UMASS AMHERST MATH 455, F. HAJIR

SOME PROBLEMS FROM HOMEWORKS 1 AND 2

Here I post some rather extended hints to selected problems from HW 1 and 2 as an aide to you. If you find errors and or points that are unclear, feel free to email me.

1.1.2:2. If G has one vertex and no edges, then its degree sequence is just 0: this case is (tacitly) excluded from the statement of the problem. If a connected component of G has a repetition in its degree sequence, then so does G; also if the graph has more than one degree 0 vertex, then the degree sequence has a repetition. Thus, it suffices to consider the case where G is connected and has more than one vertex. If the vertices of G are $v_1, \ldots, v_n$, then for each j, $\deg(v_j)$ is an element of $\{1, 2, 3, \ldots, n-1\}$. This is because none of the vertices is isolated and of course any v_j can be adjacent to at most $n-1$ other vertices. But now we have n vertices (pigeons) being placed in $n-1$ boxes, so by the Pigeonhole principle, there must be a repetition.

1.1.2:3. (a) There are no paths of length 5 or more since there are only 5 vertices. It's helpful to organize the counting by length of path. There is just one path of length 0, namely c. There are 4 paths of length 1, namely xc where x is either a, b, d, or e. Choosing a path of length 2 means choosing x, y a pair of distince vertices not equal to c. There are $4 \cdot 3$ such choices. Note that we are in K_5 here so every potential edge is an actual edge, making life easier. The total number of paths ending in c is $1 + 4 + 12 + 24 + 24 = 65$.

1.1.2:5. (a) Suppose $\delta(G) \geq k$. Prove that G has a path of length at least k.

Let v_0 be a vertex of G. Since $\deg(v_0) \geq k$, we can choose $v_1 \in N(v_0)$. There are at least $k-1$ ways to choose v_2 in $N(v_1)$ but not equal to v_0. Continuing in this manner, if v_j is chosen (for any $0 \leq j \leq k-1$), there are at least $k-j$ ways to choose v_{j+1} in $N(v_j)$ but not equal to $v_0, v_1, \ldots, v_j$. Thus, we have found a path $v_0 v_1 \ldots v_k$, of length k.

5(b) Suppose $\delta(G) \geq k \geq 2$.

Idea: To make a long *cycle*, we make as long a *path* as possible, and we try to close it up by hoping that the final vertex of the long walk is adjacent to one of the vertices in the early portion of the walk.

Let P be a path in G of longest length. Say P has length ℓ and is $v_0 v_1 \ldots v_\ell$. Let $W = \{v_0, v_1, \ldots, v_\ell\}$. Note that W has size $\ell + 1$. Consider the neighborhood of v_ℓ.

We claim that $N(v_\ell) \subseteq W$. If this were false, then there would exists a vertex u in G but not in W such that $v_\ell u$ is an edge. This would mean that $v_0 v_1 \ldots v_\ell u$ would be a path of length $\ell + 1$, contradicting the fact that P was chosen to be a path of maximal length.

So, every neighbor of v_ℓ occurs in the list $v_0, v_1, \ldots, v_{\ell-1}$. Let $r \geq 0$ be the smallest integer such that v_r is a neighbor of v_ℓ. Thus, **all** the neighbors of v_ℓ occur in the list

$v_r, v_{r+1}, v_{r+2}, \ldots, v_{\ell-1}$. Since $\deg(v_\ell) \geq k$, that means the path $v_r v_{r+1} v_{r+2} \ldots v_\ell$ has length at least k. Therefore the cycle $v_r v_{r+1} v_{r+2} \ldots v_\ell v_r$ has length at least $k+1$.

1.1.2: 6. Let $P(k)$ be the statment: If W is a closed walk of odd length $2k+1$, then W contains an odd cycle. We want to prove $P(k)$ is true for all $k \geq 1$.

We will use strong (or complete) induction. Accordingly, we will check the base case $P(1)$ and then show that for all $r \geq 1$, if $P(1), P(2), \ldots, P(r)$ all hold, then $P(r+1)$ also holds.

To prove $P(1)$, we have to show a walk of length 3 has an odd cycle. Say $W : wxyw$ is a closed walk of length 3. Since there are no loops allowed, we must have $w \neq x, x \neq y, y \neq w$. In other words, x, y, w are three distinct vertices, so W itself is an odd cycle.

Now suppose $P(1), P(2), \ldots, P(r)$ all hold, for some $r \geq 1$. Take $W : w_1 w_2 \ldots w_{2r+3} w_1$ to be a closed walk of length $2(r+1)+1 = 2r+3$. If W is a cycle itself, there is nothing further to do, so we may assume the list $w_1, w_2, \ldots, w_{2r+3}$ has a repetition somewhere. Thus, for some integer $i \geq 1$ and $k \geq 2$, we have $w_i = w_{i+k}$. Note that k is not 1 because there are no loops. We then have that $w_i w_{i+1} \ldots w_{i+k} w_i$ is a close walk withtin W, of length k. Also, $w_{i+k+1} w_{i+k+2} \ldots w_{2r+2} w_1 w_2 \ldots w_{i-1}$ is a closed walk within W of length $\ell = 2r+3-k$. Since $2r+3$ is odd and equals $\ell + k$, we must have either ℓ or k is odd. Thus, we have shown that W contains a closed odd *walk* W' of length less than $2r+3$. Since $P(1), P(2), \ldots, P(r)$ are hold by the induction hypothesis, we have shown that W' contains an odd cycle. Since W' is contained in W, this odd cycle is also contained in W. We are now done by complete induction.

1.1.2:10. Leaving aside the trivial graph with one vertex, we suppose $n \geq 2$. The graph G can't have any vertices of degree 0, else it would not be connected. Let r be the number of degree one vertices in G. Thus, there are $n-r$ vertices of degree 2 or higher. Adding up all the degrees and applying the first theorem of graph theory, we have

$$2|E| = \sum_{v \in V} \deg(v) \geq r + 2(n-r).$$

In other words, $|E| > n - r/2$. We proceed by contradiction. Suppose $|E| < n-1$¿ Then

$$n - r/2 \leq |E| < n-1 \Rightarrow 2 < r.$$

In other words, there are at least 3 vertices of degree 1. This means at least one of the connect components of G is an isolated edge, which implies that G is not connected, a contradiction. Thus, we must have $|E| \geq n-1$.

1.1.2: 15 (a) $\kappa(G) \leq \delta(G)$.

All we have to do is to show that we can disconnect the graph with $\delta(G)$ vertex deletions. Let v be a vertex of minimal degree, i.e. $\deg(v) = \delta(G)$. If we remove the $\deg(v)$ vertices in $N(v)$, then v becomes isolated from the rest of the graph, making the graph disconnected. This argument works as long as the graph is not composed only of v and its neighbors! So what do we do in the latter case? Well, in that case, since v has minimal degree, the $\deg(v)$ vertices in $N(v)$ must all have degree at least $\deg(v)$. But there is a total of $\deg(v)+1$ vertices all together, so every vertx must be connected to every other vertex, i.e. G is a complete graph. For the complete graph, κ is by definition the same as δ, so we are done.

(b) If $\delta(G) = n-1$, then G is complete, so by definition, κ and δ are both $n-1$ in that case.

Now suppose $\delta(G) = n-2$. We already know from (a) that $\kappa(G) \le n-2$, so we have to show that we *cannot* disconnect G with fewer than $n-2$ vertex deletions.

Let $P(n)$ be the statement: If a graph G on n vertices satisfies $\delta(G) \le n-1$, then $\kappa(G) = \delta(G)$.

If $n = 3$, it is easy to see that if $\delta(G) = 1$, then the graph is two sides of a triangle, for which $\kappa(G) = 1$, so that establishes the base case $P(3)$.

Now suppose $P(3), P(4), \ldots, P(r-1)$ are all true for some $r \ge 4$. We want to show these imply $P(r)$. If G is a graph on r vertices with minimal degree $r-1$, then G is complete, so its κ and δ coincide. If the minimal degree is $r-2$, suppose we remove k vertices, where $1 \le k \le r-3$ to get a graph H. In the graph H, each vertex has been deprived of at most k edges, so H has $r-k$ vertices and $\delta(H)$ is either $r-k-1$ or $r-k-2$. Another way to say this is: In the graph G, every vertex was adjacent either to all other vertices, or to all other vertices except one. After deleting k vertices, it is still the case that every vertex is adjacent either to all remaining vertices, or all remaining vertices except one. Either way you say it, H is a graph on j vertices with $3 \le j \le r-1$ such that $\delta(H) \ge |H| - 2$. Thus, by the induction hypothesis, H is connected. Thus, we cannot disconnect G by fewer than $r-2$ vertex deletions, proving $P(r)$. By strong induction, we are done.

1.1.2:16

(a) Suppose v is any vertex. Let $X = N(v) \cup \{v\}$ and $W = V - X$. We have

$$|X| \ge 1 + (n-1)/2 \text{ and } |W| = |V| - |X| \le n - (n+1)/2 \le (n-1)/2.$$

If W is empty, then G is connected because there is a path of length at most 2 from any vertex to any other (by passing through v).

If W is not empty, pick $w \in W$. Since $\deg(w) \ge (n-1)/2$, and $W - \{w\}$ has size at most $(n-1)/2 - 1$, there must exist $x \in X$ such that xw is an edge. We have shown that for every w not adjacent to v, there exists $x \in N(v) \cap N(w)$. Thus we have shown for every pair of vertices v, w, either vw is an edge, or there is a walk vxw from v to w. Thus, G is connected.

(b) If $n = 6$, the disjoint union of two triangles show that one can have disconnected graphs with $\delta(G) = (n-2)/2$. More generally, the disjoint union of two complete graphs on $(k+1)$ vertices has $n = 2k+2$ edges and minimal degree $k = (n-2)/2$.

1.2.3:2. The degree of the top vertices is r_2 and the degree of the bottom vertices is r_1, so for the graph to be regular, we must have all vertices of same degree, so $r_1 = r_2$.

1.1.3:6 Recall that K_r has $\binom{r}{2} = r(r-1)/2$ edges. Clearly $K_{r_1,\ldots,r_k}$ has $n = r_1 + r_2 + \cdots + r_k$ vertices. To count the number of edges, we can note that if we put in every single edge between the vertices, that would amount to $\binom{n}{2}$ vertices. Now we have to erase a bunch. For each grouping, say the first r_1 vertices, we have to remove all the edges that have both endpoints in this grouping: that amounts to $\binom{r_1}{2}$ erasures. Continuing in this manner, we find the number of edges of our graph is

$$\binom{r_1 + r_2 + \cdots + r_k}{2} - \sum_{j=1}^{k} \binom{r_j}{2}.$$

1.1.3: 7(c) The number of edges of G is $m = (r_1 + \cdots + r_n)/2$ by the First Theorem, so that's the number of vertices of $L(G)$.

Each vertex v_i of G is a meeting place for r_i edges of G. A pair of such edges (call them e and e') meeting at v_i (they form a "wedge" with summit v_i) is responsible for creating an edge in $L(G)$ from the vertex that correponds to e to the vertex that corresponds to e'. Thus, the number of edges "created" by the summit v_i is the number of pairs of edges meeting at v_i. Since there are r_i such edges, the number of pairs of such edges is $\binom{r_i}{2}$. Thus, the total number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \binom{r_i}{2}.$$

1.2.1:4 Since x is peripheral, $\operatorname{ecc} x = \operatorname{diam}(G)$. Since $d(x, y) = \operatorname{diam}(G)$, $\operatorname{ecc}(y) \geq \operatorname{diam}(G)$. But the diameter is the maximal eccentricity, so $\operatorname{ecc}(y) \leq \operatorname{diam}(G)$. Thus, $\operatorname{ecc}(y) = \operatorname{diam}(G)$, so y is peripheral.

1.2.1: 6 We note that if the eccentricity of every vertex is the same, then the graph is self-centered. For a bipartite graph, the distance from v to w is 1 if v, w belong to the same partite, and 2 otherwise. So the eccentricity of every vertex is 2. For a cylce graph on n vertices, the eccentricity of every vertex is $\lfloor n/2 \rfloor$. For the complete graph, the eccentricity of every vertex is 1.

1.2.2: 6 The ij entry of A^3 is the number of length 3 walks from v_i to v_j. If $i = j$, we've seen that length 3 closed walks are cycles so we just count cycles of length 3 ending at each vertex. There are 3 triangles but they can be traversed in two directions, so it's 6 no matter which vertex you pick. If $i \neq j$, the counting is still easy: by symmetry we can just do the $i = 1, j = 2$ case to illustrate: 1212,1232,1242,1312,1342,1412,1432. So there are 7 paths. Upshot: the diagonal entries of A^3 are 6 and the others are 7.

1.2.2: 3 Walks of length 2 from v_j to v_j are $v_j v v_j$ where v is in the neighbordhood of v_j, so there are $\deg(v_j)$ such walks.

1.2.2: 5 The only paths from v_1 to v_5 have length 4 and 6 respectively. For any walk from v_1 to v_5, once the repeated edges are eliminated, what remains must be a path of length either 4 or 6. Since deletion of repeated edges come in pairs, the length of the original walk is an even number plus 4 or 6 hence is even. Thus, no odd walk starting at v_1 ends at v_5 so the 1,5 entry of A^{2k+1} is always 0.

For another proof, note that our graph is bipartite (the partition coming from the parity of the index of our vertices). Note that a step of length 1 in any walk always changes the parity of the index of the vertex. In other words, since our graph is bipartite, walks of odd length always go from one partite to the other partite, thus the number of walks of odd length from a vertex in one partite to another vertex in the same partite is always 0.

1.1.2: 7 We compute S_1, S_2 etc and at each step, say k, look for a zero-less row. If none exist, we compute S_{k+1} etc. Say the smallest k for which we find a zeroless row in S_k is r. Then r is the radius of G and the center of G are the ones corresponding to the zeroless rows.