MATH 621 COMPLEX ANALYSIS, HOMEWORK 3

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Part I. Stein-Shakarchi Chapter 1, pp. 24ff: 24,25,26.

Stein-Shakarchi Chapter 2, pp. 64ff: 6.

Part II.

1. Compute the following integrals directly (by parametrizing the curve). (As usual, unless otherwise stated, curves are taken with the counterclockwise orientation and traversed once).

(a) $\int_\gamma \overline{z} dz$ where γ is the square with vertices at the points $\pm 1 \pm i$ (with the four choices of sign combinations).

(b) $\int_\gamma (z^2 + 2z + 3) dz$ where γ is the line segment joining $-1 - i$ to $1 + i$.

(c) $\int_\gamma 1/(1+z) dz$ where γ is the circle of radius 3 centered at 0.

(d) Compute $\int_\gamma x dz$ where $z = x + iy$, i.e. $x = \mathrm{Re}(z)$ and γ is the circle $|z| = r$ traversed once counterclockwise, in two ways, first by parametrization, and then by the trick $x = (z + \overline{z})/2 = (z + r^2/z)/2$.

2. Let $\gamma : [a, b] \to \mathbb{C}$ be a continuously differentiable curve. Suppose $\{f_n\}$ is a sequence of $\mathbb{C}$-valued functions, defined and continuous on an open set $\Omega \subset \mathbb{C}$ containing $\gamma([a, b])$. Prove that if $\{f_n\}$ converges uniformly on $\gamma([a, b])$ to a function f, then

$$\lim_{n\to\infty} \int_\gamma f_n(z) dz = \int_\gamma f(z) dz.$$

3. (a) Describe in words and in a picture the curve γ parametrized by $\gamma(t) = a\cos(t) + bi\sin(t)$, $t \in [0, 2\pi]$. Compute

$$\int_\gamma \frac{dz}{z}.$$

(b) Compute

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}.$$

The due date is Thursday Feb. 10.

4. For a continuous function f on an open set Ω containing a smooth curve γ, define

$$\int_\gamma f\overline{dz} = \overline{\int_\gamma \overline{f}dz},$$

i.e. integrating with respect to $\overline{dz}$ is defined to be the complex conjugate of the integral with respect to dz of the complex conjugate of f. We then define

$$2\int_\gamma fdx = \int_\gamma fdz + \int_\gamma f\overline{dz}, \qquad 2i\int_\gamma fdy = \int_\gamma fdz - \int_\gamma f\overline{dz}, \qquad .$$

Show that for $f = u + iv$ with u, v real-valued as usual, we have

$$\int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx) = \int_\gamma f(z)dz.$$