Limits and Continuity for Multivariate Functions

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Outline

Definition of a Limit in two Variables

Definition

Given a function of two variables $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ such that D contains points arbitrarily close to a point (a, b), we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) exists and has value L if and only if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\,.$$

We then write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L\,.$$

Interpretation

- Thus, to say that L is the limit of $f(x, y)$ as (x, y) approaches (a, b) we require that for any given positive "error" $\varepsilon > 0$, we can find a bound $\delta > 0$ on the distance of an input (x, y) from (a, b) which ensures that the output falls within the error tolerance around L (that is, $f(x, y)$ is no more than ε away from L).

- Another way to understand this is that for any given $\varepsilon > 0$ defining an *open metric neighborhood* $(L - \varepsilon, L + \varepsilon)$ of L on the number line $\mathbb{R}$, we can ensure there is a well defined $\delta(\varepsilon)$ such that the image of any (possibly *punctured*) *open disk* of radius $r < \delta$ centered at (a, b) is contained in the ε-neighborhood.

Limits along paths

Recall, for functions of a single variable, one has notions of *left and right one-sided limits*:

$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)\,.$$

But in $\mathbb{R}^2$ there's not merely left and right to worry about; one can approach the point (a, b) along myriad different *paths*! The whole limit $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if and only if the limits along all paths agree and equal L.

Defining Limits along paths

To write a limit along a path, we can parameterize the path as some vector valued function $\mathbf{r}(t)$ with $\mathbf{r}(1) = \langle a, b\rangle$, and then we can write

$$\lim_{t\to 1^-} f(\mathbf{r}(t)) = L$$

if for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(\mathbf{r}(t)) - L| < \varepsilon$ whenever $1 - \delta < t < 1$. Similarly we may define a "right" limit along $\mathbf{r}(t)$, $\lim_{t\to 1^+} f(\mathbf{r}(t))$ if $\mathbf{r}(t)$ exists and describes a continuous path for $t > 1$. The two sided limit along the path is then defined in the natural way:

$$\lim_{t\to 1} f(\mathbf{r}(t)) = L \iff \forall\varepsilon > 0\ \exists\delta > 0 :$$

$$|f(\mathbf{r}(t)) - L| < \varepsilon \text{ whenever } 0 < |1 - t| < \delta\,.$$

A Classic Revisted

Example

Let $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$. Then find

$$\lim_{(x,y)\to(0,0)} f(x,y)\,.$$

Solution:

We can compute the limit as follows. Let $r^2 = x^2+y^2$. Then along any path $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ such that as $t \to 1$, $\mathbf{r}(t) \to \mathbf{0}$, we have that $r^2 = \|\mathbf{r}\|^2 \to 0$. It follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r^2\to 0} \frac{\sin r^2}{r^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1\,.$$

A Surface of Revolution

The previous example has a geometric solution as well: the graph for $z = f(x, y) = \sin(r^2)/r^2$ is a surface of revolution. What is the curve revolved, and what is the axis?

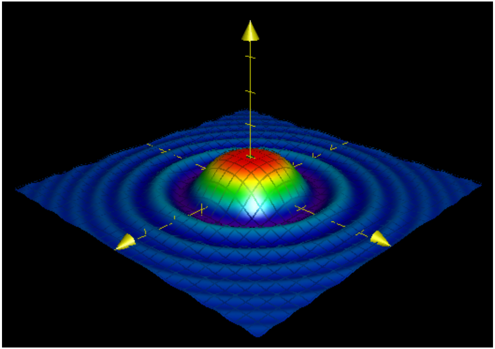

Figure: The graph of $z = \sin(r^2)/r^2$.

A Non-Existent Limit

Example

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist for $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Solution: We will show that the limits along the x and y axes are different, thus showing that the limit cannot exist.

The Axial Limits

Example

Along the x-axis, $y = 0$, and so $f(x,y) = f(x,0) = \dfrac{x^2 - 0}{x^2 + 0} = 1$, whence

$$\lim_{x\to 0} f(x,0) = 1\,.$$

Along the y-axis, $x = 0$ and $f(x,y) = f(0,y) = \dfrac{0 - y^2}{0 + y^2} = -1$, whence

$$\lim_{y\to 0} f(0,y) = -1\,.$$

Since $\lim\limits_{x\to 0} f(x,0) \neq \lim\limits_{y\to 0} f(0,y)$, the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Seeing the Crease

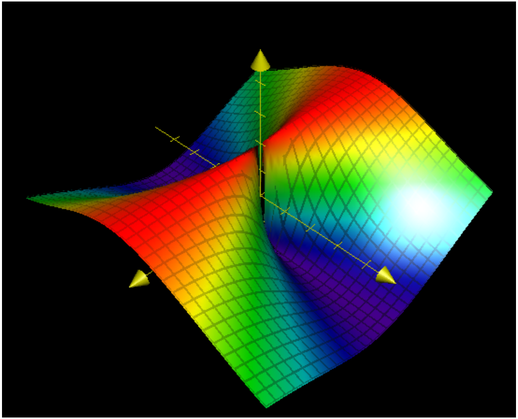

Figure: The graph of $z = (x^2 - y^2)/(x^2 + y^2)$

A Curious Wrinkle

Example

Does the limit of $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ as $(x,y) \to (0,0)$ exist, and if yes, what is it?

Solution: A simple reapplication of the method of the previous example is not sufficient. Indeed, you can check that

$$\lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} f(0,y)\,,$$

and in fact, we can show that for any line of approach through $(0,0)$, the limit is 0.

All the (defined) slopes

Example

Indeed: let m be any real number and consider the line $y = mx$, which passes through $(0,0)$ as $x \to 0$. Then

$$\begin{aligned} \lim_{x\to 0} f(x, mx) &= \lim_{x\to 0} \frac{x(mx)^2}{x^2 + (mx)^4} \\ &= \lim_{x\to 0} \frac{m^2x^3}{x^2 + m^4x^4} \\ &= \lim_{x\to 0} \frac{m^2x}{1 + m^4x^2} = 0\,, \end{aligned}$$

regardless of the value of m. (In the case of a line of undefined slope, we simply have the y-axis, which shares this limit)

So is the limit 0?

Precarious Parabolae

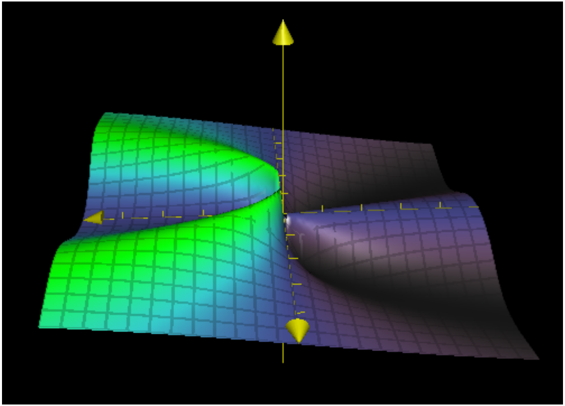

Figure: The graph of $z = (xy^2)/(x^2 + y^4)$

A new path

Example

If we instead approach $(0,0)$ along the parabola $x = y^2$, we find that

$$\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^4}{y^4 + y^4} = \frac{1}{2} \neq 0\,.$$

Thus the limit does not exist!

Key moral: Given the complexity of surface discontinuities (creases, wrinkles, precipitous slopes, etc), one cannot trust that the limit exists just from testing some small family of curves. When a limit does exist, proving it via curves is impractical, and one must resort to the definition (working with ε's and δ's.)

Using the Definition to Prove a Limit

Example

Consider the function $f(x,y) = \dfrac{3xy^2}{x^2+y^2}$.

An intuition for this one might be that the limit is zero as $(x,y) \to (0,0)$. After all, the numerator is cubic, and the denominator quadratic, so we can guess who should win in a fight.

After testing out lines, parabolas, and even some cubics approaching $(0,0)$, one gets that the limits along these curves all go to 0. How can we show that the limit is indeed zero?

All Paths Lead to. . .

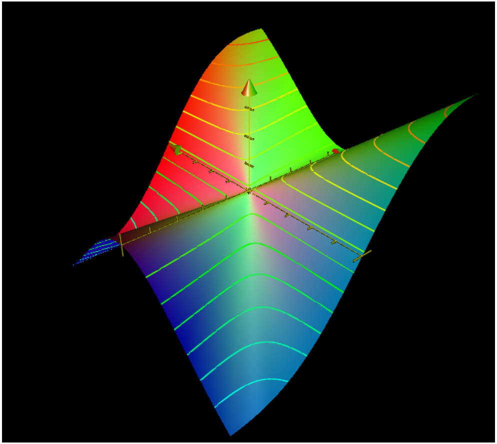

Figure: The graph of $z = 3xy^2/(x^2 + y^2)$.

To Acquire a Delta, Fix and Epsilon

Example

Suppose we are given an $\varepsilon > 0$, and we know that $|f(x, y) - 0| < \varepsilon$ for some (x, y).

We wish to work backwards to figure out how close to $(0, 0)$ the point (x, y) must be to ensure that this inequality is true.

The resulting bound on distance, δ, will depend on ε.

Inequalities and Algebra

From the assumed bound

$$|f(x,y) - 0| < \varepsilon \implies \left|\frac{3xy^2}{x^2+y^2}\right| = 3|x|\frac{y^2}{x^2+y^2} < \varepsilon$$

and the following inequalities

$$x^2 \leq x^2 + y^2 \text{ and } 0 \leq y^2/(x^2+y^2) \leq 1$$

we have that

$$3|x|\frac{y^2}{x^2+y^2} \leq 3|x| = 3\sqrt{x^2} \leq 3\sqrt{x^2+y^2}\,.$$

We see the distance between (x,y) and $(0,0)$ appearing on the far right in the inequality.

Epsilong Proofs: When's the punchline?

Since 3 times this distance is an upper bound for $|f(x, y) - 0|$, we simply choose δ to ensure $3\sqrt{x^2 + y^2} < \varepsilon$. Thus, we may take $\delta = \varepsilon/3$.

Then provided $\delta = \varepsilon/3$, we have that whenever $0 < \sqrt{x^2 + y^2} < \delta$, the inequality

$$3\sqrt{x^2 + y^2} < \varepsilon$$

holds, whence

$$|f(x, y) - 0| = 3|x|\frac{y^2}{x^2 + y^2} \leq 3\sqrt{x^2 + y^2} < \varepsilon\,,$$

which proves that

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2 + y^2} = 0\,.$$

Local Continuity

Definition

A function of two variables $f : D \to \mathbb{R}$ is continuous at a point $(x_0, y_0) \in D$ if and only if

$$f(x_0, y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x, y)\,,$$

i.e., the function is defined at (x_0, y_0), its limit exists as (x, y) approaches (x_0, y_0), and the function's value there is equal to the value of the limit.

Continuity Throughout the Domain

A function is said to be *continuous throughout its domain*, or simply is called *continuous*, if it is continuous at every point (x_0, y_0) of its domain.

A function $f : D \to \mathbb{R}$ is continuous throughout D if and only if the pre-image of any open interval $(a, b) = \{t : a < t < b\} \subseteq \mathbb{R}$ is an open subset of the domain. In this context, an open set $E \subset \mathbb{R}^2$ is one for which around every point $p \in E$ there is some open disk centered at p contained fully in E, and an open subset of D is a set which can be made as the intersection of D with an open set in $\mathbb{R}^2$. For technical reasons, the empty set and the whole of the domain D are considered open subsets of the domain D.

Well Behaved Friends

Polynomials in two variables are continuous on all of $\mathbb{R}^2$. Recall a polynomial in two variables is an expression of the form

$$p(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}x^i y^j\,.$$

Rational functions are also continuous on their domains. Rational functions of two variables are just quotients of two variable polynomials $R(x,y) = p(x,y)/q(x,y)$. Observe that $\mathrm{Dom}\big(p(x,y)/q(x,y)\big) = \{(x,y)\in\mathbb{R}^2 : q(x,y)\neq 0\}$.

A removable discontinuity

If a function f has a discontinuity at a point (a,b), but $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals L, then the function

$$\tilde{f}(x,y) = \lim_{(u,v)\to(x,y)} f(u,v) = \begin{cases} f(x,y) & \text{if } (x,y)\in D \\ L & \text{if } (x,y)=(a,b) \end{cases}$$

is continuous at (a,b).
E.g., the function

$$R(x,y) = \lim_{(u,v)\to(x,y)} \frac{3uv^2}{u^2+v^2} = \begin{cases} \frac{3xy^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at $(0,0)$, and as it is elsewhere rational with only $(0,0)$ as a zero of its denominator, $R(x,y)$ is in fact continuous throughout $\mathbb{R}^2$.

Example

Where is $f(x, y) = \arctan\left(\frac{y}{x}\right)$ continuous?

Solution: Since arctan is continuous throughout its domain, this function is continuous provided the argument y/x is continuous. Since this argument is a rational function, it is well defined everywhere in $\mathbb{R}^2$ except for points (x, y) such that the denominator is zero.

Thus, we conclude that $\arctan\left(\dfrac{y}{x}\right)$ is continuous on $\mathbb{R}^2 - \{x = 0\}$, i.e., the whole plane except the y-axis.

Angular Surface

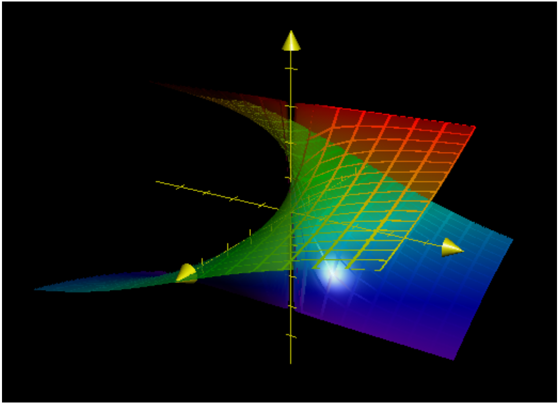

Figure: The graph of $z = \arctan(y/x)$.

Definition of a Limit in Several Variables

For a function $f : D \to \mathbb{R}$ of several variables, regard the input $(x_1, x_2, \ldots, x_n) \in D \subseteq \mathbb{R}^n$ as a vector $\mathbf{r} = \langle x_1, x_2, \ldots, x_n \rangle$.

Definition

Given a function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, we say that the limit of $f(\mathbf{r})$ as $\mathbf{r}$ approaches $\mathbf{a}$ exists and has value L if and only if for every real number $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(\mathbf{r}) - L| < \varepsilon$$

whenever

$$0 < \|\mathbf{r} - \mathbf{a}\| < \delta\,.$$

We then write

$$\lim_{\mathbf{r}\to\mathbf{a}} f(\mathbf{r}) = L\,.$$

Multivariate Continuity

Definition

A function of many variables $f : D \to \mathbb{R}$ is continuous at a point $\mathbf{r}_0 \in D \subseteq \mathbb{R}^n$ if and only if

$$f(\mathbf{r}_0) = \lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r})\,,$$

i.e., the function is defined at $\mathbf{r}_0$, its limit exists as $\mathbf{r}$ approaches $\mathbf{r}_0$, and the function's value there is equal to the value of the limit. The function is said to be continuous throughout its domain if it is continuous for every point $\mathbf{r}_0 \in D$.

Topological Definition

As before there is a topological reframing of the definition: a function $f : D \to \mathbb{R}$ is continuous throughout its domain if and only if the pre-images of open sets of $\mathbb{R}$ are open subsets of the domain (possibly empty, or all of the domain). The definition of openness involves being able to find an *open ball* around every point.

The open δ-balls appearing in the limit definition are neighborhoods of the approached point, lying in the pre-image of an ε-neighborhood. Thus, we can rephrase the limit definition as follows: $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r})$ exists and equals L if and only if for *any* small open neighborhood $\mathscr{U}$ of L, we can always find a suitable open neighborhood $\mathscr{N}$ of $\mathbf{r}_0$ for which $f(\mathscr{N}) \subseteq \mathscr{U}$.

Discontinuity along a surface

Example

Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2-1} = \dfrac{1}{\mathbf{r}\cdot\mathbf{r}-1}$, $\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$.
Where is the function discontinuous? Where is it continuous?

Solution: The function is rational, and so it is defined and continuous except where the denominator is 0. The denominator is zero when $\mathbf{r}\cdot\mathbf{r} = \|\mathbf{r}\|^2 = 1$, i.e., the denominator of f vanishes precisely along the unit sphere.

Thus $f(\mathbf{r})$ is discontinuous (and undefined) on the unit sphere $\mathbb{S}^2 = \{\mathbf{r}\in\mathbb{R}^3 : \|\mathbf{r}\| = 1\}$, and is continuous throughout $\mathbb{R}^3 - \mathbb{S}^2$.