# MATH 621 COMPLEX ANALYSIS, HOMEWORK 9

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Due date: Thursday April 21. It's a longer assignment, but you have ten days; don't procrastinate!

**I.** Stein-Shakarchi Chapter 8: 1,4,5,9,10,11,12,13,15,16

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**II.**

1. (a) Determine all FLTs for which $\infty$ is a fixed point.

(b) Show that if $\varphi$ is an FLT which fixes three distinct points on the Riemann sphere $\widetilde{\mathbb{C}}$, then $\varphi$ is the identity.

Hint: First do the case where $\infty$ is a fixed point.

(c) In class we proved that if $z_1, z_2, z_3$ are three distinct points in $\widetilde{\mathbb{C}}$ as are $w_1, w_2, w_3$, then there is an FLT $\varphi$ that satisfies $\varphi(z_i) = w_i$. Prove the uniqueness of such a $\varphi$.

2. (a) Find the FLT which maps $1, i, -1$ to $i, -1, 1$ in that order.

(b) Find the FLT which maps $0, 1, \infty$ to $-1, 0, 1$ in that order.

3. Verify that if $\varphi(z) = (az + b)/(cz + d)$ is an FLT, then $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ where $\varphi_1(z) = z + d/c$, $\varphi_2(z) = 1/z$, $\varphi_3(z) = (bc - ad)z/c^2$ and $\varphi_4(z) = z + a/c$.

Hint: if you're feeling algebraically lazy, you could use 1. Does it actually save you any time?

4. Let $\varphi(z) = (z + 1)/(z - 1)$. Determine the image of the line $\Re(z) = c$ where $c$ is a real number. Which $c$ are "exceptional?"

5. Let $f(z) = (z - 1)/(z + 1)$. What is the image under $f$ of:

(a) the real line?

(b) the circle with center 0 and radius 2?

(c) the circle with center 0 and radius 1?

(d) the imaginary axis?

(e) Show that $f$ maps $\{z : |z| > 1, |z - 1| < 2\}$ conformally to $\{z : 0 < \Re(z) < 1/2\}$.

6. Recall that the cross ratio of $z_0, z_1, z_2, z_3$ is defined to be

$$[z_0 : z_1 : z_2 : z_3] := \frac{(z_0 - z_1)(z_3 - z_2)}{(z_0 - z_2)(z_3 - z_1)}.$$

(a) Let $\varphi$ be an FLT and put $w_i = \varphi(z_i)$. Show that $[w_0 : w_1 : w_2 : w_3] = [z_0 : z_1 : z_2 : z_3]$. Hint: Use 3.

(b) Prove that four distinct points of $\mathbb{C}$ lie on a straight line or on a circle if and only if their cross ratio is a real number.

7. Write down (explicitly) an automorphism of the unit disc that takes 1/2 to 1/3.

8. Find a conformal map of the set $\Omega = \{z : |z-1| < \sqrt{2}, |z+1| < \sqrt{2}\}$ onto the (open) first quadrant.

9. Is it possible to find a conformal map from the unit disc to the exterior of the unit disc ($\{z : |z| > 1\}$)? What about the punctured unit disc (i.e. $\mathcal{D} - \{0\}$)?