MATH 621 COMPLEX ANALYSIS, SAMPLE FINAL EXAM

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You may use any theorem we proved in class, or from the book, that you can state precisely. Unless otherwise stated, $\log z$ denotes the principal branch (i.e. standard) logarithm and $z^{1/2} = e^{\log(z)/2}$ is the principal (i.e. standard) branch of the square root of z.

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1. Let n be a positive integer, and $0 < \alpha < \pi$. Prove that

$$\frac{1}{2\pi i}\int_C \frac{z}{1 - 2z\cos\alpha + z^2}dz = 1.$$

where C is the circle $|z| = 2$ traversed once counterclockwise.

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2. For an integer m, define

$$F(m) = \frac{1}{2\pi i}\int_C \frac{z^{m-1}}{z^7 - 1}dz.$$

Compute $F(14)$ and $F(13)$.

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3. (a) State and prove the Schwarz Lemma.

(b) Let f be holomorphic on the right half plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$ and suppose $|f| \leq 1$. Suppose also that $f(1) = 0$. Determine, with proof, the largest possible value of $|f'(1)|$.

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4. (a) State, but do not prove, Morera's Theorem.

(b) Prove that if f is holomorphic on $\{z \in \mathbb{C} \mid \mathrm{Im}(z) \neq 0\}$ and has a continuous extension to all of $\mathbb{C}$, then f extends to an entire function.

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5. Prove that an entire function f is a polynomial if and only if it has $\lim_{z\to\infty} |f(z)| = \infty$.

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6. How many zeros (counting multiplicities) does the function

$$f(z) = 5z^{10} - e^z$$

have inside the unit disc? Are these all simple zeros of f? Explain.

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7. Define the open disc Ω and open right-half plane H has follows:

$$\Omega = \{z \in \mathbb{C} : |z - 1| < 1\}, \qquad H = \{z \in \mathbb{C} : \Re(z) > 1\}.$$

(a) Write down explicitly a conformal map (holomorphic bijection) f from Ω onto H with the additional property that $f(2) = \infty$.

(b) Write down explicitly a harmonic function $u(x, y)$ on the disc

$$\Omega = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1\}$$

(now thought of as a subset of $\mathbb{R}^2$) which takes the constant value 1 on the boundary of Ω.

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8. Determine the singularities of $f(z) = \cot(z)/z$ inside $\mathcal{D}$, For each such singularity, compute the principal part of the Laurent expansion of f there.
