CURVATURE, NATURAL FRAMES, AND ACCELERATION FOR PLANE AND SPACE CURVES

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0. Prelude

We explore here the relationships between curvature, natural frames, and acceleration for motion along space and plane curves, described mathematically by vector-valued functions. In the first three sections (pages 1-7) we discuss generalities for space curves. As such, for these sections we consider a continuous, several-times differentiable vector-valued function $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ where $I \subseteq \mathbb{R}$ is a connected interval which may be either closed, open or half open. The image of such a vector-valued function, i.e. the set of points of $\mathbb{R}^3$ whose position vectors are $\boldsymbol{\gamma}(t)$ for some $t \in I$, determines a space curve: $\mathcal{C} = \{\boldsymbol{\gamma}(t) \in \mathbb{R}^3 | t \in I\}$. We will often simply refer to $\boldsymbol{\gamma}$ as the curve, though in truth it is a parameterization of the curve $\mathcal{C}$.

Throughout the notes, we use the idea of "time" interchangeably with an arbitrary motion parameter $t \in I$ for a particle undergoing motion along the space curve $\mathcal{C} = \{\boldsymbol{\gamma}(t) \in \mathbb{R}^3 | t \in I\}$. Unless explicitly noted we assume regularity: the component functions $x, y, z : I \to \mathbb{R}$ of $\boldsymbol{\gamma} = x(t)\,\hat{\mathbf{\imath}} + y(t)\,\hat{\mathbf{\jmath}} + z(t)\,\hat{\mathbf{k}}$ are differentiable and $\dot{\boldsymbol{\gamma}} := \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t} \neq \mathbf{0}$. This condition ensures that there are no cusps/kinks along the curve (while one may have cause to consider such curves in application, the existence of kinks doesn't play nicely with the definitions of curvature and natural frames). In fact, we sometimes need these component functions to be at least three-times continuously differentiable, and so one should assume that for any arbitrary vector-valued functions in these notes. When a curve $\boldsymbol{\gamma}$, or any other function is (re)parameterized by a unit-speed parameter s, the domain is denoted $\tilde{I}$, and we use the prime notation for derivatives, e.g. $\boldsymbol{\gamma}'(s) := \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s}$. As explained shortly, we interpret such a parameter s as being arc-length along the curve from some pre-ordained starting position on the curve.

In the fourth section (pages 8-13), we study other frames which are "natural" in some context, such as the polar frame of polar coordinates on $\mathbb{R}^2$, and frames for cylindrical and spherical coordinates in $\mathbb{R}^3$. In each case we derive expressions for velocity and acceleration of motion along curves. The goal here is to prepare for a discussion of Kepler's Laws of planetary motion, in the fifth section.

The fifth section examines a special case of the two-body problem of celestial mechanics, called the *Keplerian Problem*, and arrives at Kepler's laws of planetary motion. There's much more that could be added here, but as this is not a course in celestial mechanics, I've omitted many interesting topics. A beautiful treatment can be found in John Milnor's article *On the Geometry of the Kepler Problem*, appearing in *The American Mathematical Monthly*, vol. 90, 1983, pp. 353-365. One can also consult David M. Bressoud's *Second Year Calculus*, which inspired the structuring of the final problem in the sixth section.

The sixth section contains problems to work on for extra credit, including a chance to re-derive Kepler's first law using vector algebra and the calculus of curves, and to derive the expression for escape velocity from an orbit (which should really be called escape speed, as one will see by doing the exercise!)

There are many things I'd like to add to these notes (for one, I have a list of figures I'd like to draw, scan and include), but until such under-construction ambitions can be met, if you need visual aides, you are welcome to consult me in office hours, draw your own, or to glance at wikipedia, which has some nice animations of Frenet-Serret frames, and good diagrams in the pages on celestial mechanics.

1. Curvature

1.1. **The Unit Tangent Vector and Arc-Length Parameterizations.** Recall that the *arc-length element* for a curve $\boldsymbol{\gamma}: I \to \mathbb{R}^3$ is given in rectangular coordinates by

$$\mathrm{d}s = \|\dot{\boldsymbol{\gamma}}(t)\|\ \mathrm{d}t = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2}\,\mathrm{d}t\,.$$

We assert that $0 \in I$ (one may always reparameterize so that this is true) and define the *arc-length function*:

$$s(t) := \int_0^t \mathrm{d}s(\tau) = \int_0^t \|\dot{\boldsymbol{\gamma}}(\tau)\|\ \mathrm{d}\tau = \int_0^t \sqrt{[\dot{x}(\tau)]^2 + [\dot{y}(\tau)]^2 + [\dot{z}(\tau)]^2}\,\mathrm{d}\tau\,.$$

We now state a well known result relating unit-speed parameterizations and arc-length parameterizations:

Theorem. *If $\boldsymbol{\eta} : \tilde{I} \to \mathbb{R}^3$ is a reparameterization of the curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ by an arc-length parameter $s(t) \in \tilde{I}$, then $\|\boldsymbol{\eta}'(s)\| = 1$, i.e. any arc-length parameterization is a unit-speed parameterization. Conversely, given a unit speed reparameterization $\boldsymbol{\eta}(\tau(t)) = \boldsymbol{\gamma}(t)$, $\tau(t)$ at most differs from the arc-length parameter $s(t)$ (as defined above for $\boldsymbol{\gamma}(t)$) by a constant:*

$$\textit{for all } t \in I : \tau(t) - s(t) = C \textit{ for some constant } C \in \mathbb{R}\,.$$

A unit speed/arc-length parameterization exists for a curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ if and only if the curve $\boldsymbol{\gamma}$ is regular throughout its domain I.

The intuition for this theorem is simple, and the explanation elementary: if the speed of $\boldsymbol{\eta}(\tau)$ is constantly equal to one, then $\frac{\mathrm{d}s}{\mathrm{d}\tau} = 1$, so the differentials satisfy $\mathrm{d}s = \mathrm{d}\tau$ which implies that s and τ differ only by a constant. The existence statement is only slightly deeper: if $\boldsymbol{\gamma}$ is regular throughout I, then $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\boldsymbol{\gamma}}(t)\| \neq 0$ for any $t \in I$, and so either s increases or decreases monotonically in t. But this means that the arc-length function s passes the horizontal line test! Therefore, s is invertible in t, though there may be no known elementary expression for this inverse function. The converse, that regularity is necessary, follows from the chain rule: if $\boldsymbol{\eta}(s(t)) = \boldsymbol{\gamma}(t)$ is an arc-length reparameterization, then $1 = \|\boldsymbol{\eta}'(s)\| = \|\dot{\boldsymbol{\gamma}}(t)\|\,\frac{\mathrm{d}t}{\mathrm{d}s} \implies \|\dot{\boldsymbol{\gamma}}(s)\| \neq 0$.

Henceforth, we will assume regularity, and thus we may opt to use a unit-speed parameterization as convenient. We will therefore also dispense with re-labeling re-parameterizations, and use the same symbol to represent various parameterizations of a fixed regular space curve (thus, we symbolically conflate $\boldsymbol{\gamma}$ with the curve $\mathcal{C} = \{\boldsymbol{\gamma}(t) \in \mathbb{R}^3 | t \in I\}$ itself).

If we are given a unit-speed curve $\boldsymbol{\gamma} : \tilde{I} \to \mathbb{R}^3$ then we know that its tangent vector $\boldsymbol{\gamma}'(s)$ has unit length. Otherwise, assuming regularity, we may fashion a unit tangent vector $\mathbf{T} : I \to \mathbb{R}^3$ by normalizing:

$$\mathbf{T} := \frac{\dot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\|}\,.$$

1.2. **Defining Curvature.** Intuitively, the unit tangent vector captures the direction of motion along the curve, but ignores the rate of travel. Thus, its derivative with respect to arc-length (or any other unit speed parameter) should tell us purely about how much the curve is bending. We define the *curvature* to be the (non-negative) scalar measure of this change:

$$\kappa(t) := \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right\|\,.$$

Of course, if $\boldsymbol{\gamma}(s)$ is an arc-length parametrized curve, then $\kappa(s) = \|\boldsymbol{\gamma}''(s)\|$, i.e. the magnitude of the acceleration of a particle along a unit speed trajectory is precisely a measure of the curvature of the particle's path, and the direction of acceleration tells us which way this trajectory bends in space.

We thus define also the *curvature vector*, which is the vector quantity capturing both the amount of bending and the direction:

$$\boldsymbol{\mathcal{K}} := \mathbf{T}'(s) = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\,.$$

For a unit speed curve this is precisely the acceleration vector:

$$\boldsymbol{\mathcal{K}}(s) = \boldsymbol{\gamma}''(s)\,.$$

Observe that $\mathbf{T}(s)\cdot\boldsymbol{\mathcal{K}}(s) = 0$ for any s, and thus the curvature vector is orthogonal to the curve (meaning it is always perpendicular to the tangent vector). Indeed, if placed at the position $\boldsymbol{\gamma}(s)$, $\boldsymbol{\mathcal{K}}(s)$ points purely in the direction that the curve $\boldsymbol{\gamma}$ is locally bending at $\boldsymbol{\gamma}(s)$.

For planar curves, the curvature vector points either to the left or the right of the motion, and we can capture the direction with something more simple than a two dimensional vector: sign. We can define a notion of *signed curvature* as follows: for an arc-length parametrized curve $\boldsymbol{\gamma} : \tilde{I} \to \mathbb{R}^2$, let $\mathbf{n}_s$ be the signed normal vector obtained by rotating $\mathbf{T} = \boldsymbol{\gamma}(s)$ by an angle of $\pi/2$ counterclockwise, and let k_s be the quantity satisfying $\boldsymbol{\gamma}''(s) = k_s\mathbf{n}_s$. Convince yourself of the following facts:

(*i*) $k_s = \boldsymbol{\gamma}''(s)\cdot\mathbf{n}_s$,

(*ii*) $k_s = \pm\kappa$, with the sign depending on whether the curve is bending to the left $(+)$ or to the right $(-)$ relative to $\mathbf{T}$,

(*iii*) Given a fixed planar vector $\mathbf{u}$, let $\varphi(s) \in (-\pi, \pi]$ be the measure of the angle between $\mathbf{T}$ and $\mathbf{u}$ (taken counterclockwise from $\mathbf{u}$). Then $k_s = \varphi'(s)$.

1.3. **Computing Curvature.** Computing curvature given an arc-length parameterization is as easy as taking second derivatives and vector magnitudes, but what about for a general parameterized curve? The abstraction necessary to define curvature might leave one wondering whether it is generally practical to compute; after all, there is no guarantee that we can explicitly write down the arc-length s as an elementary function of some original parameter t, and the prospect of taking the derivative with respect to some variable defined by an impossible-to-analytically-compute integral is rather discouraging. Thankfully, the chain rule comes to the rescue, together with some elementary vector geometry to give the fantastic formula

$$(\star) \qquad \kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3}\,.$$

Try to prove the above formula! Check that curvature meets our expectations in the following sense:

- the curvature of a line is identically 0,
- the curvature of a circle is the reciprocal of its radius; this is sensible, since a tight circle requires more bending than a loose or huge circle (consider that "great circles" on a sphere, e.g. circles of longitude on the earth, appear straight on small scales, such as within the view of an earthbound person),
- the curvature of a standard helix $\boldsymbol{\rho}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + s\hat{\mathbf{k}}$ is constant,
- the maximum curvature of a parabola happens at its vertex.

The following result is certainly helpful. If we assume our curve is a graph in the plane given by $y = f(x)$, then we have a vector parameterization $\boldsymbol{\gamma}(x) = x\hat{\imath} + f(x)\hat{\jmath}$ and the formula $(\star)$ reduces to

$$\kappa(x) = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}\,.$$

2. Natural frames

2.1. **The Normal and Binormal Vectors.** Note that the curvature vector $\boldsymbol{\mathcal{K}}$ is normal to the unit tangent vector $\mathbf{T}$, since $\|\mathbf{T}\|^2 = \mathbf{T}\cdot\mathbf{T} = 1 \implies \mathbf{T}\cdot\mathbf{T}' = 0$. Thus we may normalize $\boldsymbol{\mathcal{K}}$ to obtain a *unit normal vector*:

$$\mathbf{N}(s) := \frac{\boldsymbol{\mathcal{K}}(s)}{\|\boldsymbol{\mathcal{K}}(s)\|} = \frac{\mathbf{T}'(s)}{\kappa(s)}\,.$$

The vector $\mathbf{N}$ is sometimes called a *principal normal vector*, and the directed line it spans at a given time the *principal normal direction*.

Given a point $\boldsymbol{\gamma}(s_0)$ on a space curve, the tangent and unit normals span a plane, called the *osculating plane*. The name comes from the following idea: given any point on the curve of consideration, there is a unique circle, called the *osculating circle*, tangent to that point on the curve such that the curvature of the circle equals that of the curve at the point of tangency (namely, the radius of the osculating circle to $\boldsymbol{\gamma}(s)$ is $1/\kappa(s)$. The name 'osculating circle' comes from the latin term *circulus osculans*[1] meaning 'kissing circle'. The osculating plane for the point $\boldsymbol{\gamma}(s_0)$ is precisely the plane containing the osculating circle. We will see shortly that acceleration is always parallel to the osculating plane.

The (right-handed) unit normal vector to the osculating plane is then $\mathbf{B} := \mathbf{T}\times\mathbf{N}$, which is called the *binormal vector*. This determines another plane, together with $\mathbf{N}$, called the *normal plane*, since the tangent direction along the curve is always perpendicular to the normal plane at the point of contact.

2.2. **Torsion and Frenet-Serret Equations.** The derivative of the binormal allows us to measure the twisting of the osculating plane as we move along the curve. In particular, there exists some scalar function $\tau : \tilde{I} \to \mathbb{R}$ such that

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}\,,$$

which is called the *torsion* of the curve $\boldsymbol{\gamma}(s)$. As an exercise, convince yourself that such a τ exists, and that τ satisfies

$$\tau(s) = -\mathbf{B}'\cdot\mathbf{N}\,.$$

The triple of mutually orthogonal unit vectors $(\mathbf{T},\mathbf{N},\mathbf{B})$ is called a *natural frame*[2] for motion along the space curve $\boldsymbol{\gamma}(s)$. This is motivated by the fact that the natural frame, if known as a function of either an arc-length parameter s or a time parameter t, captures all the information necessary to completely describe the shape of the curve.

There is a famous result relating the change in the frame along the curve with the two local invariants, curvature and torsion:

Theorem (Frenet-Serret Equations)**.** *Given a regular, arc-length parameterized space curve $\boldsymbol{\gamma} : \tilde{I} \to \mathbb{R}^3$, with natural frame $(\mathbf{T},\mathbf{N},\mathbf{B})$, curvature $\kappa(s)$ and torsion $\tau(s)$, the following equations hold for all s in $\tilde{I}$:*

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s), \qquad \mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \qquad \mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)\,.$$

In matrix form, this is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}.$$

The Frenet-Serret equations express the derivatives of the natural frame in terms of linear combinations of the natural frame vectors, using just curvature and torsion (up to sign) as the scalar weights. Thus how the natural frame changes along the curve is controlled by just two scalar functions! Curvature tells us about bending, while torsion tells us how much the curve twists out of the osculating plane, infinitesimally. It follows from the Frenet-Serret equations that the curvature and torsion determine a space curve up to orientation preserving rigid motions of $\mathbb{R}^3$ (i.e. up to compositions of spatial rotations and translations). As a particularly simple example, if the torsion is 0 you should be able to convince yourself the curve must be confined to a plane.

Like the curvature, the torsion may be computed by a formula using only derivatives with respect to a given parameter and vector algebra:

$$(\diamond) \qquad \tau(t) = \frac{\big(\dot{\gamma}(t)\times\ddot{\gamma}(t)\big)\cdot\dddot{\gamma}(t)}{\big\|\dot{\gamma}(t)\times\ddot{\gamma}(t)\big\|^2}\,.$$

[1]The term *circulus osculans* was probably coined by Leibniz in the 1686 paper *Meditatio nova de natura anguli contactus et osculi.*

[2]Natural frames are sometimes called *Frenet-Serret* frames in honor of Jean Frédéric Frenet and Joseph Alfred Serret, who are the two french mathematicians credited with independently discovering the relationships between curvature, torsion, and the equations of motion. They expressed these equations without the use of modern vector algebra, which was not developed at the times of discovery in 1847 by Frenet and 1851 by Serret.

One way to prove the formula $(\diamond)$ is to express everything on the right-hand side of $(\diamond)$ in the natural frame, and then apply Frenet-Serret equations. Doing this involves first expressing the acceleration $\ddot{\boldsymbol{\gamma}}$ and the jerk $\dddot{\boldsymbol{\gamma}}$ in terms of natural frames. Though computing this is a routine application of chain rule, the resulting formulae have a physically meaningful interpretation.

3. Acceleration and Jerk in Natural Frames

3.1. **Tangential and Normal Components of Acceleration.** Let us compute the acceleration in a natural frame, using a time parameter t and employing the chain rule. Recall that $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\boldsymbol{\gamma}}(t)\|$ by definition.

$$\begin{aligned}
\ddot{\boldsymbol{\gamma}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\dot{\boldsymbol{\gamma}}(t)\|\,\frac{\dot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\|}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\mathbf{T}(t)\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)\mathbf{T}(t) + \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t)\\
&= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{T}(t) + \kappa(t)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\mathbf{N}(t)\,. \qquad (\dagger)
\end{aligned}$$

Thus, the acceleration naturally decomposes into two parts: the linear acceleration (in the tangential direction $\mathbf{T}$) which is measured by the change in speed $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$, and the centripetal acceleration (in the principal normal direction $\mathbf{N}$) which is proportional to the curvature via the square of the speed. Using a notation common in physics, one writes speed as $v(t)$, and the acceleration as $\mathbf{a}(t)$, and $(\dagger)$ can then be rewritten as

$$\mathbf{a}(t) = \dot{v}(t)\mathbf{T}(t) + \kappa(t)[v(t)]^2\mathbf{N}(t)\,.$$

Observe that the natural frame expression for acceleration is a linear combination of just the unit tangent vector and unit principal normal vector, and so in particular it lacks any binormal component. Further, the tangential and normal components of acceleration can be computed directly via vector algebra from the velocity and acceleration of an arbitrary time-dependent parameterization $\boldsymbol{\gamma}(t)$:

$$\begin{aligned}
\mathbf{a}(t) = \ddot{\boldsymbol{\gamma}}(t) &= \left(\mathbf{T}(t)\cdot\ddot{\boldsymbol{\gamma}}(t)\right)\mathbf{T}(t) + \|\mathbf{T}(t)\times\ddot{\boldsymbol{\gamma}}(t)\|\,\mathbf{N}(t)\\
&= \frac{\dot{\boldsymbol{\gamma}}(t)\cdot\ddot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\|}\,\mathbf{T}(t) + \frac{\|\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|}\,\mathbf{N}(t)\,.
\end{aligned}$$

Written more briefly in the preferred symbols of a modern physicist:

$$\mathbf{a}(t) = \frac{\mathbf{v}\cdot\mathbf{a}}{v}\,\mathbf{T} + \frac{\|\mathbf{v}\times\mathbf{a}\|}{v}\,\mathbf{N}$$

Let us consider a simple physical application. Suppose an object is undergoing circular motion with constant speed v around a circle of radius r. Since the curvature of a circle is the reciprocal of its radius, we can write $\kappa = 1/r$. Since v is constant, $\dot{v} \cong 0$, and it follows that the acceleration is given as

$$\mathbf{a} = \frac{v^2}{r}\mathbf{N}\,,$$

where $\mathbf{N}$ is the unit vector in the radial direction (pointing inward). Later, we'll describe this vector in coordinates when we discuss planetary motion. Newton's second law of motion then gives the *centripetal force* as

$$\mathbf{F}_c = \frac{mv^2}{r}\mathbf{N}\ .$$

We can use this example to intuit the meaning of the natural frame expression for acceleration. In the absence of turning, there is no curvature term, and acceleration is purely linear, in the tangential direction along the path of motion. Applying Newton's law, we get a force expression that depends on how our *speed* is changing in the direction of linear motion, and this force is what we experience when we ride in an elevator as it accelerates up, or when we undergo free fall due to gravity, or when we accelerate a car or bicycle on a straight stretch of road. On the other hand, if we undergo purely centripetal acceleration, as one would going around a curve at constant speed, we experience a force that increases linearly as the radius decreases (while the curvature increases), and increases quadratically as velocity increases.

3.2. **Jerk à la Frenet-Serret.** The third time derivative, jerk, is important in the context of motion with non-constant acceleration, and in paticular, the context of true spatial motion (motion not confined to a plane). The analogous expression for jerk in the natural frame is a bit more complex than for acceleration:

$$(\ddagger) \qquad \dddot{\boldsymbol{\gamma}} = \left(\frac{\mathrm{d}^3 s}{\mathrm{d}t^3} - \kappa^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\right)\mathbf{T} + \left(3\kappa\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mathrm{d}\kappa}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right)\mathbf{N} + \kappa\tau\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\mathbf{B}\,.$$

One can obtain (‡) by applying the Frenet-Serret equations and the chain rule to compute the time-derivative of the right-hand side of (†). Again, using the common notation from physics, we can remember this as

$$\dddot{\boldsymbol{\gamma}} = (\ddot{v} - \kappa^2 v^3)\mathbf{T} + (3\kappa v\dot{v} + \dot{\kappa}v^2)\mathbf{N} + \kappa\tau v^3\mathbf{B}\,.$$

Observe that the arc-length parametrized third derivative in the natural frame is much simpler than the time-dependent expression for jerk:

$$\boldsymbol{\gamma}'''(s) = -[\kappa(s)]^2\mathbf{T}(s) + \kappa'(s)\mathbf{N}(s) + \kappa(s)\tau(s)\mathbf{B}(s)\,,$$

from which it follows that torsion can be computed quite simply for a unit speed curve $\boldsymbol{\gamma} : \tilde{I} \to \mathbb{R}^3$ as the binormal component of $\boldsymbol{\gamma}'''(s)$ divided by the curvature:

$$\tau(s) = \frac{\boldsymbol{\gamma}'''(s)\cdot\mathbf{B}(s)}{\kappa(s)}\,,$$

while the tangential component of $\boldsymbol{\gamma}'''(s)$ encodes curvature:

$$\kappa(s) = \sqrt{-\boldsymbol{\gamma}'''(s)\cdot\mathbf{T}}\,,$$

and the normal component encodes *the arc-length derivative of curvature*:

$$\kappa'(s) = \boldsymbol{\gamma}'''(s)\cdot\mathbf{N}\,.$$

One would need to examine the fourth derivative $\boldsymbol{\gamma}''''(s)$ to gather information about the arc-length derivative of torsion, $\tau'(s)$.

3.3. **Proving the Torsion Formula.** Here, we assemble the pieces to prove the torsion formula ($\diamond$). For ease, we use the notation $v := \frac{\mathrm{d}s}{\mathrm{d}t}$, and omit writing the independent variable t. First, note that the cross product $\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}$, which when re-expressed in natural frames can be computed as

$$\begin{aligned}\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= v\mathbf{T} \times (\dot{v}\mathbf{T} + \kappa v^2\mathbf{N})\\ &= \kappa v^3(\mathbf{T}\times\mathbf{N})\\ &= \kappa v^3\mathbf{B}\,.\end{aligned}$$

Thus, its magnitude is given as $\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| = |\kappa v^3|$. Then, by writing the jerk in natural frames, we see that the scalar triple product $(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}$ is

$$\begin{aligned}(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}} &= \kappa v^3\mathbf{B}\cdot\big((\ddot{v} - \kappa^2 v^3)\mathbf{T} + (3\kappa v\dot{v} + \dot{\kappa}v^2)\mathbf{N} + \kappa\tau v^3\mathbf{B}\big)\\ &= \kappa^2 v^6\tau\\ &= \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2\,\tau\,.\end{aligned}$$

Solving for τ gives the desired formula:

$$\tau = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2}\,.$$

If $\boldsymbol{\gamma}(t) = x(t)\,\hat{\mathbf{\imath}} + y(t)\,\hat{\mathbf{\jmath}} + z(t)\,\hat{\mathbf{k}}$, we can express this via determinants as

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2 + (\dot{x}\ddot{z} - \ddot{x}\dot{z})^2 + (\dot{y}\ddot{z} - \ddot{y}\dot{z})^2}\,.$$

4. Other Curvilinear Frames

4.1. **The Polar Frame.** In applications to the geometry of various plane curves, and as well to the study of celestial mechanics, it is often helpful to use polar coordinates. The constant rectangular frame $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}})$ is not a particularly natural choice to work with when one wishes to use polar coordinates (as one might choose when there is a central, attractive force such as gravity, which acts *radially*). Thus, in this section we introduce a new frame which is adapted to polar coordinates and proves useful when discussing planetary motion in a two body problem.

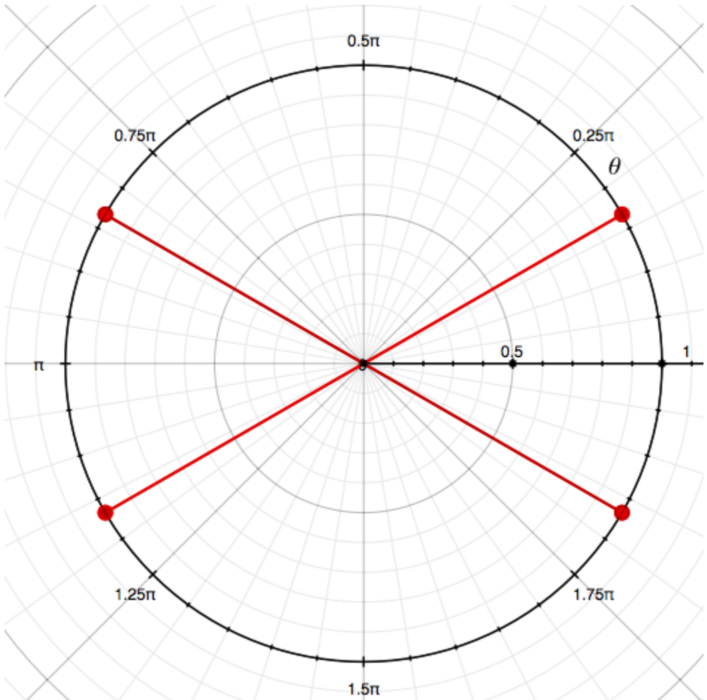

Figure 1. Pictured are the points with polar coordinates $(1, \pi/6)_{\mathscr{P}}$,$(1, -\pi/6)_{\mathscr{P}}$, $(-1, \pi/6)_{\mathscr{P}}$, and $(-1, -\pi/6)_{\mathscr{P}}$. Can you find a few other representations for each of these points? For those with positive r values, try to find a representation with negative r values, and for those with negative r values, try to find representations with positive r values.

Recall, in polar coordinates one measures the position of a point P relative to some origin O by specifying an oriented reference ray from the origin O, an angle θ from this reference ray to a ray containing the line segment OP, and a real number r measuring displacement along the ray containing OP, with $r > 0$ corresponding to displacement along the ray in agreement with the orientation, $r < 0$ corresponding to displacement along the "negative ray" (equivalently, positively along the ray's image under rotation by π), and $r = 0$ gives $P = O$. To distinguish rectangular coordinate pairs from polar ones we introduce a subscript of $\mathscr{R}$ for rectangular coordinate pairs, and a subscript of $\mathscr{P}$ for polar coordinate pairs. One should readily deduce that the following relations hold between rectangular coordinates $(x, y)_{\mathscr{R}}$ and polar coordinates $(r, \theta)_{\mathscr{P}}$:

$$r^2 = x^2 + y^2\,, \qquad \tan\theta = \frac{y}{x}.$$

From these facts, one can extract the relevant equations to transform between rectangular and polar coordinates:

$$P(x, y)_{\mathscr{R}} = (r\cos\theta, r\sin\theta)_{\mathscr{R}}\,, \qquad P(r, \theta)_{\mathscr{P}} = \Big((-1)^{\varepsilon}\sqrt{x^2+y^2}, \operatorname{atan2}(y, x) + \varepsilon\pi + 2\pi n\Big)_{\mathscr{P}}\,.$$

Here, $\varepsilon \in \{0, 1\}$ and $n \in \mathbb{Z}$ are parameters that account for the many-to-one nature of polar representations of a given point. The $\operatorname{atan2}(y, x)$ function takes values in $(-\pi, \pi]$ according to the following:

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0\,, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0\,, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0\,, \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0\,, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0\,, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0\,. \end{cases}.$$

We introduce now a new notation for an arbitrary position vector of a point of the plane $\mathbb{R}^2$:

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} = r\cos(\theta)\hat{\mathbf{\imath}} + r\sin(\theta)\hat{\mathbf{\jmath}}\,.$$

Observe that $\|\mathbf{r}\| = |r|$. If we insist that $\varepsilon = 0 = n$, then $r \geq 0$ and $\theta \in (-\pi, \pi]$, whence we may define

$$\hat{\mathbf{u}}_r := \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}}{r} = \cos(\theta)\hat{\mathbf{\imath}} + \sin(\theta)\hat{\mathbf{\jmath}}\,.$$

Since counter-clockwise is the direction of increasing θ, we form a right handed frame at any point $(r,\theta)_{\mathscr{P}}$ by defining $\hat{\mathbf{u}}_\theta$ to be the vector obtained by rotating $\hat{\mathbf{u}}_r$ by $\pi/2$ counter-clockwise:

$$\hat{\mathbf{u}}_\theta := -\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}\,.$$

The frame $(\hat{\mathbf{u}}_r, \hat{\mathbf{u}}_\theta)$ is called the *polar frame* for $\mathbb{R}^2$.

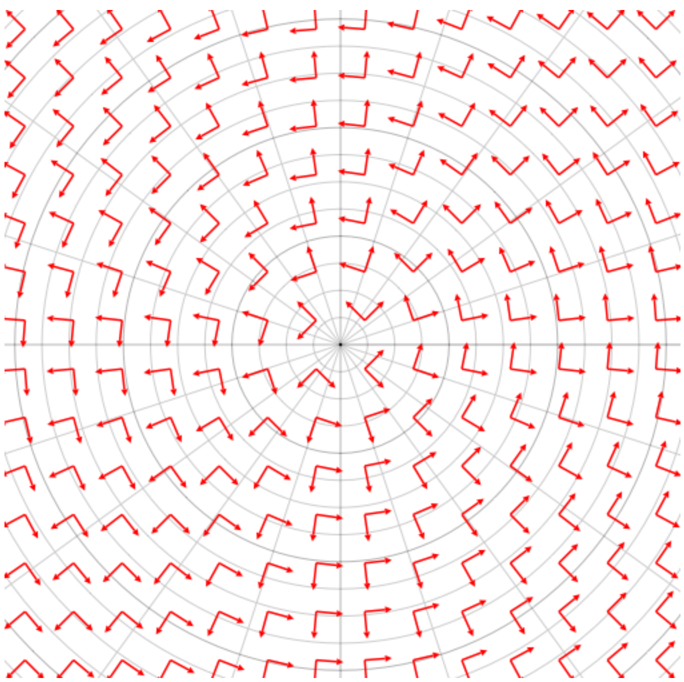

FIGURE 2. The polar frame, visualized as a a pair of orthogonal vector fields. Note that the frame is undefined at the origin, as neither $\hat{\mathbf{u}}_r$ nor $\hat{\mathbf{u}}_\theta$ can be defined there. The *flow-lines* for the vector field $\hat{\mathbf{u}}_r$ are rays from the origin, while the flow-lines for the vector field $\hat{\mathbf{u}}_\theta$ are concentric origin centered circles. Together they form a web of orthogonal curves which define the constant sets for the polar coordinate system; the rays and circles play the same roles as the gridlines of the rectangular Cartesian coordinate system on $\mathbb{R}^2$.

In the polar frame, position is given simply by

$$\mathbf{r} = r\hat{\mathbf{u}}_r\,,$$

where it is implicit that $\hat{\mathbf{u}}_r$ depends on θ. To emphasize this, one may write $\mathbf{r} = r\hat{\mathbf{u}}_r(\theta)$, or in the case where one is examining a plane curve that can be described in polar coordinates by r as a function of θ, one writes $\mathbf{r}(\theta) = r(\theta)\hat{\mathbf{u}}_r(\theta)$. A general parametrization of a plane curve $\mathbf{r} : I \to \mathbb{R}^2$ using a time parameter t necessitates regarding each of r and θ as time-dependent functions, whence one may write $\mathbf{r}(t) = r(t)\hat{\mathbf{u}}_r\big(\theta(t)\big)$.

4.2. **Velocity and Acceleration in the Polar Frame.** Suppose a particle undergoes planar motion according to a vector valued function $\mathbf{r} : I \to \mathbb{R}^2$, and we wish to study this motion using the polar frame. We need to know how the polar frame changes along the curve. A first step is to observe how the frame changes with respect to θ:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\mathbf{u}}_r &= \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos(\theta)\hat{\mathbf{\imath}} + \sin(\theta)) = -\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}} = \hat{\mathbf{u}}_\theta\,,\\ \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\mathbf{u}}_\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}(-\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}) = -\cos(\theta)\hat{\mathbf{\imath}} - \sin(\theta)\hat{\mathbf{\jmath}} = -\hat{\mathbf{u}}_r\,.\end{aligned}$$

We can now apply the chain rule to express velocity $\dot{\mathbf{r}}(t)$ in the polar frame:

$$\begin{aligned}\dot{\mathbf{r}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) &= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{u}}_r(t) + r(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_r(t)\\ &= \dot{r}(t)\hat{\mathbf{u}}_r(t) + r(t)\hat{\mathbf{u}}_\theta(t)\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= \dot{r}(t)\hat{\mathbf{u}}_r(t) + r(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta(t)\,.\end{aligned}$$

When it is understood that all quantities above are functions of time t, it is convenient and customary to abbreviate the velocity equation, as a physicist might, by excluding the restatements of the parameter:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta\,.$$

We can interpret the components of velocity in this frame as follows: scaling $\hat{\mathbf{u}}_r$ by $\dot{r}$ tells us that the velocity in the radial direction is precisely given by the time derivative of the radial coordinate. We should expect this since $\dot{r}$ measures the (signed) speed at which the particle moves away from (if $\dot{r} > 0$) or toward (if $\dot{r} < 0$) the origin, and $\dot{r} = 0$ if and only the distance locally remains constant (in which case the curve is either tangent to an origin centered circle at one point, or locally contains an arc of such a circle). For the $\hat{\mathbf{u}}_\theta$-component, we see $\dot{\theta}$ measuring the rate of change of the angle with time, along the $\hat{\mathbf{u}}_\theta$ direction, scaled by r (so the magnitude of this component increases with distance from the origin). If you consider circular motion with a given angular velocity, a larger radius requires a larger component tangential to the motion (i.e. a larger linear velocity for the same angular velocity but a larger radius). Indeed, the $\hat{\mathbf{u}}_\theta$ component measures motion in a direction tangent to curves of constant radius. These are of course circles or arcs thereof (r = a constant), and along such curves, the $\hat{\mathbf{u}}_\theta$-component simplifies to the arc-length element of the radius r circle parametrized by the corresponding time dependent angle function $\theta(t)$. A geometer might expect that $\hat{\mathbf{u}}_\theta \cdot \dot{\boldsymbol{\gamma}} = r\dot{\theta}$, since the arc-length of an arc with total angle $\Theta = \int_0^t \theta(\tau)\,\mathrm{d}\tau$ grows linearly with radius: $s(\Theta) = r\Theta$, and any change in r is reflected in the radial component. Had Newton possessed our language of vectors, he might well have drawn a picture and used the idea of "infinitesimals" to arrive at such a formula for these components (I encourage you to try this, and then see how the modern language of limits would yield a more rigorous proof giving precisely the terms we expect!)

Let us now express the acceleration in the polar frame:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \ddot{\mathbf{r}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{r}(t)\hat{\mathbf{u}}_r(t) + r(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta(t)\Big)\\ &= \ddot{r}(t)\hat{\mathbf{u}}_r(t) + \dot{r}(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_r(t) + \dot{r}(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta(t) + r(t)\ddot{\theta}(t)\hat{\mathbf{u}}_\theta(t) + r(t)\dot{\theta}(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_\theta(t)\\ &= \ddot{r}(t)\hat{\mathbf{u}}_r(t) + \dot{r}(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta + \dot{r}(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta(t) + r(t)\ddot{\theta}(t)\hat{\mathbf{u}}_\theta(t) - r(t)[\dot{\theta}(t)]^2\hat{\mathbf{u}}_r\\ &= \Big(\ddot{r}(t) - r(t)[\dot{\theta}(t)]^2\Big)\hat{\mathbf{u}}_r(t) + \Big(2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t)\Big)\hat{\mathbf{u}}_\theta(t)\end{aligned}$$

As with velocity, it is more pleasant to remember the polar-frame expression for acceleration in the more compact notation:

$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = (\ddot{r} - r\ddot{\theta})\hat{\mathbf{u}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{u}}_\theta\,.$$

Observe that

$$\mathbf{a}(t)\cdot\hat{\mathbf{u}}_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \frac{2}{r}\frac{\mathrm{d}\dot{\mathcal{A}}}{\mathrm{d}t}\,,$$

where $\mathcal{A}(t)$ is the area, at time t, swept out in the plane by $\mathbf{r}(t)$ as it follows the particle's motion, measured from some initial position vector (e.g. one might choose $\mathbf{r}(0)$.)

4.3. **The Cylindrical Frame.** For $\mathbb{R}^3$, the rectangular frame $(\hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}})$ and the corresponding rectangular coordinates $(x, y, z)_{\mathscr{R}}$ do not provide the most natural description of positions for studying objects which are radially symmetric about the z-axis, such as cylinders or other surfaces of revolution. It is however not difficult to generalize polar coordinates to $\mathbb{R}^3$ to give *cylindrical coordinates* $(r, \theta, z)_{\mathscr{C}}$. The first two coordinates are defined as in the $r\theta$-plane, and the third coordinate is the usual z coordinate from the rectangular system. Observe that in these coordinates, the equation of a right circular cylinder of radius a centered on the z-axis is just $r = a$, and a point upon such a cylinder can then be located by its θ and z coordinates.

The cylindrical frame extends the polar frame into three dimensions in the obvious way compatible with cylindrical coordinates: just as before we define $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ by

$$\hat{\mathbf{u}}_r := \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\,,$$

$$\hat{\mathbf{u}}_\theta := -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\,,$$

and our third vector in the frame is $\hat{\mathbf{k}}$. It is simple to verify that $\hat{\mathbf{u}}_r \times \hat{\mathbf{u}}_\theta = \hat{\mathbf{k}}$, and so the frame $(\hat{\mathbf{u}}_r, \hat{\mathbf{u}}_\theta, \hat{\mathbf{k}})$ is a right handed coordinate frame for cylindrical coordinates.

As we did in $\mathbb{R}^2$ with the polar frame, we can express velocity and acceleration in the cylindrical frame. Writing as before $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} = r\hat{\mathbf{u}}_r$, we can write a general curve as

$$\boldsymbol{\gamma}(t) = \mathbf{r}(t) + z(t)\hat{\mathbf{k}} = r(t)\hat{\mathbf{u}}_r(t) + z(t)\hat{\mathbf{k}}\,.$$

This gives the velocity as

$$\dot{\boldsymbol{\gamma}}(t) = \dot{\mathbf{r}}(t) + \dot{z}(t)\hat{\mathbf{k}} = \dot{r}(t)\hat{\mathbf{u}}_r(t) + r(t)\dot{\theta}(t)\hat{\mathbf{u}}_\theta(t) + z(t)\hat{\mathbf{k}}\,,$$

and the acceleration as

$$\ddot{\boldsymbol{\gamma}}(t) = \left(\ddot{r}(t) - r(t)[\dot{\theta}(t)]^2\right)\hat{\mathbf{u}}_r(t) + \left(2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t)\right)\hat{\mathbf{u}}_\theta(t) + \ddot{z}(t)\hat{\mathbf{k}}\,.$$

If $\mathbf{v}(t) = \dot{\boldsymbol{\gamma}}(t)$ and $\mathbf{a}(t) = \dot{\mathbf{v}}(t)$, we can write velocity and acceleration cleanly as

$$\mathbf{v}(t) = \dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta + \dot{z}(t)\hat{\mathbf{k}}\,,$$
$$\mathbf{a}(t) = (\ddot{r} - r\ddot{\theta})\hat{\mathbf{u}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{u}}_\theta + \ddot{z}\hat{\mathbf{k}}\,.$$

4.4. **Spherical Coordinates and the Spherical Frame.** In many situations it is more natural for one to use coordinates which are symmetric in some way with respect to the problems one is considering. We've described polar and cylindrical coordinates, which are radially symmetric about the origin of $\mathbb{R}^2$ or the z-axis of $\mathbb{R}^3$, respectively, and thus convenient for classes of problems that are more simply expressed with a variable that gives distance to the origin of $\mathbb{R}^2$ or the z-axis. But what if we want a $3D$ system which is radially symmetric about the *origin*?

Such a coordinate system is not so far removed from a geographer's understanding of coordinates. Consider how we usually locate a point on the globe: by latitude and longitude, which are angular measurements (given in degrees). Here's how one can define a longitude/latitude coordinate system mathematically for any sphere (say, of radius ϱ):

(1) View the sphere as origin centered in $\mathbb{R}^3$, and let $\boldsymbol{\rho} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ be the position vector of a point (x, y, z) on the sphere $x^2 + y^2 + z^2 = \varrho^2$, where $\varrho := \|\boldsymbol{\rho}\|$.

(2) Choose an equatorial plane $\Pi_{\mathscr{E}}$ (a plane through the origin), e.g. the xy-plane $\{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. The circle of intersection of that plane with the sphere is called the sphere's *equator* by geographers. To a mathematician, any great circle[3] can be an equator, but once we choose one, we'll specify that as *the equator* of the sphere for our coordinates. Denote the equator by $\mathscr{E}$. One chooses a *northern* or positive hemisphere, and a *southern* or negative hemisphere. For us, the northern hemisphere will be the one intersecting the positive z axis, and *the north pole* has position vector $\mathbf{N} = \varrho\hat{\mathbf{k}}$, which is the length ϱ normal to $\Pi_{\mathscr{E}}$ in the positive/north direction.

(3) Choose a *prime meridian*: a meridian is a semi-circle of a great circle that lies in a plane orthogonal to the chosen equatorial plane, and the prime meridian is one which we distinguish to build our coordinates. We can, for example, choose the meridian in the xz-plane with $x \geq 0$. Denote the meridian by $\mathscr{M}$. The meridional plane $\Pi_{\mathscr{M}}$ divides the sphere into two pieces: the *eastern hemisphere*, and the *western hemisphere*. The intersection of $\mathscr{M}$ with $\mathscr{E}$ determines a vector $\mathbf{m}_0$. In our example, $\mathbf{m}_0 = \varrho\hat{\mathbf{\imath}}$, and the eastern hemisphere is the hemisphere where the y coordinate is positive.

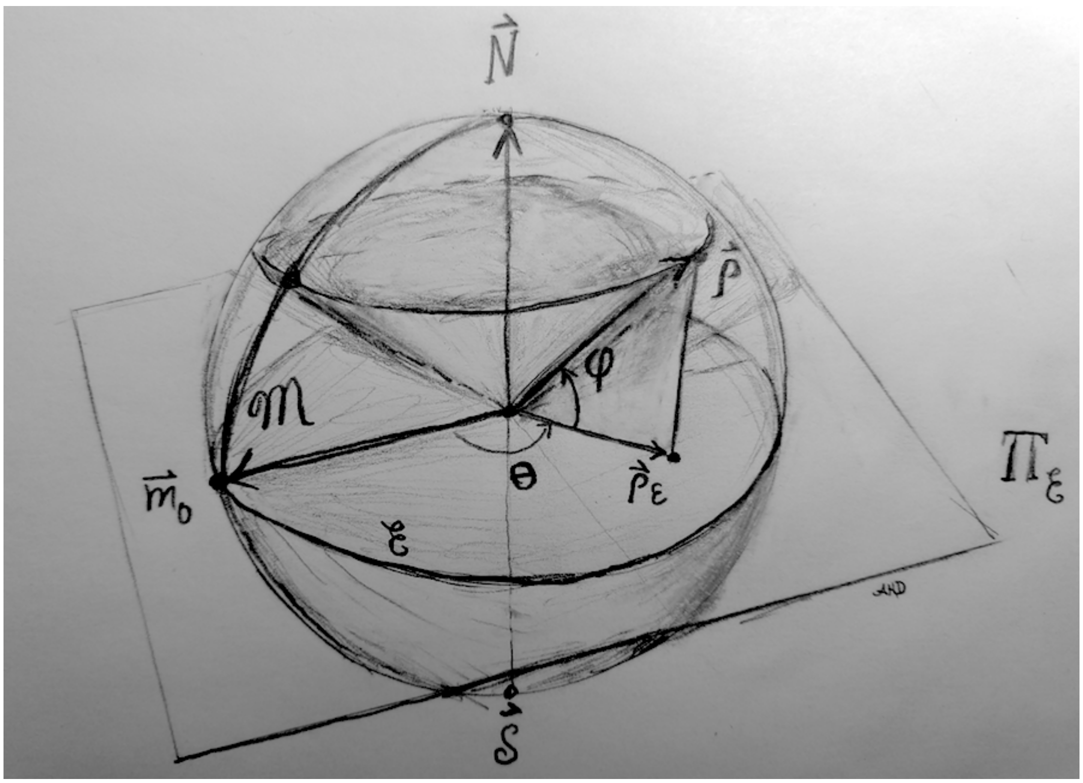

FIGURE 3. Spherical coordinates modeled loosely on geographic coordinates by longitude and latitude.

(4) The vector $\boldsymbol{\rho}$ decomposes into a sum of a vector parallel to the equatorial plane $\boldsymbol{\rho}_{\mathscr{E}}$, and a vector $\boldsymbol{\rho}_{\mathscr{M}}$ that is perpendicular to the equatorial plane and parallel to the prime meridional plane. $\boldsymbol{\rho}_{\mathscr{E}}$ is in fact the "shadow" (i.e. the orthogonal projection) of $\boldsymbol{\rho}$ on the equatorial plane, and so with our choices, $\boldsymbol{\rho}_{\mathscr{E}} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} = \mathbf{r}$ and $\boldsymbol{\rho}_{\mathscr{M}} = z\hat{\mathbf{k}}$, which corresponds to the coordinate decomposition of cylindrical coordinates.

[3]A *great circle* on a sphere S is any circle obtained by intersecting a plane with the sphere S such that the plane contains the sphere's center. These are also the *geodesics* of a round sphere in its usual metric, meaning the curves that locally minimize distance, and whose curvature is purely *normal* to the surface at every point. In particular, the shortest path on a sphere between two points is a segment of the great circle containing those two points., and the great circles minimize the component of the curvature vector $\boldsymbol{\mathcal{K}}$ tangent to the sphere.

(5) The *longitude* is the angle θ made by $\boldsymbol{\rho}_{\mathscr{E}}$ with the $\mathbf{m}_0$, which in our example is the usual angle θ measured in polar coordinates, taking values in $(-\pi, \pi]$ if we use radians. Geographers instead use degrees and the interval $[-180°, 180°]$, and by convention, avoid negatives by clarifying *East* (abbrv. E.) for counterclockwise angular increase or *West* (abbrv. W.) for clockwise angular increase, relative to the north pole. The angle θ is called the *azimuthal angle*.

(6) Geographers define the *latitude* to be the angle above or below the equator, i.e. the angle between the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_{\mathscr{E}}$, and it takes values in the interval $[-90°, 90°]$. Again, the geographical convention is to write *South* (abbrv. S.) for negative values, and *North* (abbrv. N.) for positive values.[4].

By φ we will denote the angle of longitude, also called *the angle of inclination*, as measured in radians. Thus, φ will denote the angle of separation of $\boldsymbol{\rho}$ and its equatorial-plane component $\boldsymbol{\rho}_{\mathscr{E}}$, taking values[5] in the interval $[-\pi/2, \pi/2]$.

We have now defined a coordinate system (θ, φ) on any sphere of radius ϱ. By allowing ϱ to take on any non-negative real value, we can now locate any point with position vector $\boldsymbol{\rho} \in \mathbb{R}^3$ by specifying its distance $\varrho = \|\boldsymbol{\rho}\|$ from the origin, and then specifying the latitude φ and the longitude θ on the sphere of radius ϱ containing the point. The relevant relations from our definitions give

$$\varrho = \|\boldsymbol{\rho}\| = \sqrt{x^2 + y^2 + z^2}\,, \quad \cos(\theta) = \frac{\mathbf{m}_0 \cdot \boldsymbol{\rho}_{\mathscr{E}}}{\varrho\,\|\boldsymbol{\rho}_{\mathscr{E}}\|}\,, \qquad \cos(\varphi) = \frac{\boldsymbol{\rho} \cdot \boldsymbol{\rho}_{\mathscr{E}}}{\varrho\,\|\boldsymbol{\rho}_{\mathscr{E}}\|}\,.$$

Thus, if $(\varrho, \theta, \varphi)$ satisfy the above relations for a point with position vector $\boldsymbol{\rho}$, we say that its spherical coordinates are given by the triple $(\varrho, \theta, \varphi)_{\mathscr{S}}$, where the $\mathscr{S}$ distinguishes that the triple gives spherical, rather than cylindrical or rectangular coordinates.

The transformation between spherical and cylindrical coordinates is given by

$$r = \varrho \cos\varphi\,, \quad \theta = \theta\,, \quad z = \varrho \sin\varphi\,,$$

which implies that the transformation between spherical and rectangular coordinates is then given by

$$x = \varrho\cos\theta\cos\varphi\,, \quad y = \varrho\sin\theta\cos\varphi\,, \quad z = \varrho\sin\varphi\,.$$

We can now describe the spherical frame. Consider a point $P \in \mathbb{R}^3$ with position $\boldsymbol{\rho}$ and coordinates $(\varrho, \theta, \varphi)_{\mathscr{S}}$. We define the frame at this point as follows: the spherically radial direction, given by the position vector $\boldsymbol{\rho} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$ can be normalized to give a vector $\hat{\mathbf{u}}_\varrho := \boldsymbol{\rho}/\|\boldsymbol{\rho}\|$. The rectangular expression for $\hat{\mathbf{u}}_\varrho$ is

$$\hat{\mathbf{u}}_\varrho = \frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} = \cos(\theta)\cos(\varphi)\,\hat{\mathbf{\imath}} + \sin(\theta)\cos(\varphi)\,\hat{\mathbf{\jmath}} + \sin(\varphi)\,\hat{\mathbf{k}}\,.$$

Observe that $\hat{\mathbf{u}}_\varrho = \cos(\varphi)\hat{\mathbf{u}}_r + \sin(\varphi)\hat{\mathbf{k}}$ when expressed in the cylindrical frame. We can take $\hat{\mathbf{u}}_\theta$ to be defined in the same manner as in cylindrical coordinates, and so in particular, it has the same rectangular coordinate expression, and gives a unit tangent at P to the line of latitude on the origin centered sphere containing P.

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[4]E.g., the coordinates of the Lederle Graduate Research Tower (LGRT) at UMass are 42.340382° N latitude and 72.496819 W longitude. The convention for latitude and longitude on earth is to list latitude first, then longitude, which is the reverse of the mathematical convention we are using. The reason we use the differing convention is so that the spherical frame will be right handed, and at a point on the positive x-axis, $\hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{u}}_\varphi$ point in positive directions in the rectangular system, along $+\hat{\mathbf{\jmath}}$ and $+\hat{\mathbf{k}}$ respectively. Thus, at points along the positive x-axis, our spherical frame $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$ matches the rectangular frame $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$.

[5]Many mathematicians prefer to instead define φ as the angle between the positive polar axis and the vector $\boldsymbol{\rho}$, and use the interval $[0, 2\pi)$ for the domain of θ. With our choices of equator and meridian for an abstract sphere, the azimuthal angle θ still agrees with polar angle up to a shift of domain, but φ in this convention is defined as a *polar angle*, taking values[6] in $[0, \pi]$, given as the angle between $\mathbf{N} = \varrho\hat{\mathbf{k}}$ and $\boldsymbol{\rho}$.

[6]Observe that in the geographer's definitions, the point with coordinates $(0°, 0°)$ corresponds to the point with position $\mathbf{m}_0$. On earth, this point lies in the gulf of Guam, about 500 miles southwest of Lagos, Nigeria. In the mathematical coordinates using the polar angle for φ, the point with $\varphi = 0$ and θ undefined is the north pole. Finally, note, θ is not well-defined at either the north or south poles for either version of the coordinates.

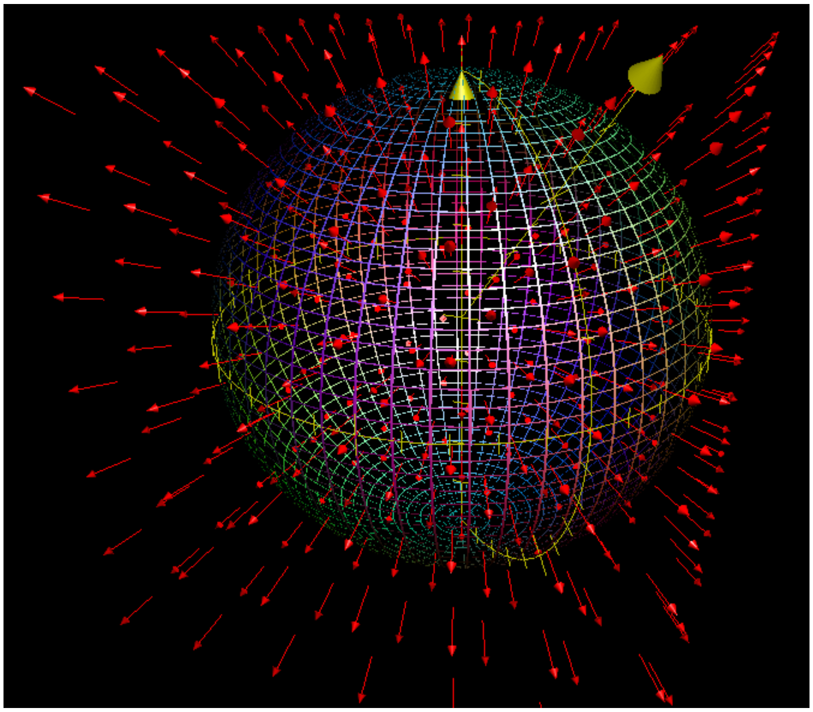

FIGURE 4. The spherical frame element $\hat{\mathbf{u}}_\varrho$ can be regarded as a *unit radial vector field* on $\mathbb{R}^3 - \{\mathbf{0}\}$.

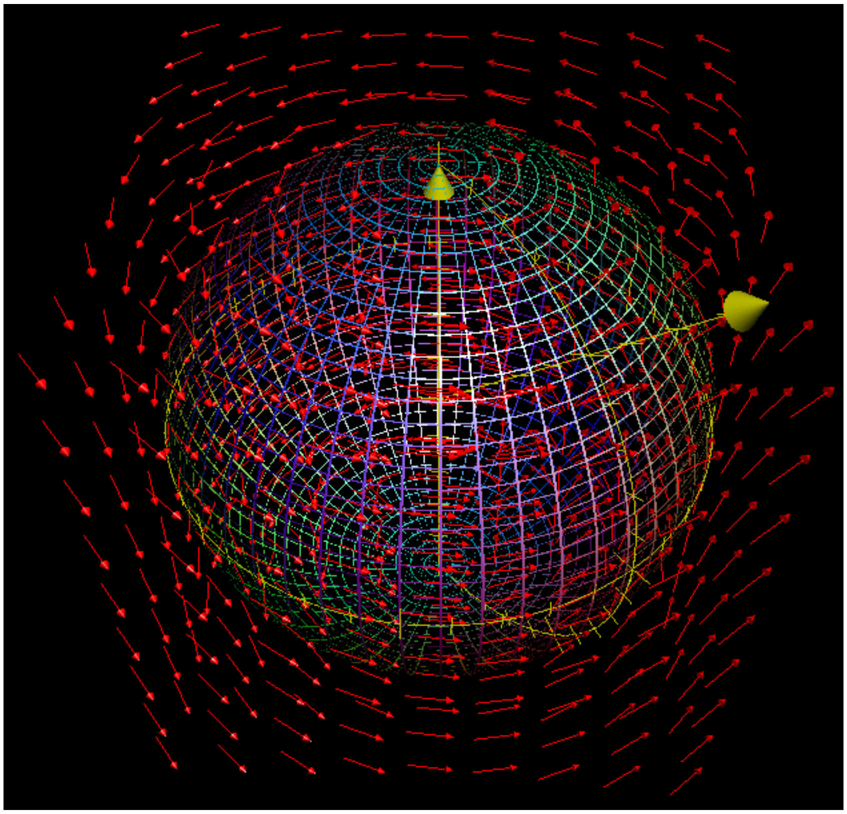

FIGURE 5. The frame element $\hat{\mathbf{u}}_\theta$ of polar/cylindrical and spherical coordinates, as a *unit spin vector field* on $\mathbb{R}^3 - \{x = 0 = y\}$, with central axis the z-axis.

Finally, we have a vector $\hat{\mathbf{u}}_\varphi$ which is tangent to the meridian through P on this sphere, and which is given by $\hat{\mathbf{u}}_\varphi = \hat{\mathbf{u}}_\varrho \times \hat{\mathbf{u}}_\theta$. Thus, it can be expressed as

$$\begin{aligned}\hat{\mathbf{u}}_\varphi &= \Big(\cos(\varphi)\,\hat{\mathbf{u}}_r + \sin(\varphi)\,\hat{\mathbf{k}}\Big) \times \hat{\mathbf{u}}_\theta \\ &= \cos(\varphi)\,\hat{\mathbf{u}}_r \times \hat{\mathbf{u}}_\theta + \sin(\varphi)\,\hat{\mathbf{k}} \times \hat{\mathbf{u}}_\theta \\ &= \cos(\varphi)\,\hat{\mathbf{k}} - \sin(\varphi)\,\hat{\mathbf{u}}_r \\ &= -\cos(\theta)\sin(\varphi)\,\hat{\mathbf{\imath}} - \sin(\theta)\sin(\varphi)\,\hat{\mathbf{\jmath}} + \cos(\varphi)\,\hat{\mathbf{k}}\,.\end{aligned}$$

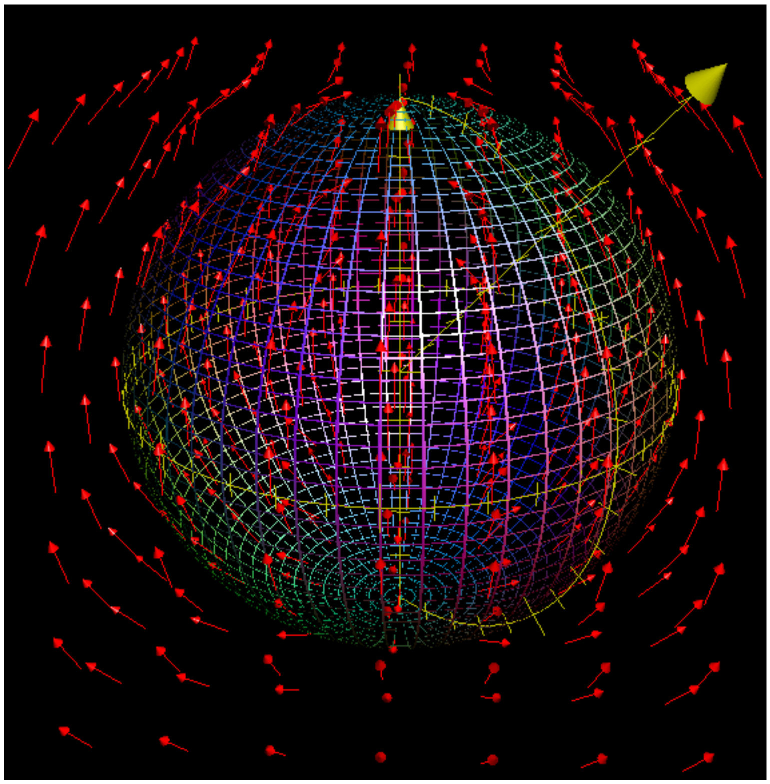

FIGURE 6. The spherical frame element $\hat{\mathbf{u}}_\varphi$ visualized as a *unit vector field* on $\mathbb{R}^3 - \{x = 0 = y\}$. Note that it is everywhere tangent to meridians of spheres.

As P was arbitrary, we've now defined a new frame $\big(\hat{\mathbf{u}}_\varrho(\theta,\varphi), \hat{\mathbf{u}}_\theta(\theta,\varphi), \hat{\mathbf{u}}_\varphi(\theta,\varphi)\big)$ in $\mathbb{R}^3$, which is non-constant as it depends on the angles θ and φ. We will suppress the appearance of these angular variables when writing the vectors in this frame, much as we have done for $\hat{\mathbf{u}}_r = \hat{\mathbf{u}}_r(\theta)$. This frame $(\hat{\mathbf{u}}_\varrho, \hat{\mathbf{u}}_\theta, \hat{\mathbf{u}}_\varphi)$, which is adapted to our choice of spherical coordinates in $\mathbb{R}^3$, will be called *the spherical frame*. Its flow-lines are rays leaving the origin (sets where both θ and φ are constant), and longitudinal and latitudinal circles of origin centered spheres (respectively, sets of constant ϱ and θ, and sets of constant ϱ and φ.) The whole frame is constant along rays leaving the origin. See figures (4), (5), and (6) for visualizations of $\hat{\mathbf{u}}_\varrho$, $\hat{\mathbf{u}}_\theta$, and $\hat{\mathbf{u}}_\varphi$ respectively.

4.5. **Velocity and Acceleration in the Spherical Frame.** Before computing the spherical coordinate expressions for velocity and acceleration of a particle undergoing motion, we give the time differentials of the frame elements[7]:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_\varrho &= \dot\theta\cos(\varphi)\,\hat{\mathbf{u}}_\theta + \dot\varphi\,\hat{\mathbf{u}}_\varphi\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_\theta &= -\dot\theta\cos(\varphi)\,\hat{\mathbf{u}}_\varrho + \dot\theta\sin(\varphi)\,\hat{\mathbf{u}}_\varphi\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}_\varphi &= -\dot\varphi\,\hat{\mathbf{u}}_\varrho - \dot\theta\sin(\varphi)\,\hat{\mathbf{u}}_\theta\,.\end{aligned}$$

Now, consider motion along the curve $\boldsymbol{\gamma}(t) = \boldsymbol{\rho}(t) = \varrho(t)\hat{\mathbf{u}}_\varrho(t)$. We can now easily obtain the velocity as

$$\begin{aligned}\dot{\boldsymbol{\gamma}}(t) &= \frac{\mathrm{d}\varrho}{\mathrm{d}t}\hat{\mathbf{u}}_\varrho + \varrho\frac{\mathrm{d}\hat{\mathbf{u}}_\varrho}{\mathrm{d}t}\\ &= \dot{\varrho}\,\hat{\mathbf{u}}_\varrho + \varrho\dot{\theta}\cos(\varphi)\,\hat{\mathbf{u}}_\theta + \varrho\dot{\varphi}\,\hat{\mathbf{u}}_\varphi\,.\end{aligned}$$

The expression for acceleration in our version of spherical coordinates is then given by

$$\begin{aligned}\ddot{\boldsymbol{\gamma}}(t) = &\left(\ddot{\varrho} - \varrho\dot{\varphi}^2 - \varrho\dot{\theta}^2\cos^2(\varphi)\right)\hat{\mathbf{u}}_\varrho\\ &+ \left(\varrho\ddot{\theta}\cos(\varphi) + 2\dot{\varrho}\dot{\theta}\cos(\varphi) - 2\varrho\dot{\theta}\dot{\varphi}\sin(\varphi)\right)\,\hat{\mathbf{u}}_\theta\\ &+ \left(\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi} + \varrho\dot{\theta}^2\sin(\varphi)\cos(\varphi)\right)\,\hat{\mathbf{u}}_\varphi\,.\end{aligned}$$

Observe that if $r = \varrho$ and $\varphi \equiv 0$, then the expressions for velocity and acceleration reduce to those given in the polar frame.

[7]Differentiating the spherical frame with respect to time seems like a routine calculation with the chain rule, but it hides that we are implicitly using *the chain rule for partial derivatives*, as the frame elements themselves depend on both of the angles θ and φ. One can first take partial derivatives of each frame element with respect to either θ or φ, and then employ the chain rule to arrive at the appropriate expressions. This is truly a multivariable frame, and so differentiating it by using partial derivatives is an instructive exercise in the calculus of multivariable vector valued functions. We leave the details of the calculation as an exercise.

5. Planetary Motion and Kepler's Laws

5.1. **Gravitation and the Two Body Problem.** To study the motion of the planets, we need to understand the forces that act upon them. The principal force that dominates the dynamics of celestial bodies is *the force of gravity*, which is an attractive force between massive bodies. Here, mass is an intrinsic quantity, which for reasons of simplicity, we'll model as constant for a given body. Frequently, we will refer to an object's *center of mass*, a concept which is more readily defined when we have a working theory of integration in multiple variables. For now, it suffices to know that planets have mass, and centers of mass, and that at the scales and distances involved, it is sufficient to model interactive forces between such objects as being forces acting between point masses, placed at the centers of mass. Thus, if an object has mass m, and a force acts upon this object, we can model the dynamics induced by this force by examining how they would act on a volume-less particle of mass m located at the center of mass of the original object.

We study here the celestial two-body problem, which concerns the motion of two celestial bodies which are attracted to each other by the force of gravity alone. We neglect the gravitational influence of additional bodies, and consider only two bodies at once. One wishes, in particular, to calculate the trajectories of each of the bodies, knowing only the initial configuration (positions and velocities for each in 3-space), and knowing how gravity acts upon the bodies. This force of gravity, as described by Sir Isaac Newton, acts along the axis connecting the centers of mass of each of the bodies, with a magnitude which is proportional to the product of their masses, and inversely proportional to the square of the distance between the centers of mass. Let us describe this in the language of vectors. If the objects have masses m_1 and m_2, and are positioned at points O and P in space, then the force of gravitational attraction of the second body upon the first body is given by

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{\|\overrightarrow{OP}\|^2}\frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|} = \frac{Gm_1m_2}{\|\overrightarrow{OP}\|^3}\,\overrightarrow{OP}\,,$$

while the force of the first body upon the second is

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{\|\overrightarrow{PO}\|^3}\,\overrightarrow{PO} = -\mathbf{F}_{21}\,,$$

where G is the constant of proportionality. This G is universal for this model, in the sense that its value depends only upon our choices of units, and not upon the choice of coordinates, nor the objects themselves.

One might simplify the problem one of two different ways. One simplification is to use a coordinate system with its origin at the center of mass of one of the objects, say the first one. Thus, though each object moves under the influence of gravitational attraction exerted by the other, our coordinates follow the movement of one, and so we can solve for the *relative motion* of the second body in the reference frame of the first. This is a sensible approach especially if the mass of the object used to define the origin is much larger than the mass of the other object, since in this case the more massive object undergoes less motion relative to an observer in an independent coordinate system (not based upon the motion of either object). The alternate strategy of simplification is to place the origin at the center of mass of the system, which better captures the symmetry of the gravitational attraction between objects, and is very well suited to the case where the masses are of comparable magnitude.

We are especially interested in *orbits*, which are trajectories that are closed in some natural reference frame, meaning that in a coordinate system centered around the more massive object, the position function of the less massive object is periodic. The smaller mass whose trajectory is an orbit is then called a *satellite* of the larger mass. This special case of the two body problem is called *the Keplerian problem*, and was motivated by the study of the motion of the Earth around the Sun, and the moon around the earth.

Suppose the more massive of the two bodies has mass M. We will use spherical coordinates with the origin at the center of mass of the more massive body. Suppose the smaller body has mass m and is located at a point P with position vector $\boldsymbol{\rho}$ with magnitude $\varrho = \|\boldsymbol{\rho}\|$. Then the force of gravity acting on the smaller body is

$$\mathbf{F}_m = -\frac{GMm}{\varrho^2}\,\hat{\mathbf{u}}_\varrho\,.$$

In our coordinates, this force is a *central force*, acting purely in the $\hat{\mathbf{u}}_\varrho$ direction. If we divide this force by the mass of our second object and consider the position $\boldsymbol{\rho}$ to be variable, we can describe a *force field*, which gives the force of gravity, at every point of space, of the object of mass M on any "test mass" (a particle of unit mass). Mathematically, this is a *vector field*, which we can regard as a function assigning vectors to points of space. In our case, the map giving the vector field is well defined on $\mathbb{R}^3 - \{(0,0,0)\}$, and exhibits radial symmetry. The magnitude of the force field increases as one approaches the origin.

We can rephrase our special case of the two body problem as follows: given a central gravitational force field

$$\mathbf{F}_G(\boldsymbol{\rho}) = -\frac{GM}{\varrho^2}\,\hat{\mathbf{u}}_\varrho\,,$$

determine the trajectory $\varrho(t)$ of an object of mass m placed into the field at any position $\boldsymbol{\rho}_0 = \boldsymbol{\rho}(0)$ with initial velocity $\mathbf{v}_0 = \dot{\boldsymbol{\rho}}(0)$. The object of mass m whose trajectory we wish to discover will be called a *satellite* of the more massive body, which we will call the *central mass*.

5.2. **Planarity of Two-Body Orbits.** One of the first steps towards a solution of the two-body problem for a satellite orbiting a central mass is to demonstrate that the motion is in fact confined to some plane. By adjusting our coordinates, we can assume that $\mathbf{v}_0$ and $\boldsymbol{\rho}_0$ both lie in the equatorial plane $\Pi_{\mathscr{E}}$ of our spherical coordinates. We now aim to show that the motion of the satellite occurs in this plane.

To do this, we must make use of Newton's second law of motion: the net force on a massive object is equal to the acceleration of the object, times the mass. This allows us to compute $\mathbf{a}(t) = \ddot{\boldsymbol{\rho}}(t)$:

$$\mathbf{a}(t) = \frac{\mathbf{F}_m}{m} = \mathbf{F}_G\big(\boldsymbol{\rho}(t)\big) = -\frac{GM}{[\varrho(t)]^2}\,\hat{\mathbf{u}}_\varrho(t)\,.$$

Thus, the only component of acceleration is radial (in the $\hat{\mathbf{u}}_\varrho$ direction).

Now, let $\mathbf{h}(t) = \boldsymbol{\rho}(t) \times \mathbf{v}(t)$, which is $[\varrho(t)]^2$ times the angular velocity associated to the motion of the satellite. Observe that $\dot{\mathbf{h}}(t) = \boldsymbol{\rho}(t) \times \dot{\mathbf{v}}(t) = \boldsymbol{\rho}(t) \times \mathbf{a}(t)$. Since $\mathbf{a}(t)$ is a scalar multiple of $\hat{\mathbf{u}}_\varrho(t)$, and is thus parallel to $\boldsymbol{\rho}(t)$, it follows that $\dot{\mathbf{h}}(t) = \mathbf{0}$ for all times t. But then, $\mathbf{h}$ is independent of t! Unless $\mathbf{h} \equiv \mathbf{0}$, it necessarily gives a normal to the plane containing $\boldsymbol{\rho}_0$ and $\mathbf{v}_0$, i.e. $\mathbf{h} = \boldsymbol{\rho}_0 \times \mathbf{v}_0$ is a constant vector giving a normal of $\Pi_{\mathscr{E}}$. But then, the velocity and position are confined in this equatorial plane since their cross product always gives a normal to $\Pi_{\mathscr{E}}$. Henceforth we will call $\Pi_{\mathscr{E}} = \{(\varrho,\theta,\varphi) \in \mathbb{R}^3 \,|\, \varphi = 0\}$ the *plane of orbit*, or simply the *orbital plane* and since $\varphi \equiv 0$ along the trajectory, we may pass to polar coordinates $\mathbf{r}(t) := \boldsymbol{\rho}(t)$ within this plane.

Observe also that we've proven that the torsion of such a planetary trajectory is identically zero, without ever having to fuss with the torsion formula! In fact, what we are seeing is *conservation of angular momentum*: the angular momentum is

$$\mathbf{L} := I\boldsymbol{\omega} = r^2 m\frac{\mathbf{r}\times\mathbf{v}}{r^2} = m\mathbf{h}\,,$$

where $I := r^2 m$ is the point-mass moment of inertia, and $\boldsymbol{\omega} = (\mathbf{r}\times\mathbf{v})/r^2$ is the angular velocity vector. Thus in proving the constancy of $\mathbf{h}$, we've shown that under the influence of a single radially acting force where one only has centripetal acceleration, the angular acceleration $\boldsymbol{\alpha} := \dot{\boldsymbol{\omega}}$ is identically zero, which implies that $\mathbf{L}$ is conserved.

5.3. **Orbital Velocity and Orbital Area.** We will now study in detail the velocity of the orbital motion of a satellite, as this will be the key to proving Kepler's laws of planetary motion. We begin with a geometric consequence of the planarity of two-body motion, which reveals a constraint on the geometry of the velocity function. In particular we will uncover another geometric interpretation of the normal vector $\mathbf{h}$ to the plane of orbit. Let $h = \|\mathbf{h}\|$. Calculating $\mathbf{h}$ explicitly in cylindrical coordinates yields

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{u}}_r \times (\dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta) = r^2\dot{\theta}\,\hat{\mathbf{k}} \implies \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2}\,.$$

Note that the differential for the area swept out by the motion is

$$\mathrm{d}\mathcal{A} = \frac{1}{2}r^2\,\mathrm{d}\theta = \frac{1}{2}r^2\dot{\theta}\,\mathrm{d}t = \dot{\mathcal{A}}\,\mathrm{d}t\,.$$

Therefore $h = r^2\dot{\theta} = 2\dot{\mathcal{A}}$, so h can be interpreted as twice the speed at which area accumulates as $\mathbf{r}$ sweeps out the path of the satellite's orbit.

Next, recalling the acceleration expression in cylindrical coordinates

$$\mathbf{a}(t) = (\ddot{r} - r\ddot{\theta})\hat{\mathbf{u}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{u}}_\theta + \ddot{z}\hat{\mathbf{k}}\,,$$

and using that $\dot{z}(t) \equiv 0$ and $\mathbf{a}$ is purely radial in the plane of orbit, we find that

$$0 = \mathbf{a}(t)\cdot\hat{\mathbf{u}}_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \frac{2}{r}\frac{\mathrm{d}\dot{\mathcal{A}}}{\mathrm{d}t}\,.$$

Since we presume finite distance $r(t)$ at all times, $1/r$ is always nonzero, and thus $\ddot{\mathcal{A}} \equiv 0$, which implies $\dot{\mathcal{A}}$ is constant. This is just a restatement of the fact that $\mathbf{h}$ has constant length, but it has a nice interpretation. If we calculate the net change in area swept out by $\mathbf{r}$ for any time interval of length $T \geq 0$ starting at time t, we get

$$\mathcal{A}(t,T) = \int_t^{t+T} \dot{\mathcal{A}}(\tau)\,\mathrm{d}\tau = \int_t^{t+T} \frac{1}{2}h\,\mathrm{d}\tau = \frac{1}{2}hT\,,$$

which depends only on the duration T of motion, and not on the starting time t. That is, *the position vector sweeps out equal areas over equal time intervals.* This is Kepler's second law! Stated more carefully:

Theorem (Kepler's Second Law of Planetary Motion)**.** *For any given time periods* $[t_0, t_1]$ *and* $[t_2, t_3]$ *with* $t_1 - t_0 = t_3 - t_2$*, the areas*

$$\mathcal{A}[t_0,t_1] = \int_{t_0}^{t_1} \frac{1}{2}r^2\dot{\theta}\,\mathrm{d}t \quad \textit{and} \quad \mathcal{A}[t_2,t_3] = \int_{t_2}^{t_3} \frac{1}{2}r^2\dot{\theta}\,\mathrm{d}t$$

swept out in the orbital plane by the position vector $\mathbf{r}(t)$ *of the satellite are equal.*

Observe that since $\dot{\theta} = h/r^2$, we can now re-express velocity using h. From the chain rule, we can write

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\,.$$

And so the velocity function in terms of the invariant h is

$$\mathbf{v} = \frac{h}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\hat{\mathbf{u}}_r + \frac{h}{r}\hat{\mathbf{u}}_\theta = \frac{h}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\hat{\mathbf{u}}_r + r\hat{\mathbf{u}}_\theta\right) = \frac{h}{r^2}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\,.$$

By using the expression for acceleration from the law of gravitation, we can get a more explicit formula for velocity, and in particular, deduce formulae for both $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ and $r(\theta)$. This work was carried out in the mid nineteenth century by the mathematician William Rowan Hamilton, who illuminated the geometry of satellite motion by proving the following characterization of the velocity:

Theorem (Hamilton, 1846)**.** *The velocity function* $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ *of a trajectory* $\mathbf{r}(t)$ *for the two-body problem with a centralized mass traces out a circle or an arc thereof in the plane of motion, centered at some point* C *with position* $\mathbf{c}$ *in the plane of motion; and to any such circle or arc* $\mathcal{C}$ *in a plane through the central mass, there corresponds a possible trajectory whose velocity traces out* $\mathcal{C}$*. Thus, the* velocity circle *determines the trajectory completely.*

Observe that the theorem does not presume a closed orbit. In fact, a consequence of the proof, discussed below, is that any connected component of a conic section can arise as a trajectory. For non-closed orbits, the velocity only traces out an arc of a circle, while for a closed orbit, it traces the full circle each time the motion completes one period. Though the velocity may only trace out an arc of a circle, we'll still refer to this geometric figure as *a velocity circle*, with the understanding that perhaps only a portion of the circle corresponds to velocities of the object for real times. The first part of the theorem follows from straightforward integration. The intuition for the proof that any velocity circle determines an orbit is to use the constancy of $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ to algebraically find $\mathbf{r}$ as a function of θ. Though at first it seems we can avoid an integration in this way, one still must integrate in order to express position as a function of time. We now commence with the proof of Hamilton's theorem.

Proof. From our above velocity considerations and Newton's expression for gravitational acceleration, we have

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{u}}_r = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \frac{h}{r^2}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\theta}\,.$$

Thus, $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\theta} = \frac{-GM}{h}\,\hat{\mathbf{u}}_r$, and integrating with respect to θ one has

$$\mathbf{v}(\theta) = \frac{GM}{h}\hat{\mathbf{u}}_\theta + \mathbf{c}\,,$$

where $\mathbf{c}$ is some constant vector arising from indefinite integration. As $\hat{\mathbf{u}}_\theta\big(\theta(t)\big)$ takes values in the unit circle, we've established that $\mathbf{v}(t)$ parameterizes a circle or an arc of a circle $\mathcal{C}$ of radius GM/h centered at some point C with position vector $\mathbf{c}$.

Given any velocity circle $\mathbf{v}(\theta) = R\hat{\mathbf{u}}_\theta + \mathbf{c}$, we can define a vector $\mathbf{h} = \frac{GM}{R}\hat{\mathbf{k}}$, and seek a function $\mathbf{r}(t)$ such that $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, and $\dot{\mathbf{r}}(t) = \mathbf{v}(\theta(t))$ with $\dot{\theta} = \|\mathbf{h}\|/\|\mathbf{r}(t)\|^2 = GM/(R[r(t)]^2)$. Indeed, such an $\mathbf{r}$ is determined uniquely. Let ϕ be the angle made between $\mathbf{c}$ and the ray $\theta = 0$, and let $\varepsilon := \|\mathbf{c}\|/R$. Then

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{u}}_r \times (R\hat{\mathbf{u}}_\theta + \mathbf{c}) = Rr\Big(1 + \varepsilon\sin(\phi - \theta)\Big)\hat{\mathbf{k}}$$

Using that $h = GM/R$, and solving for $r(\theta)$, one obtains

$$r(\theta) = \frac{GM}{R^2\Big(1 + \varepsilon\sin(\phi - \theta)\Big)}\,.$$

The condition that $r(\theta) \geq 0$ and finite implies that we should restrict our values of θ to ensure that $1 + \varepsilon\sin(\phi - \theta) > 0$ if $\varepsilon \geq 1$.

To get $\mathbf{r}$ as a function of t, we need to express θ as a function of t. Since $r^2\dot{\theta} = h = GM/R$, we have a differential equation relating θ to t:

$$\frac{(GM)^2}{R^4[1 + \varepsilon\sin(\phi - \theta)]^2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{GM}{R} \implies \frac{\mathrm{d}\theta}{[1 + \varepsilon\sin(\phi - \theta)]^2} = \left(\frac{R^3}{GM}\right)\mathrm{d}t\,.$$

The integral on the left-hand side is difficult, as it involves a half-angle substitution (also called an Euler-Weierstrass substitution), but it suffices to note that, as the integrand itself is positive, the integral of the left hand side is a monotonic function of θ, while the right hand side becomes a linear function of t. Thus, the result on the left is invertible, and there exists a function $\theta(t)$ satisfying the differential equation. There's no hope of explicitly writing $\theta(t)$ in terms of elementary functions, though one can write t in terms of θ (it's not pretty!).

The constant of integration is determined by an initial value $\mathbf{v}_0 = \mathbf{v}(0) = R\hat{\mathbf{u}}_\theta(\theta_0) + \mathbf{c}$, from which there is a unique $\theta_0 = \theta(0)$, and this value also determines the initial position $\mathbf{r}_0$. Thus, the velocity circle, together with an initial velocity on it, uniquely determine a solution to a corresponding initial value problem for satellite motion. Conversely, given an initial position $\mathbf{r}_0 = r_0\hat{\mathbf{u}}_r(\theta_0)$ and initial velocity $\mathbf{v}_0 = \mathbf{v}(\theta_0)$, there is a unique velocity circle suited to the initial conditions. Indeed, since $\mathbf{h} = \mathbf{r}_0 \times \mathbf{v}_0$ and $\|\mathbf{h}\| = h = GM/R$, one has $R = GM/\|\mathbf{r}_0 \times \mathbf{v}_0\|$ and

$$\mathbf{c} = \mathbf{v}_0 - \frac{GM}{\|\mathbf{r}_0 \times \mathbf{v}_0\|}\hat{\mathbf{u}}_\theta(\theta_0).$$

□

5.4. **Orbital Shape and Period.** In the course of proving Hamilton's theorem, we encountered a general solution for the orbital trajectory as a polar curve, which we can now express in terms of G, M, and h:

$$r(\theta) = \frac{h^2/(GM)}{1 + \varepsilon \sin(\phi - \theta)}\,.$$

Perhaps you already recognize this type of equation for a polar curve. There's a more standard way to write $r(\theta)$: choose coordinates so that the center of the velocity circle, $\mathbf{c}$, lies on the ray with $\theta = \pi/2$, and let $\Lambda := h^2/(GM)$. Then the polar expression for the distance of the satellite from the central mass becomes

$$r(\theta) = \frac{\Lambda}{1 + \varepsilon \cos(\theta)}\,.$$

It is a standard exercise to see that this is a conic section of eccentricity ε. In particular, it is a circle if $\varepsilon = 0$, an ellipse if $0 < \varepsilon < 1$, a parabola if $\varepsilon = 1$ and a hyperbola if $\varepsilon > 1$. While parabolae and hyperbolae can occur as trajectories for a general two-body problem, the Keplerian problem we are studying concerns only orbits, and thus the solutions that concern us have eccentricity $\varepsilon \in [0, 1)$, and are ellipses or circles. We have arrived at Kepler's first law: orbits (in the sense of closed trajectories) are elliptical.

Theorem (Kepler's First law of Planetary Motion)**.** *An orbital (periodic) solution to the two-body problem for a satellite in the reference frame of a central mass is either an ellipse or a circle. In particular, there are constants $\varepsilon \in [0, 1)$ and $\Lambda > 0$ such that the satellite traverses an ellipse of eccentricity ε, with semi-latus rectum Λ, and with the central mass at one focus. The semi-major axis has length $a = \Lambda/(1 - \varepsilon^2)$, and the semi-minor axis has length $b = a\sqrt{1 - \varepsilon^2} = \Lambda/\sqrt{1 - \varepsilon^2}$.*

While we cannot write an explicit formula for the orbital position as a function of time, from the orbital shape, we can determine the *period*, which is the time it takes for the satellite to return to a position. Since the orbit is an ellipse or circle, we can easily compute the *orbital area*, i.e. the area enclosed in the orbital plane by the closed trajectory. For an ellipse with semi-major axis of length a, and semi-minor axis of length b, the area is $\mathcal{A} = \pi ab$, which reduces to πa^2 for the circular case $b = a$. Let T denote the orbital period. We will give a formula for T in terms of the length of the semi-major axis, and the constants G and M. Observe that since $a = \Lambda/(1 - \varepsilon^2)$, $1 - \varepsilon^2 = \Lambda/a$, and $= b = a\sqrt{1 - \varepsilon} = a\sqrt{\Lambda/a} = h\sqrt{\frac{a}{GM}}$. Thus

$$\mathcal{A} = \pi ab = \pi ah\sqrt{\frac{a}{GM}} = \frac{\pi a^{3/2} h}{\sqrt{GM}}\,.$$

Since $h = 2\dot{\mathcal{A}}$, we have

$$\mathcal{A} = \int_0^T \mathrm{d}\mathcal{A} = \int_0^T \dot{\mathcal{A}}\,\mathrm{d}t = \frac{h}{2}T\,,$$

and equating these two expressions for area gives Kepler's Third law:

Theorem (Kepler's Third Law of Planetary Motion)**.** *The square of the period T of an elliptical orbit of a satellite is proportional to the cube of the length a of the semi-major axis of the elliptical trajectory:*

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}\,.$$

6. Problems

(1) A *logarithmic spiral* is a plane curve given by a polar equation of the form

$$\ln(r/a) = b\theta \quad \text{or} \quad r = ae^{b\theta}\,.$$

Note that if $b = \ln c$ one can write this as

$$r = a(c^{\theta})\,.$$

(a) Give a parameterization of a logarithmic spiral as a vector valued function in rectangular coordinates.

(b) Sketch the spiral

$$r = \varphi^{2\theta/\pi}\,,$$

where φ is the golden ratio, which is the positive root of the equation $\varphi^2 - \varphi - 1 = 0$. This is called the *golden spiral.* Hint: you can use the property of a "golden rectangle", whose sides are in ratio $\varphi : 1$, that if you cut a square off of the rectangle, the remaining rectangle is also a golden rectangle. For the interested, it is possible to construct a golden rectangle with a compass and straight edge.

(c) Compute the arc-length function for a general logarithmic spiral, in terms of a and b, and give an arc-length parameterization of the spiral as a vector valued function in both rectangular and polar coordinates.

(d) What is the total length of the golden spiral from the point $(r,\theta)_{\mathscr{P}} = (1,0)_{\mathscr{P}}$ to the origin?

(2) Why is the curvature vector $\boldsymbol{\mathcal{K}}$ perpendicular to the unit tangent vector? Give an algebraic argument as well as a geometric interpretation, in terms of derivatives.

(3) Verify that signed curvature satisfies properties (i), (ii) and (iii) as listed in 1.2.

(4) Prove the curvature formula $(\star)$. Use the formula to check that the curvature of a line is identically 0, that the curvature of a circle is the reciprocal of its radius, that the curvature of a circular helix is constant, and that the maximum curvature of a parabola happens at the parabola's vertex.

(5) Show from a vector parameterization $\boldsymbol{\gamma}(x) = x\hat{\mathbf{\imath}} + f(x)\hat{\mathbf{\jmath}}$ of the graph of a function $y = f(x)$ that the curvature is given by

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}\,.$$

(6) Let $\boldsymbol{\gamma}(t)$ be a space curve with non-vanishing curvature at $t = t_0$. Let

$$\Pi_{\mathcal{O}}(t_0) = \{\boldsymbol{\gamma}(t_0) + u\mathbf{T}(t_0) + v\mathbf{N}(t_0)\,|\,u, v \in \mathbb{R}\}$$

be the osculating plane to $\boldsymbol{\gamma}(t)$ at $t = t_0$.

(a) Why is it necessary that $\kappa(t_0) \neq 0$ to define $\Pi_{\mathcal{O}}$?

(b) Find (as a function of t) the osculating plane of the helix $\boldsymbol{\gamma}(t) = a\Big(\mathbf{i}\cos(t) + \mathbf{j}\sin(t)\Big) + \mathbf{k}bt$, and give scalar equations for the osculating planes at the points with $t = \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}$. By scalar equations of the plane, recall we mean equations of the form $ax+by+cz+d = 0$ for some constants $a, b, c, d \in \mathbb{R}$.

(7) Argue that the torsion function exists for for any thrice differentiable vector valued function $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, i.e. show that that there exists a scalar function τ such that

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau(s)\mathbf{N}\,,$$

where s is an arc-length parameter, $\mathbf{N}$ is the principal normal vector, and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is the binormal vector.

(8) In this problem you will give another proof of the torsion formula ($\diamond$). First, show from the definition $\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$ that if $\boldsymbol{\gamma}(s)$ is a unit speed parameterization, then

$$\tau(s) = \frac{\big(\boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s)\big) \cdot \boldsymbol{\gamma}'''(s)}{\|\boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s)\|}\,,$$

and then use the chain rule to show that this formula is invariant under change of variables, i.e. that re-parameterizing to a variable t gives

$$\frac{\big(\boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s)\big) \cdot \boldsymbol{\gamma}'''(s)}{\|\boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s)\|} = \frac{\big(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\big) \cdot \dddot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}\,.$$

(9) Interpret the torsion in terms of a volume and an area related to the velocity, acceleration, and jerk. What linear relation between velocity, acceleration, and jerk must be true if torsion is zero?

(10) Perform the necessary calculations using the chain rule and the Fenet-Serret equations to obtain the formula (‡) expressing the jerk in the Fenet-Serret frame.

(11) Recall, the first and third equations of Frenet and Serret are essentially the definitions of the curvature and torsion, respectively. Prove the remaining Frenet-Serret equation

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)\,.$$

(12) Given a unit-speed curve $\boldsymbol{\gamma}(s)$ with natural frame $\big(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\big)$, find a vector $\mathbf{D}(s)$ such that

$$\mathbf{D}(s) \times \mathbf{T}(s) = \mathbf{T}'(s)\,, \quad \mathbf{D}(s) \times \mathbf{N}(s) = \mathbf{N}'(s)\,, \quad \mathbf{D}(s) \times \mathbf{B}(s) = \mathbf{B}'(s).$$

What is $\mathbf{D} \times \dot{\boldsymbol{\gamma}}$?

(13) If a space curve has constant, nonzero curvature and *no torsion*, what kind of curve must it be?

(14) If a space curve has constant, nonzero curvature and constant torsion, what kind of curve must it be?

(15) This problem concerns the geometry and topology of closed curves. Let $I = [0, 1]$. Then a *closed curve* has a parametrization $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1)$.

Recall, a curve is *regular* at a point if the tangent vector is defined and nonzero there. We say for a closed curve $\boldsymbol{\gamma}(t)$ that it is regular at the point $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1)$ if one can define a nonzero tangent vector $\dot{\boldsymbol{\gamma}}(0)$ and a nonzero tangent vector $\dot{\boldsymbol{\gamma}}(1)$, and these tangent vectors are equal. We say a simple closed curve is regular if it is regular at all points.

A closed curve is called *simple* if there are no self-intersections.

(a) Show that if $\ddot{\boldsymbol{\gamma}}(t) = 0$ for all $t \in I$ then $\boldsymbol{\gamma}$ is a line segment/line in $\mathbb{R}^3$. Show in particular that a regular closed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ has nonzero acceleration around at least two distinct values (meaning there exists a pair of points $t_1, t_2 \in I$ and some sufficiently small $\varepsilon > 0$ such that $(t_1 - \varepsilon, t_1 + \varepsilon), (t_2 - \varepsilon, t_2 + \varepsilon) \subset I$ are disjoint intervals for which $\ddot{\boldsymbol{\gamma}}(t)$ is everywhere nonzero.)

(b) Justify or falsify the following statements about regular closed curves:

(i.) If $\ddot{\boldsymbol{\gamma}}(t) \neq 0$ for any $t \in I$, then $\dot{\boldsymbol{\gamma}}$ and $\ddot{\boldsymbol{\gamma}}$ are each regular closed curves.

(ii.) For any regular closed curve $\boldsymbol{\gamma}(t)$ there exists a re-parametrization $\boldsymbol{\xi}\big(s(t)\big) = \boldsymbol{\gamma}(t)$ such that $\boldsymbol{\xi}'(s) := \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\xi}(s)$ is a closed curve on the unit sphere.

(iii.) Any simple regular closed space curve has at least one pair of points $\boldsymbol{\gamma}(t_1), \boldsymbol{\gamma}(t_2)$ such that $\dot{\boldsymbol{\gamma}}(t_1)$ and $\dot{\boldsymbol{\gamma}}(t_2)$ are parallel but point in opposite directions, i.e. there's some constant $\lambda > 0$ such that $\dot{\boldsymbol{\gamma}}(t_1) = -\lambda\dot{\boldsymbol{\gamma}}(t_2)$.

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(16) For θ the angle coordinate in polar coordinates on the plane, show that

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}\,.$$

(17) Express $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$ in the spherical frame, i.e. give each as a linear combination of $\hat{\mathbf{u}}_\varrho$, $\hat{\mathbf{u}}_\theta$ and $\hat{\mathbf{u}}_\varphi$.

(18) Compute the partial derivatives $\frac{\partial \hat{\mathbf{u}}_\varrho}{\partial \theta}$, $\frac{\partial \hat{\mathbf{u}}_\varrho}{\partial \varphi}$, $\frac{\partial \hat{\mathbf{u}}_\theta}{\partial \theta}$, $\frac{\partial \hat{\mathbf{u}}_\theta}{\partial \varphi}$, $\frac{\partial \hat{\mathbf{u}}_\varphi}{\partial \theta}$, and $\frac{\partial \hat{\mathbf{u}}_\varphi}{\partial \varphi}$, and then use the chain rule for partial derivatives to compute the velocity and acceleration along a curve $\boldsymbol{\gamma}(t)$ in spherical coordinates.

(19) Express the jerk $\dddot{\boldsymbol{\gamma}}(t)$ in cylindrical and spherical coordinates.

(20) Let $f(\mathbf{r})$ be a function defined on a domain $\mathcal{D} \subseteq \mathbb{R}^2$. By $f(r, \theta)$ we mean f evaluated at the point with position $\mathbf{r} = r\hat{\mathbf{u}}_r(\theta) = r\cos(\theta)\hat{\mathbf{\imath}} + r\sin(\theta)\hat{\mathbf{\jmath}}$. Express the gradient of f in polar coordinates, meaning, describe the operator ∇ in terms of $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ by giving functions $u(r, \theta)$ and $v(r, \theta)$ such that

$$\nabla = u(r,\theta)\frac{\partial}{\partial r}\hat{\mathbf{u}}_r + v(r,\theta)\frac{\partial}{\partial \theta}\hat{\mathbf{u}}_\theta$$

and so that $\nabla f(r(x,y), \theta(x,y)) = \frac{\partial f}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial f}{\partial y}\hat{\mathbf{\jmath}}$ for all points $\big(r(x,y), \theta(x,y)\big)_{\mathscr{P}} = (x,y)_{\mathscr{C}}$ in $\mathcal{D}$.

(21) Express the gradient operator in spherical coordinates (see the previous problem for the two dimensional, polar version of this problem.)

(22) Compute the gradients of the coordinate functions for spherical coordinates, i.e. compute $\nabla\varrho$, $\nabla\theta$ and $\nabla\varphi$.

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(23) This problem deals with conic sections, and provides results useful to the above treatment of Kepler's laws of planetary motion. The last part presumes familiarity with complex numbers.

(a) Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be two position vectors for points F_1 and F_2 in $\mathbb{R}^2$, and let a a real number such that $2a > \|\mathbf{f}_1 - \mathbf{f}_2\|$. The locus $\mathbf{r} \in \mathbb{R}^2$ satisfying $\|\mathbf{r} - \mathbf{f}_1\| + \|\mathbf{r} - \mathbf{f}_2\| = 2a$ defines an ellipse with foci F_1 and F_2. Parametrize this ellipse in a polar frame as a function of the angle θ measured from the center of the ellipse and swept from the line containing the foci.

(b) The ellipse's eccentricity is defined to be $\varepsilon := \|\mathbf{f}_1 - \mathbf{f}_2\|/2a$. Fix a and let F_1 be the origin and $\mathbf{f}_2 \cdot \hat{\mathbf{\jmath}} = 0$. Show that the Cartesian equation (i.e. the equation in rectangular coordinates) for the ellipse is $(x \pm 2a\varepsilon)^2 + y^2 = (2a - r)^2$ where r as usual satisfies $r^2 = x^2 + y^2$.

(c) The *semi-latus rectum* is defined to be $\Lambda := (1 - \varepsilon^2)a$. Draw and label an ellipse, making note of the geometric meaning of the semi-latus rectum.

(d) Show that $r = \Lambda/(1 \pm \varepsilon\cos\theta)$, which is the polar form of an ellipse with a focus at the origin. If we place the second focus on the polar ray $\theta = 0$, and the polar curve is oriented so that increasing θ takes one counterclockwise around the origin, which sign should be used in the denominator?

(e) Note that for the ellipse, $0 \leq \varepsilon < 1$, with $\varepsilon = 0$ if and only if $F_1 = F_2$, which gives a circle. Verify that the above equation for r gives other conic sections for $\varepsilon \geq 1$.

(f) By replacing vectors with the corresponding complex numbers ($\hat{\mathbf{\imath}} \mapsto 1$, $\hat{\mathbf{\jmath}} \mapsto i = \sqrt{-1}$), show that the map $z \mapsto z^2$ takes the origin centered ellipse $z(t) = a\sin(t) + ib\cos(t)$ to an ellipse with one focus at the origin. What are the semi-major and semi-minor axis lengths of the new ellipse, in terms of a and b?

(24) Give an integral expression for the arc-length of an ellipse.

(25) In this set of problems you will use a slightly different approach to proving Kepler's first law, and then derive some additional results concerning orbits and escape velocity. For parts (a)–(e) presume an orbital trajectory $\mathbf{r}(t)$ for the Keplerian problem, and let the velocity be $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$. You may assume Kepler's second law, and let $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ as above.

(a) Show that $\mathbf{v} \times \mathbf{h} - GM\mathbf{u}_r$ is constant.

(b) Let $\boldsymbol{\epsilon} := \frac{1}{GM}(\mathbf{v} \times \mathbf{h}) - \mathbf{u}_r$, and let $\epsilon = \|\boldsymbol{\epsilon}\|$. By considering scalar triple products of $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{h}$, give an alternate proof of Kepler's first law, without using Hamilton's theorem.

(c) The vector $\boldsymbol{\epsilon}$ above is called the *eccentricity vector*. Show that the eccentricity $\epsilon = \|\boldsymbol{\epsilon}\|$ in the proof of Kepler's first law in the previous part is the same as the eccentricity that arises from the proof via Hamilton's theorem, by relating the eccentricity vector $\boldsymbol{\epsilon}$ to the velocity circle's center $\mathbf{c} = \mathbf{v}_0 - \frac{GM}{\|\mathbf{r}_0 \times \mathbf{v}_0\|}\hat{\mathbf{u}}_\theta(\theta_0)$, where $\mathbf{r}_0 = r_0\hat{\mathbf{u}}_r(\theta_0)$ is a given initial position and $\mathbf{v}_0 = \mathbf{v}(\theta_0)$ a given initial velocity for the orbit.

(d) Find the farthest distance of the satellite from its orbital focus, called the *apoapsis*, in terms of h, G, M, and ε. Similarly, give the shortest distance of the satellite from its orbital focus, called the *periapsis*.

(e) Using the area formula for an ellipse, and the expressions for apoapsis and periapsis, re-derive Kepler's third law relating the period of the orbit to the semi-major axis. Express the answer separately in terms of the eccentricity, and in terms of the semi-major axis length.

(f) Let ψ denote the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, and let $v := \|\mathbf{v}\| = \|\dot{\mathbf{r}}\|$. Express ε^2 in terms of r, v, ψ, G and M.

(g) Escape velocity is precisely the velocity necessary to break free from a periodic orbit. Show that in fact, there is only escape speed (in the sense that escaping doesn't depend on ψ so long you begin from a position where your initial velocity when attempting escape won't put you on a trajectory to impact the central mass), and that this speed is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}\,.$$