 1$.

To make explicit the correspondence, we compute the contact structure associated to the general ABC field. The contact one form is

$$\alpha_{ABC} := \iota_X g = \big(A \sin z + C \cos y\big)\, \mathrm{d}x + (B \sin x + A \cos z)\, \mathrm{d}y + (C \sin y + B \cos x)\, \mathrm{d}z\,,$$

and the plane distribution is then $\mathfrak{H}_{ABC} = \ker \alpha_{ABC}$. This gives a contact volume of

$$\alpha \wedge \mathrm{d}\alpha = \|X\|^2\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,.$$

Thus the contact volume is *not* the preserved metric volume, and so the ABC fields are not able to be realized as strict contact Reeb-Beltrami fields.

Illustrated is a numerical simulation of the flow of points on a sphere for the ABC field with parameters $A = 1$, $B = 9\sqrt{3}/20$, $C = 7/20$. The image was produced using the program Grapher for Macintosh OSX, running a fourth order Runge-Kutta integrator with a step size of 0.01. Experimentation with higher step sizes and fewer, closer flowlines gives a qualitatively similar picture but which is generally less dramatic. Increased accuracy of the numerical integrator serves to better delineate the actual separation of streamlines, but for short ranges of time a smaller step size seems sufficient to capture the wandering behavior present in this classic example of a Beltrami field.

Even in the limit as $B^2 + C^2 \to 1$, the flow is nonsingular except at countably many points, and one can view the Beltrami flow as being a nonsingular flow on $\mathbb{R}^3$ with punctures at the zeros of the vector field X. One can see the sensitive dependence on initial conditions in the following images of the flow for parameters $A = 1$, $B = \sqrt{3}/2$, $C = 1/2$. In particular, the mixing and then subsequent branching of flowlines originating from nearby initial conditions can be seen, and suggests the presence of Lagrangian turbulence, which Etnyre and Ghrist describe as a property of volume preserving flows whose "flowlines fill up regions of space ergodically."

Owing to the periodicity in the components of the ABC fields, it is perhaps more natural to consider ABC fields as a Beltrami fields on $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ with the flat metric induced by the Euclidean metric on $\mathbb{R}^3$. In this toroidal setting, it is clear that the dynamics possess closed orbits (again, by Taubes' proof of Weinstein's conjecture). Etnyre and Ghrist more fully investigate the topology of Beltrami flows and on $\mathbb{T}^3$ in [5].

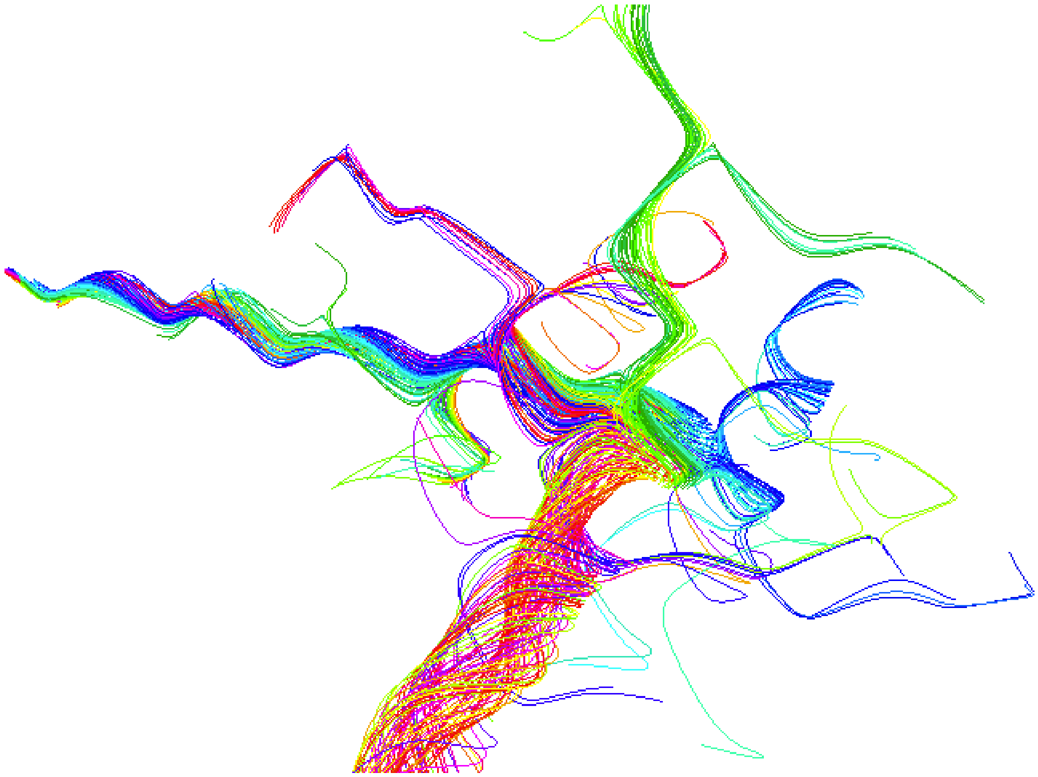

Figure 4: Integral curves for the ABC field with $A = 1$, $B = 9\sqrt{3}/20$, $C = 7/20$ originating on a sphere of radius $\pi/2$.

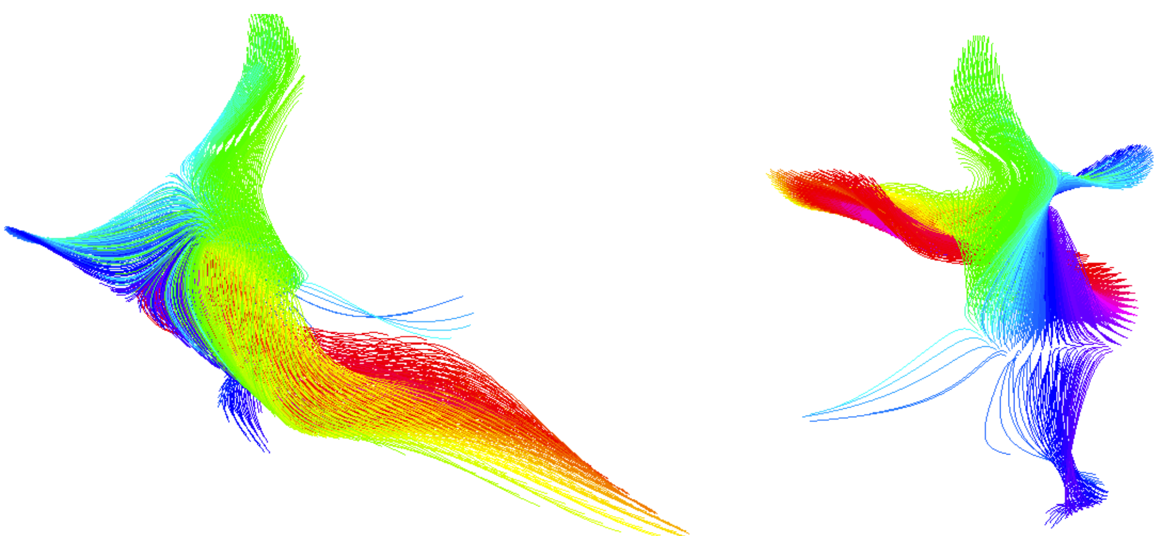

Figure 5: Above are two views of the flow of a disk of radius π centered at $(0, 0, 0) \in \mathbb{R}^3$ for $A = 1$, $B = \sqrt{3}/2$, $C = 1/2$.

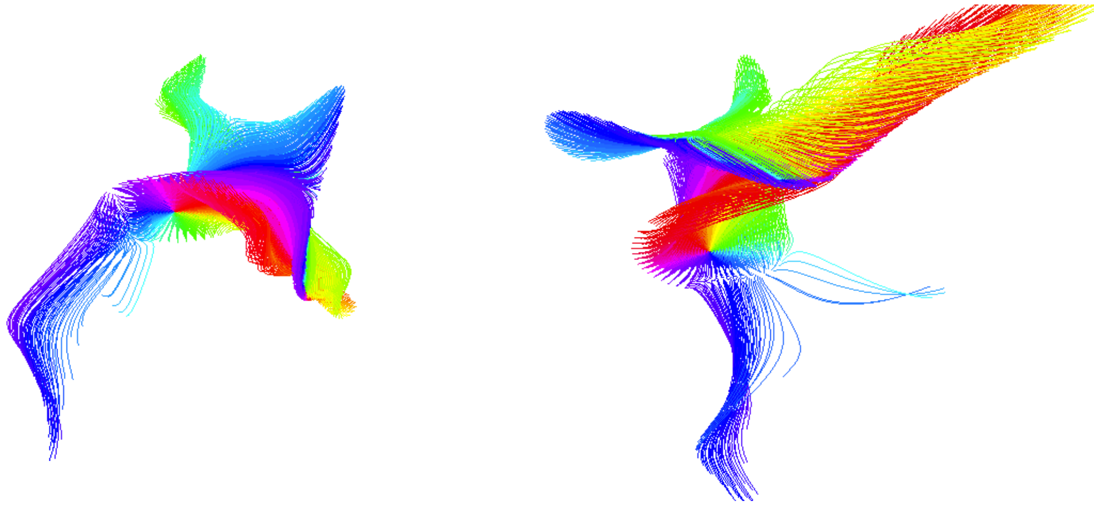

Figure 6: Two more views of the flow of a disk of radius π centered at $(0,0,0) \in \mathbb{R}^3$ for $A = 1$, $B = \sqrt{3}/2$, $C = 1/2$.

4.2 Standard Contact Structures on $\mathbb{R}^3$

We will now pursue a visual investigation of the dynamics Reeb-Beltrami fields in $\mathbb{R}^3$. For $\mathbb{R}^3$, recall we have seen three contact structures coming from three contact one forms: the standard structure $\mathfrak{H}_{std} = \ker \alpha_{std}$ with $\alpha_{std} = \mathrm{d}z + x\,\mathrm{d}y$, its radially symmetric counterpart $\mathfrak{H}_{rstd} = \ker \alpha_{rstd}$ with $\alpha_{rstd} = \mathrm{d}z + r^2\,\mathrm{d}\theta$, and the overtwisted structure $\mathfrak{H}_{o\mathcal{T}} = \ker \alpha_{o\mathcal{T}}$ with $\alpha_{o\mathcal{T}} = \cos r \mathrm{d}z + r \sin r\,\mathrm{d}\theta$. For each of these structures, we've classified Reeb fields via parametrizations by non-vanishing $\mathcal{C}^\infty(\mathbb{R}^3)$ functions. We will examine some numerical visualizations of flowlines for a few choices of f.

First we consider $(\mathbb{R}^3, \mathfrak{H}_{std}, f\alpha_{std})$ where $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ is a non-vanishing smooth function. The dynamics of the Reeb field are rather uninteresting when $f \equiv 1$, since the flow of $R_{\alpha_{std}}$ generates the obvious vertical symmetry of $\mathfrak{H}_{std}$. Let us consider the dynamics of $R_{f\alpha_{std}}$ where

$$f(x,y,z) = \exp\Big(-\sqrt{1+x^2+y^2+z^2}\Big).$$

The resulting coordinate expression for $R_{f\alpha_{std}}$ is

$$R_{f\alpha_{std}} = e^{\sqrt{1+x^2+y^2+z^2}}\left(\frac{xz-y}{\sqrt{1+x^2+y^2+z^2}}\,\partial_x + \frac{x}{\sqrt{1+x^2+y^2+z^2}}\,\partial_y + \left(1 - \frac{x^2}{\sqrt{1+x^2+y^2+z^2}}\right)\partial_z\right)$$

Let $R_{f\alpha_{std}} = P(x,y,z)\partial_x + Q(x,y,z)\partial_y + R(x,y,z)\partial_z$. Then metric for which $R_{f\alpha_{std}}$ becomes a Beltrami field is then, with respect to the basis $\{\partial_x, \partial_y, \partial_z\}$ of $\mathbb{R}^3$, given by the matrix

$$g := \begin{bmatrix} 1 & G(x,y,z) & K(x,y,z) \\ G(x,y,z) & E(x,y,z) & L(x,y,z) \\ K(x,y,z) & L(x,y,z) & F(z,y,z) \end{bmatrix},$$

where

$$E(x,y,z) = x^2[f(x,y,z)P(x,y,z)]^2 + [f(x,y,z)]^2 + [f(x,y,z)R(x,y,z)]^2$$

$$F(x,y,z) = \Big(1 + [P(x,y,z)]^2 + [Q(x,y,z)]^2\Big)[f(x,y,z)]^2$$

$$G(x,y,z) = xf(x,y,z)P(x,y,z)$$

$$K(x,y,z) = -f(x,y,z)P(x,y,z)$$

$$L(x,y,z) = \Big[x\Big(1 + [P(x,y,z)]^2\Big) - Q(x,y,z)R(x,y,z)\Big][f(x,y,z)]^2$$

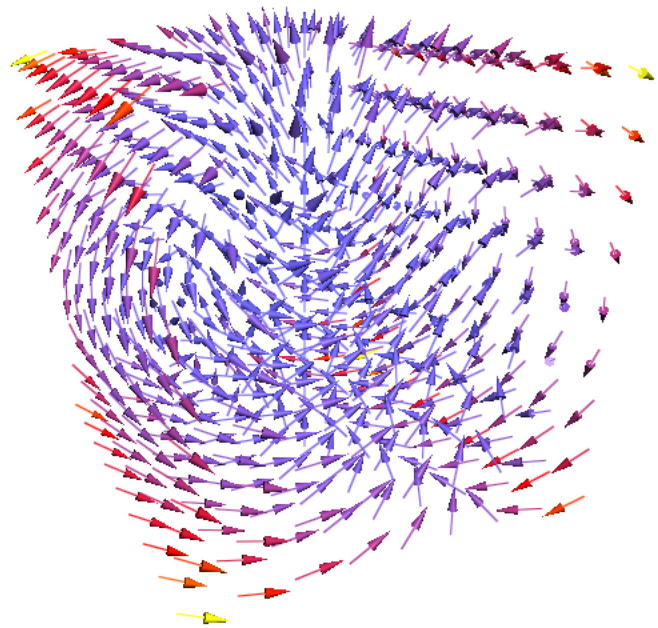

Figure 7: A visualization of the Reeb vector field $R_{f\alpha_{std}}$ for $f(x,y,z) = \exp\left(-\sqrt{1+x^2+y^2+z^2}\right)$.

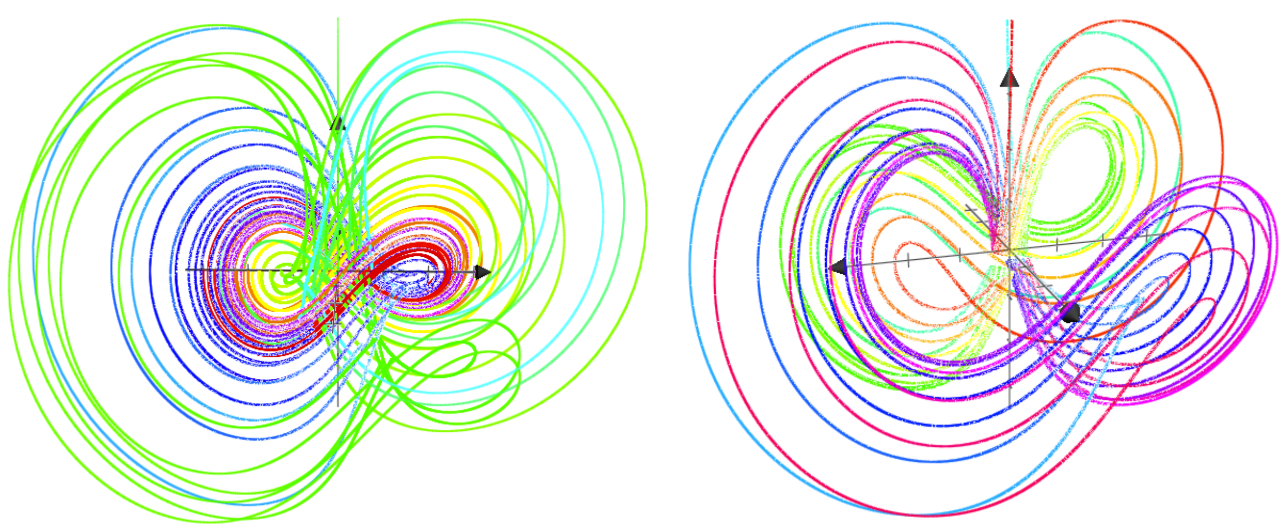

Figure 8: Visualizations of the Reeb-Beltrami flowlines for $R_{f\alpha_{std}}$ with $f(x,y,z) = \exp\left(-\sqrt{1+x^2+y^2+z^2}\right)$. On the left the inital values are along the x axis, and on the right along a circle of radius 1/3 in the plane $z=0$, centered at $(0,0,0) \in \mathbb{R}^3$.

We next consider the structure for $(\mathbb{R}^3, \mathfrak{H}_{rstd}, f\alpha_{std})$ with the same f, namely $f(r,\theta,z) = \exp\big(-\sqrt{1+r^2+z^2}\big)$. Note that $(\mathbb{R}^3, \mathfrak{H}_{std})$ is *globally contactomorphic* to $(\mathbb{R}^3, \mathfrak{H}_{rstd})$. Here we see that the flowlines of the Reeb field now foliate tori. This is still relatively tame behavior compared to the ABC fields.

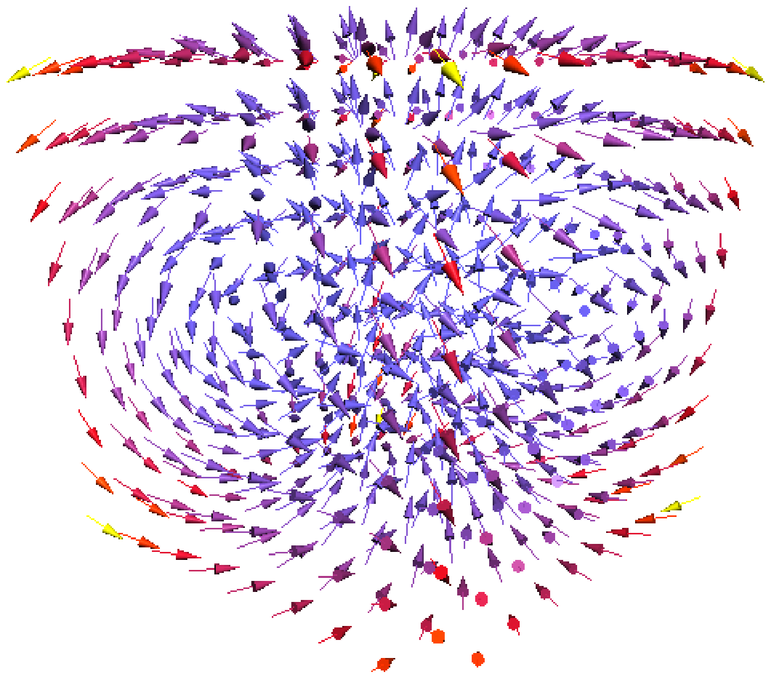

Figure 9: A visualization of the Reeb vector field $R_{f\alpha_{rstd}}$ for $f(r,\theta,z) = \exp\big(-\sqrt{1+r^2+z^2}\big)$.

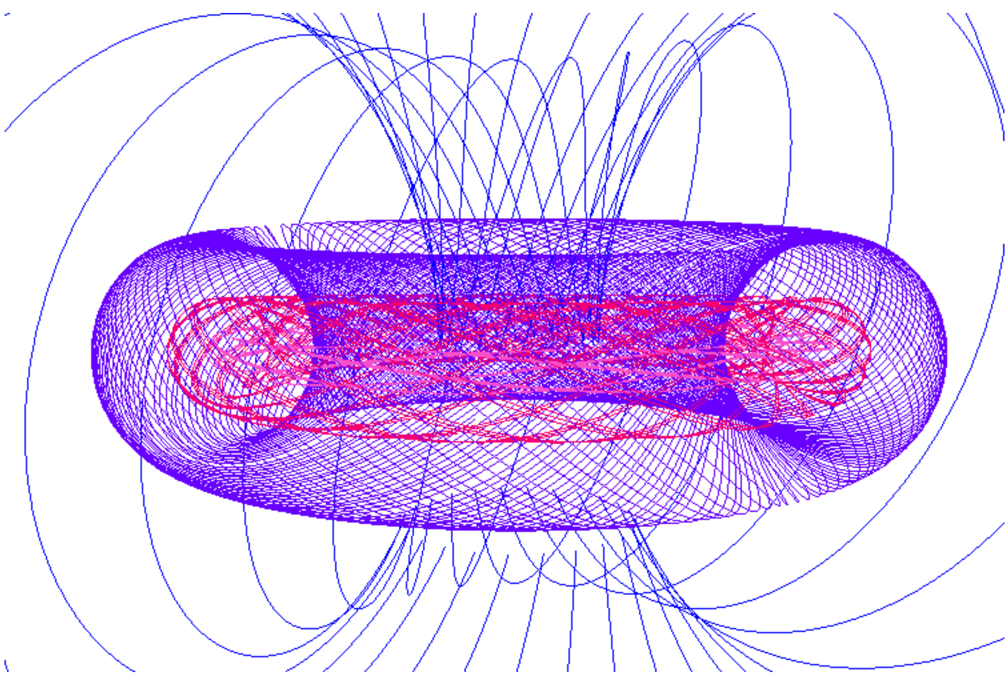

Figure 10: A visualization of the Reeb flowlines for $R_{f\alpha_{rstd}}$ for $f(r,\theta,z) = \exp\big(-\sqrt{1+r^2+z^2}\big)$.

A more interesting choice of function is $f(r,\theta,z) = \sqrt{2} + \cos(x^2+z^2)$. This yields some orbits which foliate (topological) cylinders of unbounded height above and below the plane $z=0$, and others foliating tori, as before.

Figure 11: A visualization of some Reeb flowlines of $R_{f\alpha_{rstd}}$ for $f(r, \theta, z) = \sqrt{2} + \cos(x^2 + z^2)$.

At last we come to the overtwisted structure. Recall, with the form $\alpha_{o\tau} = \cos r \, \mathrm{d}z + r \sin r \, \mathrm{d}\theta$, we have the Reeb vector field given by

$$R_{o\tau} = \frac{1}{r + \sin r \cos r}\Big(\sin r \, \partial_\theta + (\sin r + r \cos r) \, \partial_z \Big)$$

which yields the system

$$\begin{aligned} \dot{r} &= 0 \, , \\ \dot{\theta} &= \frac{\sin r}{r + \sin r \cos r} \, , \\ \dot{z} &= \frac{\sin r + r \cos r}{r + \sin r \cos r} \, . \end{aligned}$$

Since r is constant in t, the integral curves are necessarily confined to cylinders, and integration with respect to t directly yields three possibilities:

- $r = r_0$ is a root of $0 = \sin r_0 + r_0 \cos r_0$ and the integral curves are horizontal circles which foliate the vertical cylinder $r = r_0$,
- $r = k\pi$, $k \in \mathbb{Z}$ and the integral curves are vertical lines foliating the cylinder $r = k\pi$,
- otherwise the flowlines are helices of pitch which varies with the radial distance, and approaches the vertical as r tends to an integer multiple of π and tend to circles as r tends to any root r_0 of $0 = \sin r_0 + r_0 \cos r_0$.

We finish with images of the Reeb vector field and some flow lines for the overtwisted structure with the function $f(r, \theta, z) = \sqrt{2} + \sin \sqrt{1 + r^2 + z^2}$.

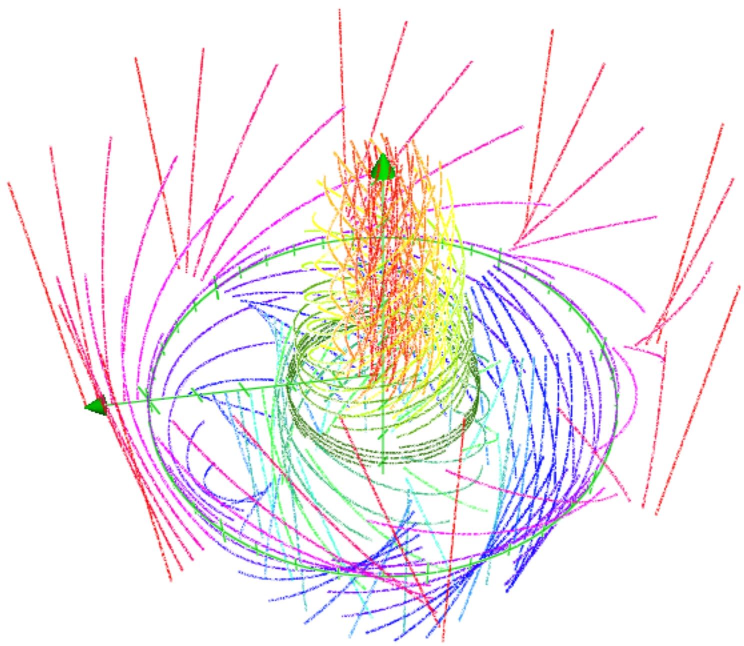

Figure 12: A visualization of the Reeb flowlines for $R_{\alpha_{o\tau}}$.

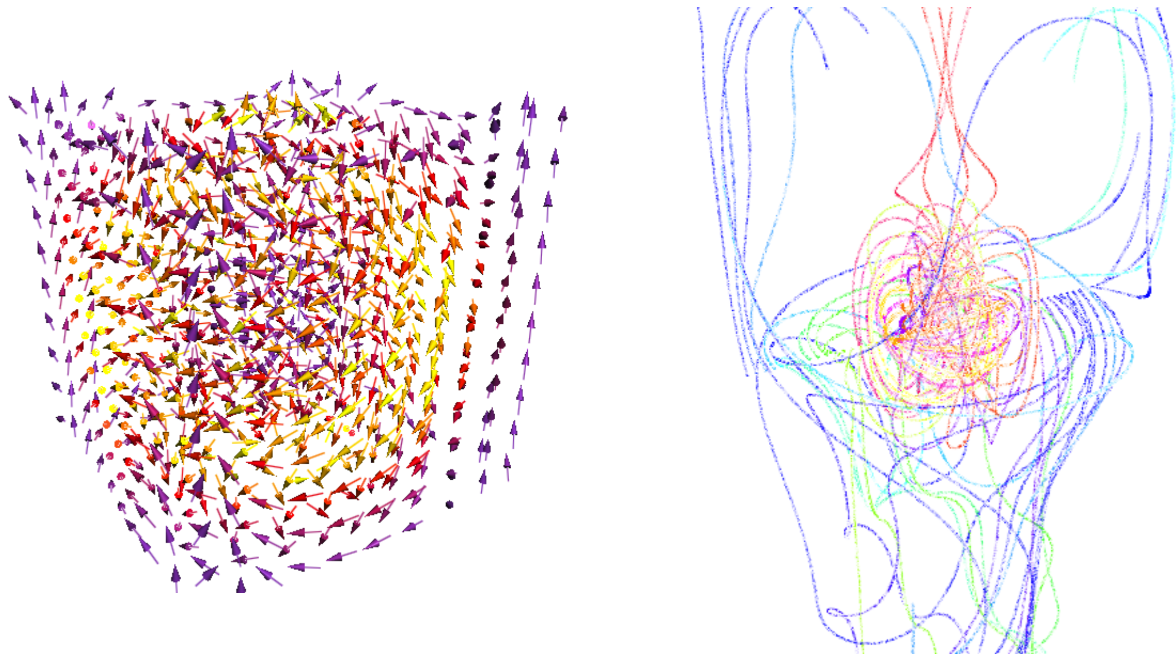

Figure 13: On the left is a visualization of the Reeb vector field for the overtwisted form $f\alpha_{o\tau}$ with $f(r,\theta,z) = \sqrt{2} + \sin\sqrt{1+r^2+z^2}$ and on the right are the flowlines for $R_{f\alpha_{o\tau}}$.

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