UMASS AMHERST MATH 455 F. HAJIR

EXAM 1

Unless otherwise stated, G denotes a SIMPLE graph with vertex set V of size n and edge set E of size m. Write your answers in the Blue Book.

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1. Definitions (3 points each)

a. The graph G is connected means that

b. The connectivity $\kappa(G)$ is defined to be

c. The eccentricity of a vertex v is defined by $\mathrm{ecc}(v) =$

d. A cycle in a graph G is

e. The incidence matrix M of G having vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$ is defined by

f. A graph isomorphism from G to H is

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2. True or False (3 points each)

You do not need to justify your answer here, simply write out the word "TRUE" or the word "FALSE" to indicate the truth value of the statement.

a. If two graphs are isomorphic, then their adjacency matrices have the same determinant.

b. Every closed walk of even length contains a cycle of even length.

c. If the periphery and center of a graph G coincide, then G must be complete.

d. If G has no bridges, then G has exactly one cycle.

e. If every vertex of a connected graph G lies on at least one cycle, then G is 2-connected.

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3. Short Answer (7 points each)

a. Draw a picture of $G = K_{3,4}$, labeling the top vertices 1,2,3 and the bottom vertices 4,5,6,7. Now write down the adjacency matrix of G with respect to this ordering. What feature of this matrix lets you "see" that the graph is bipartite?

b. Suppose a graph G has degree sequence $(4, 3, 3, 3, 3, 3, 3, 2)$. Compute the number of edges of the line graph $L(G)$ of G.

c. Consider connected graphs with six vertices and five edges. Draw six such graphs which are all essentially different, meaning there is no isomorphism between any two of them. *Hint: if you only have five edges to work with, can you afford to construct any cycles?!*

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4. Proofs (15 points each)

a. Prove that in every 2-connected graph, there is at least one cycle.

b. Suppose G has vertices $v_1, \ldots, v_n$ with associated adjacency matrix A.
i) Show that for any $j \in \{1, 2, \ldots, n\}$, the number of triangles that contain v_j is one-half the jj entry of A^3, i.e. $\frac{1}{2}(A^3)_{jj}$.
ii) Show that the number of triangles in G is $\mathrm{Tr}(A^3)/6$, where $\mathrm{Tr}(M)$ means the sum of the diagonal entries of M.

c. Suppose $\delta(G) \geq (n-1)/2$, where G has n vertices. Show that $\mathrm{diam}(G) \leq 2$.

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5. Extra Credit (5 points for a and 10 points for b)

a. Recall that for a graph G, its complement $\overline{G}$ has the same vertex set, with uv an edge in $\overline{G}$ if and only if uv is not an edge in G.
(i) Prove that if G is connected and $\mathrm{diam}(G) \geq 3$, then $\overline{G}$ is connected.
(ii) Prove that if $\mathrm{diam}(G) \geq 3$ then $\mathrm{diam}(\overline{G}) \leq 3$.
(iii) Prove that if G is regular and $\mathrm{diam}(G) = 3$, then $\mathrm{diam}(\overline{G}) = 2$.

b. Let $S = \{x_1, \ldots, x_n\}$ be a set of points in the plane such that the distance between any two distinct points is at least 1. Show that there are at most $3n$ pairs of points at distance exactly 1.