MATH 621 COMPLEX ANALYSIS, SAMPLE MIDTERM EXAM/HOMEWORK 6

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The distribution of midterm questions will be smiliar to these in terms of difficulty (note that this varies quite a bit), but there will only be about 7 or 8 questions and you will choose 5 to be graded for 20 points each. Note that there are lots of important concepts/theorems we learned about that do not make an appearance on this assignment, but which of course could appear on the midterm. My aim here is to list the types and difficulty level of the problems so you can get a sense of what to expect. For the exam, you may use any theorem we proved in class, or from the book, that you can state precisely, except of course if I am asking you to prove a theorem from class or from the book, you can't simply invoke the theorem itself.

1. Prove or Disprove:

(a) Whenever $f(z)$ is an entire function, the function $g(z) = f(z) + \overline{f(z)}$ is entire. [If true, give a proof, if false, give an entire function f and **prove** that the corresponding g is not entire.]

(b) Whenever $f(z)$ is an entire function, the function $g(z) = \overline{f(\overline{z})}$ is entire.

2. If $\sum a_n z^n$ has radius of convergence R, what (if anything) can you say about the radius of convergence of (a) $\sum a_n^2 z^n$, (b) $\sum a_n z^{2n}$?

3. State and prove Cauchy's Formula for the nth derivative of $f^{(n)}(z)$ of a function f holomorphic in an open set Ω (where z is an arbitrary point in Ω).

4. (a) State the Residue Theorem.

(b) List the poles of the function $f(z) = (9 + z^2)^{-1}$ together with their corresponding residues.

(c) Calculate the following integral by the method of contour integration (using the residue theorem):

$$\int_{-\infty}^{\infty} \frac{dx}{9 + x^2}.$$

This assignment, which also serves as a review for the midterm, will be due at the start of class on Thursday March 10.

5. Let C be the circle of radius 2 centered at the origin (going around once counterclockwise, as usual). Calculate the following integral without parametrizing the curve:

$$\int_C \frac{2z-1}{z^2-z} dz.$$

6. (a) Suppose f is holomorphic on some non-empty open disc D containing z_0. Show that there exists an integer N such that $|f^{(n)}(z_0)| < n!n^n$ for all $n \geq N$.

(b) For $f(z) = e^{z^2}$, and $z_0 = 0$, what is the least value of such an integer N?

7. Suppose f is holomorphic in an open connected set Ω. Suppose there exists $z_0 \in \Omega$ and a positive integer N such that $f^{(n)}(z_0) = 0$ for all $n \geq N$. Prove that f is a polynomial of degree at most N, i.e. there exist constants $a_0, a_1, \cdots, a_N \in \mathbb{C}$ such that for all $z \in \Omega$, we have $f(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1} + a_N z^N$.

8. Suppose Ω is a non-empty open subset of the complex plane and f is holomorphic in Ω. Let $\overline{D} = \overline{D_r(z_0)} \subset \Omega$ be a *closed* disc contained in Ω. Show that $\overline{D}$ contains only finitely many zeros of f.

9. (a) Prove the **Mean Value Property of holomorphic functions**: If f is holomorphic in an open disc of radius R, $D_R(z_0)$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \text{ for any } 0 < r < R.$$

(b) Take real parts in (a) and use Exercise 12(a) of Chapter 2 to conclude the mean value property for harmonic functions. [On the exam, if I give you this problem, I would spell out what 12(a) says].

10. SS Ch. 2, p. 103: 2, 3, 6, 8