MATH 621 COMPLEX ANALYSIS, HOMEWORK 7

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1. Suppose C_1, C_2 are simple closed curves in the complex plane. Let Ω_1 be the region **inside** C_1 and let Ω_2 be the region **outside** C_2. Assume that C_2 lies entirely in Ω_1, and let $\Omega = \Omega_1 \cap \Omega_2$ be the "quasi-annular" region between the two curves. Show that any holomorphic function f on Ω can be decomposed as a sum $f = f_1 + f_2$ where f_i is holomorphic on Ω_i for $i = 1, 2$. Moreover show that this decomposition is unique up to an additive constant, i.e. if g_i are holomorphic on Ω_i and add up to f, then there is a constant c such that $f_1 = g_1 + c$ and $f_2 = g_2 - c$.

2. Give the Laurent series expansions for the regions indicated:

a)

$$e^{1/(z-1)}, |z| > 1$$

b)

$$\frac{1}{(z-a)(z-b)}, 0 < |a| < |z| < |b|$$

c) same function as b) but for the region $|z| > b$.

Some Definitions about Behavior of functions at ∞.

How can we define a "neighborhood" of infinity? We use the intuitive idea that for z to be close to infinity, $1/z$ has to be close to 0. Thus, for a real number $R > 0$, we define a disc of radius R about ∞ to be $D_R(\infty) = \{z \in \mathbb{C} : |z| > R\}$. To be "close to ∞," therefore, means that you are outside the circle of radius R centered at 0 for a large R. Equivalently, for $z \in \mathbb{C}$, $z \in \mathcal{D}_R(\infty)$ if and only if $1/z \in D_r(0)$ where $r = 1/R$.

Here is a rough definition, to be made precise below. We say that a function f behaves a certain way at ∞ if the function $g(z) = f(1/z)$ behaves that way at $z = 0$.

So, let us suppose that f is defined on $D_R(\infty)$ for some $R > 0$. We say that f is holomorphic at ∞ if there exists a real number $r > 0$ such that $g(z) = f(1/z)$ extends to a holomorphic function on $D_r(0)$. In that case, we write $f(\infty) := g(0)$ which is of course $\lim_{z\to 0} g(z)$. We say f has a pole of order n at ∞ if $g(z) = f(1/z)$ extends to a holomorphic function on $D_r^0(0)$ with a pole of order n at 0. More generally, if $g(z) = f(1/z)$ is holomorphic on the punctured neigborhood $D_r^0(0)$, then the type of the singularity of f at ∞ is determined by the Laurent series expansion of $g(z)$ at $z = 0$.

For example, since $e^{1/z}$ has an essential singularity at $z = 0$, we say that function e^z has an essential signularity at $z = \infty$.

3. Determine the type of singularity each function have at the indicated point; show your work.

a)

$$\sin\left(\frac{1}{z-1}\right), z = 1$$

b)

$$\frac{1}{1-e^z}, z = 2\pi i$$

c)

$$\frac{1}{\sin(z)-\cos(z)}, z = \pi/4$$

The rest are at $z = \infty$

d)

$$\frac{z^2+4}{e^z}$$

e)

$$\cos(z) - \sin(z)$$

f)

$$\cot(z)$$

g)

$$e^{-1/z^2}$$

4. Here is a fact I am not asking you to prove, but you may do so for extra credit (it's not particularly hard):

(*) Suppose $z_1, \ldots, z_k \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_k$ is a "partition of unity" i.e. they are non-negative reals adding to 1, $\alpha_1 + \cdots + \alpha_n = 1$, then the number $\zeta = \sum_{\nu=1}^{k} \alpha_\nu z_\nu$ lies in the "convex hull" of the z_ν's (the smallest closed convex polygon containing these points). Moreover, if z is any complex number satisfying

$$\sum_{\nu=1}^{k} \frac{\alpha_\nu}{z - z_\nu} = 0,$$

the z lies in the convex hull of $z_1, \ldots, z_k$.

(a) What I am asking you to prove using (*) is:

If $P(z)$ is a non-constant complex polynomial with roots $z_1, \ldots, z_k$, then the roots of $P'(z)$ are contained in the convex hull of $z_1, \ldots, z_k$. This is sometimes called the Gauss-Lucas Theorem.

(b) Show that if the roots of a complex polynomial $P(z)$ are real, then so are the roots of all of its derivatives.

5. (a) Suppose f has a zero of order n at z_0. What is the residue of

$$z\frac{f'(z)}{f(z)} \text{ or more generally } \varphi(z)\frac{f'(z)}{f(z)}$$

at $z = z_0$. Here $\varphi(z)$ is any function which is holomorphic at z_0.

(b) State (with the requisite assumptions on f, C and φ) and prove a generalization of the Argument Principle for the integrals

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz, \qquad \frac{1}{2\pi i}\int_C \varphi(z)\frac{f'(z)}{f(z)}.$$

6. Suppose that f is holomorphic on an open set Ω containing the closed unit disc such that $|f(z)| < 1$ whenever $|z| = 1$. Show that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside the unit disc.

7. How many zeros does the function $f(z) = 3z^{621} - e^z$ have inside the unit disc (counting multiplicities)? Do they all have multiplicity 1?

8. (a) Suppose f is an entire function such that $|f(z)| \leq 621|\cos(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = k\cos(z)$ for some constant k of absolute value at most 621.

Hint: Riemann to the rescue!

(b) Generalize (a).

(c) Explain how you could try to nail down the exact location of a simple zero of a holomorphic function using integration if you had a good estimate on where the zero is. Do you think this would be as practical as say Newton's method for finding real roots of real polynomials?