MATH 621 COMPLEX ANALYSIS, NOTES ON A THEOREM OF WEIERSTRASS

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Here is some clarification for the proof of Theorem 5.3 from Chapter 2 of Stein-Shekarchi.

Theorem 0.1. *If $f_1, f_2, f_3, \ldots$ is a sequence of functions holomorphic on an open set $\Omega \subseteq \mathbb{C}$ which converge to a function f on Ω and if this convergence is uniform on all compact subsets of Ω, then*

(1) the function f is holomorphic on Ω, and

(2) the sequence of derivatives f_n' converges uniformly to f' on all compact subsets of Ω.

Proof of (2). . This theorem was apparently first given by Weierstrass, not Hurwitz, as I mistakenly said in class. The proof of (1) using Morera from the book or lecture is straightforward. Here is a fussy elaboration of the proof given in Stein-Shekarchi. Let Γ be a compact subset of Ω. Since Γ is compact and Ω is open, there exists a chain of sets

$$\Gamma \subsetneq \Omega' \subsetneq \Gamma' \subsetneq \Omega,$$

with Ω' open and Γ' compact. (Easy verification left to the reader). Now there exists $\delta > 0$ such that

$$\Gamma' \subset \Omega'_\delta, \text{ where } \Omega'_\delta := \{z \in \Omega' \mid \overline{D_\delta(z)} \subset \Omega'\}.$$

We will now show that $f_n' \to f'$ uniformly on Ω'_δ, which is all we need do since $\Gamma \subset \Omega'_\delta$.

Claim. If $F(z)$ is holomorphic on Ω (or even just on Ω'), then

$$\sup_{z \in \Omega'_\delta} |F(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega'} |F(\zeta)|.$$

Note that although Ω'_δ and Ω' are not compact, they are both contained in the compact set Γ', hence the two sup's above are well-defined real numbers.

To prove the claim, we apply Cauchy's formula just as in the book and lecture: For all $z \in \Omega'_\delta$, $\overline{D_\delta(z)} \subset \Omega'$, so $C_\delta(z) \subset \Omega'$, giving us

$$\begin{aligned} |F'(z)| &= \left| \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{F(\zeta)}{\zeta - z} dz \right| \\ &\leq \frac{1}{2\pi} \sup_{z \in \Omega'} |F(z)| \frac{2\pi\delta}{\delta^2} \\ &\leq \frac{1}{\delta} \sup_{z \in \Omega'} |F(z)|, \end{aligned}$$

verifying the claim.

Now suppose $\epsilon > 0$ is given. Since $f_n \to f$ uniformly on Γ' [1] there exists an integer N such that

$$|f_n(z) - f(z)| < \delta\epsilon/2 \text{ for all } z \in \Gamma' \text{ and all } n \geq N.$$

Then, for $n \geq N$ and $z \in \Omega'_\delta$, we apply the claim to $F(z) := f_n(z) - f(z)$ and find that

$$\begin{aligned} |F'(z)| = |f'_n(z) - f'(z)| &\leq \frac{1}{\delta} \sup_{z \in \Omega'} |f_n(z) - f(z)| \\ &\leq \frac{1}{\delta} \sup_{z \in \Gamma'} |f_n(z) - f(z)| \\ &\leq \frac{1}{\delta}\frac{\delta\epsilon}{2} < \epsilon. \end{aligned}$$

We have shown that $f'_n \to f'$ uniformly on Ω'_δ. □

The most common application of the above theorem is contained in:

Corollary 0.2. *If the functions $e_0(z), e_1(z), e_2(z), \ldots$ are holomorphic on the open set Ω and the series $\sum_{n\geq 0} e_n(z)$ converges to a function f on Ω and it does so uniformly on compact subsets of Ω, then f is holomorphic on Ω and $f'(z) = \sum_{n\geq 0} e'_n(z)$ for $z \in \Omega$.*

Later, after we prove the Maximum Modulus and Argument Principles, we'll be able to obtain the following theorems.

Theorem 0.3. *If $e_n(z)$ are holomorphic on $|z| < R$ and $f(z) = \sum_n e_n(z)$ converges uniformly on the* **circles** *$C_r(0)$ for all $0 < r < R$, then f is holomorphic on $|z| < R$.*

The proof follows from the Corollary once we invoke the maximum modulus principle.

Theorem 0.4 (Hurwitz). *If f_n are holomorphic and non-vanishing on Ω and converge uniformly to f on compact subsets of Ω, then f is either identically zero or non-vanishing on Ω.*

We'll be able to deduce this Hurwitz from Weierstrass after we prove the Argument Principle.

Alternative Hint for Exercise 2 from Chapter 2 of Stein-Shekarchi.

To evaluate $I = \int_0^\infty \sin(x)/x dx$, we can note that

$$I = \frac{1}{2i} \lim_{\epsilon \to 0} \left(\int_\infty^{-\epsilon} \frac{e^{ix}}{x} dx + \int_\epsilon^\infty \frac{e^{ix}}{x} dx \right).$$

Now we integrate e^{iz}/z over the indented semi-circle, consisting of a radius R semi-circle in the upper half plane and the real axis between $-R$ and R with the exception of a radius ϵ circular "bump" around the origin. The integral over the radius R semi-cirlce goes to 0 as R goes to 0, as you can bound its modulus from above by

$$\int_0^\pi e^{-R\sin(\theta)} d\theta = 2 \int_0^{\pi/2} e^{-R\sin\theta} d\theta.$$

[1] Note that we are using the uniform convergence of f_n on the *larger set* Γ' to deduce the uniform convergence of f'_n on the proper subset Γ.

For $0 \leq \theta \leq \pi/2$, there exists a constant $C > 0$ such that $\sin(\theta) \geq C\theta$, so that the integral above is bounded above by

$$2\int_0^{\pi/2} e^{-RC\theta} d\theta,$$

which you can evaluate explicitly. For the integral on the circle of radius ϵ, $e^{iz}/z = 1/z + G(z)$ where $G(z)$ is holomorphic at 0. Since $G(z)$ is bounded by a constant, say B, near $z = 0$, we get for all sufficiently small $\epsilon > 0$,

$$\left| \int_{C_\epsilon} e^{iz}/z dz - \int_{C_\epsilon dz/z} dz/z \right| < B\pi\epsilon$$

so upon letting $\epsilon \to 0$, we get what we want.