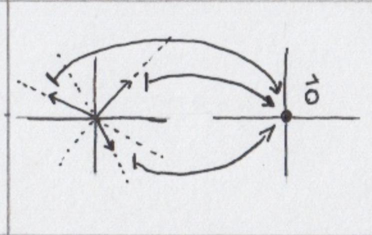
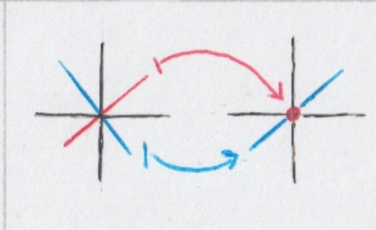
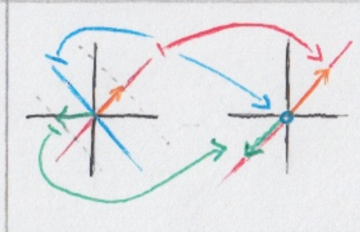
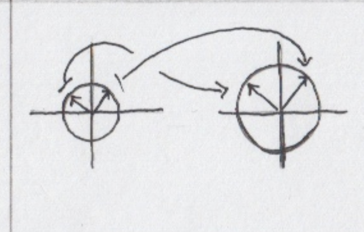
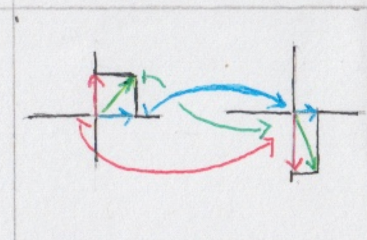
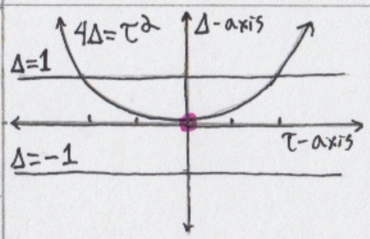
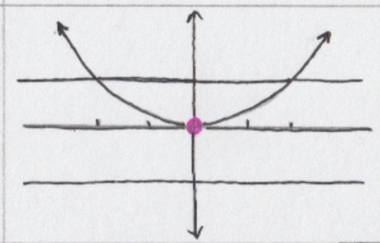
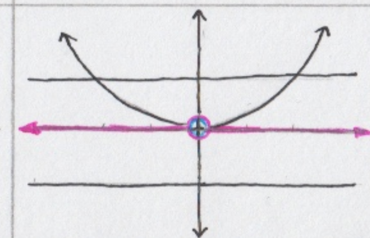
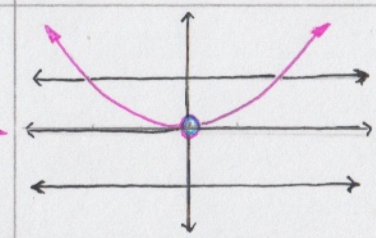
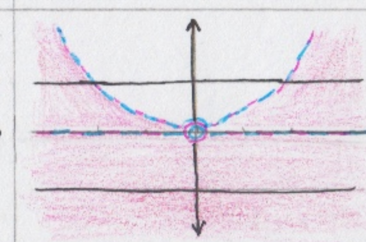
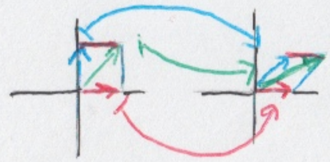
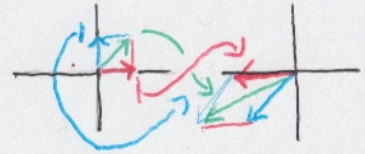
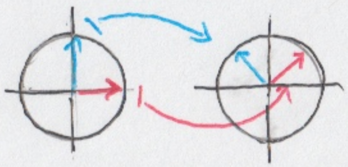
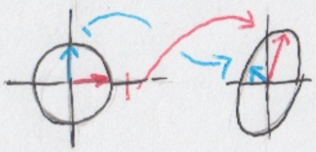
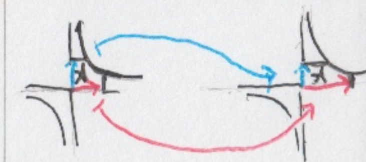
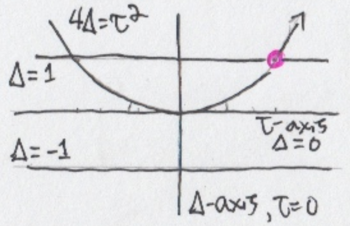
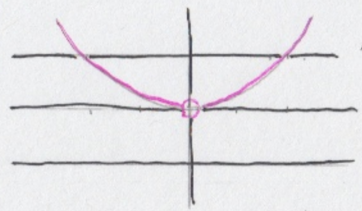
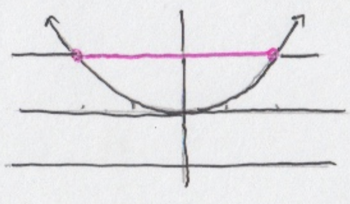
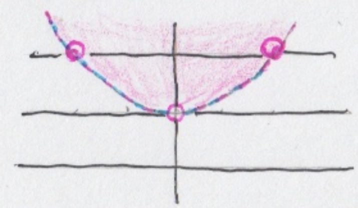
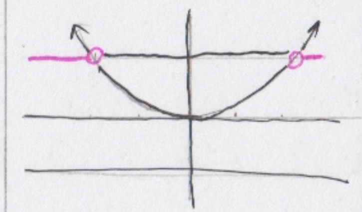
| Invertibility: | Non-invertible (Singular) | | | Invertible (Non-Singular) | |
|---|---|---|---|---|---|
| Type of Planar Endomorphism | Trivial Map | Nilpotent Map | (Stretched) Projection | Scaling: Dilations and Contractions | Stretch Mappings: Unequal Scaling |
| Geometric action Illustrated |  |  |  |  |  |
| $(\tau, \Delta)$-plane Regions closed open $\tau$ = trace $\Delta$ = determinant |  |  |  |  |  |
| Effect on Areas | Crushes whole plane $\mathbb{R}^2$ to $\vec{0}$... $\Rightarrow$ Destroys all areas | Crushes the whole plane $\mathbb{R}^2$ to a line $\Rightarrow$ all areas collapse to collections of line segments (Areas of Images are 0). | | Area scaled by a factor of $\Delta = \upsilon^2$ if $\upsilon \neq \pm 1$ | Area scaled by a factor of $\Delta = \lvert \upsilon_1 \upsilon_2 \rvert$ if $\Delta \neq \pm 1$. |
| Prototypical Matrix | Only $\vec{O}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | Similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ | Similar to $\begin{bmatrix} \tau & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & \tau \end{bmatrix}$ | Equals $\upsilon I_2 = \begin{bmatrix} \upsilon & 0 \\ 0 & \upsilon \end{bmatrix}$ Dilation if $\lvert\upsilon\rvert > 1$ Contraction if $\lvert\upsilon\rvert < 1$ | Similar to $\begin{bmatrix} \upsilon_1 & 0 \\ 0 & \upsilon_2 \end{bmatrix}$ $\upsilon_1, \upsilon_2 \in \mathbb{R}\setminus\{0\}$ distinct |
| Characteristic Polynomial | $\chi(\lambda) = \lambda^2$; the only eigenvalue is $\lambda = 0$ | | $\chi(\lambda) = \lambda(\lambda - \tau)$ Eigenvalues $\lambda = 0$ and $\lambda = \tau \neq 0$. | $\chi(\lambda) = (\lambda - \upsilon)^2$, the only eigenvalue is $\lambda = \upsilon \neq 0$. | $\chi(\lambda) = (\lambda - \upsilon_1)(\lambda - \upsilon_2)$ Distinct eigenvalues $\upsilon_1$ & $\upsilon_2$, both real |
| Algebraic Multiplicities | $m(0) = 2$ | | $m(0) = 1$ $m(\tau) = 1$ | $m(\upsilon) = 2$ | $m(\upsilon_1) = 1$ $m(\upsilon_2) = 1$ |
| Geometric Multiplicities | $\mu(0) = 2$ | $\mu(0) = 1$ | $\mu(0) = 1$ $\mu(\tau) = 1$ | $\mu(\upsilon) = 2$ | $\mu(\upsilon_1) = 1$ $\mu(\upsilon_2) = 1$ |
| Reflections and orientation reversals | No nontrivial reflection (This map destroys everything) | $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ | Reverses $E_\tau$ if $\tau < 0$ | Only origin reflection/rotation by $\pi$, if $\upsilon < 0$ | Can reverse one or both eigenspaces, if $\upsilon_1$ or $\upsilon_2$ or both negative |
| Remarks | Total Annihilation. | Squares to the trivial map | Stretch factor $\lvert\tau\rvert$ $\tau = 1 \Rightarrow$ orthogonal or oblique projection. Limiting case as $\tau \to 0$ is the trivial map | $\upsilon = 1 \Rightarrow$ the identity map. Limiting case as $\upsilon \to 0$ is the trivial map. | Limiting cases $\upsilon_2 \to \upsilon_1$: Scaling map $(\upsilon_1, \upsilon_2) \to (0,0)$: Trivial $\Delta \to 0, \tau \neq 0$: Projection |

| Invertibility: | All Non-Singular / All Invertible | | | | |
|---|---|---|---|---|---|
| Type of Planar Endomorphism | Shear Map | Generalized / scaled Shear Map | Rotation | Scaled Rotation | Squeeze Mapping / Hyperbolic rotation |
| Geometric action Illustrated |  |  |  |  |  |
| $(\tau, \Delta)$-plane Regions<br>$\tau$ = trace<br>$\Delta$ = determinant |  |  |  |  |  |
| Effect on Areas | Area preserving | Area scaled by a factor of $\Delta = v^2$, $v \neq \pm 1$. | Area preserving | Area scaled by $\Delta = a^2+b^2 = \lvert a \pm bi \rvert^2$. | Area Preserving |
| Prototypical Matrix | $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$, $k \in \mathbb{R}-\{0\}$ | Similar to $\begin{bmatrix}v & v\\0 & v\end{bmatrix}$ | Equals $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ for some $\theta \in [0, 2\pi)$. | Similar to $\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$, $a,b \in \mathbb{R}$, $b \neq 0$. | Equals $\pm\begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}$<br>$t \in \mathbb{R}$, $\cosh t = \frac{e^t+e^{-t}}{2}$, $\sinh t = \frac{e^t-e^{-t}}{2}$ |
| Characteristic Polynomial | $\chi(\lambda) = (\lambda-1)^2$<br>$\lambda = 1$ only eigenvalue | $\chi(\lambda) = (\lambda - v)^2$<br>$\lambda = v$ only eigenvalue | $\chi(\lambda) = (\lambda - e^{i\theta})(\lambda - e^{-i\theta})$<br>$= \lambda^2 - 2\cos\theta\,\lambda + 1$<br>$\lambda = e^{\pm i\theta} \in S^1 \hookrightarrow \mathbb{C}$ | $\chi(\lambda) = (\lambda - a - bi)(\lambda - a + bi)$<br>$= \lambda^2 - 2a\lambda + a^2 + b^2$<br>$\lambda = a \pm bi \in \mathbb{C}$ | $\chi(\lambda) = \pm(\lambda - e^t)(\lambda - e^{-t})$<br>$= \pm(\lambda^2 - 2\cosh t\,\lambda + 1)$ |
| Algebraic Multiplicities | $m(1) = 2$ | $m(v) = 2$ | $m(e^{i\theta}) = 1$<br>$m(e^{-i\theta}) = 1$ | $m(a+ib) = 1$<br>$m(a-ib) = 1$ | $m(e^t) = 1$<br>$m(e^{-t}) = 1$ |
| Geometric Multiplicities | $\mu(1) = 1$ | $\mu(v) = 1$ | $\mu(e^{\pm i\theta}) = 0$ in $\mathbb{R}^2$, 1 in $\mathbb{C}^2$ | $\mu(a \pm ib) = 0$ in $\mathbb{R}^2$, 1 in $\mathbb{C}^2$ | $\mu(e^t) = 1$<br>$\mu(e^{-t}) = 1$ |
| Reflections and Orientation Reversals | Can be rotated by $\pi$ to $\begin{bmatrix}-1 & -k\\0 & -1\end{bmatrix}$ | Reverses $v$-eigenspace if $v < 0$. | Orientation preserving Decomposes as composition of two reflections. | Orientation preserving Decomposes as product of two reflections and scaling | Replacing $e^t$ by $-e^t$ or $e^{-t}$ by $-e^{-t}$ reverses corresponding eigenspace. |
| Remarks | For $k \in \mathbb{N}$, $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^k$. | [redacted] Censored for your protection. | None worth making. | $b=0$ gives [redacted] simple scaling by $a$, a real set of eigenvalues. | Called hyperbolic rotation because it preserves a hyperbola |