UMASS AMHERST MATH 411 SECTION 2, FALL 2009, F. HAJIR

HOMEWORK 4: DUE OCT. 29 2009

1. If G is a group and H is a subgroup of it, then a right coset of H in G is a subset of the form $Hg = \{hg | h \in H\}$ for some $g \in G$. In other words, this is just like left cosets but the element is multiplied on the right. We denote by $H \setminus G$ the set of right cosets of H (recall that G/H is the set of left cosets of H). Give an example of a group G and a subgroup H and an element $g \in G$ such that $Hg \neq gH$. Hint: don't pick a commutative group!!

2. In the following examples, for each group G, give a list of the different left cosets of H as a subset of G, then a list of the different right cosets of H. For example, if $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$, then $G/H = \{\{2k | k \in \mathbb{Z}\}, \{2k+1 | k \in \mathbb{Z}\}\}$ has two elements, the set of even numbers and the set of odd numbers. The left and right cosets coincide so $H \setminus G$ is the same as G/H.

a. $G = \mathbb{Z}$, $H = 6\mathbb{Z}$.

b. $G = S_3$, $H = \{e, \sigma, \sigma^2\}$ where $\sigma = (123)$ is the 3-cycle sending 1 to 2, 2 to 3, 3 to 1.

c. $G = S_3$, H is the set of permutations that fix the number 2.

d. $G = GL_2(\mathbb{Z})$; this is the group of 2×2 matrices with integer entries which have determinant either $+1$ or -1. Verify that G is a group and that $H = SL_2(\mathbb{Z})$ is a subgroup of G. Now list G/H.

3. **Conjugation.** If G is a group, and $g \in G$, then we define a map $c_g : G \to G$ by $c_g(x) = gxg^{-1}$. Show that c_g is always an isomorphism. What is the relationship between c_g and $c_{g^{-1}}$? Is it correct to say that the kernel of c_g is the set of elements of G that commute with g?

4. Suppose G is a group, and H, K are two subgroups of G. Show that their intersection $H \cap K$ is also a subgroup of G. Show that if H and K are normal subgroups of G, then $H \cap K$ is a normal subgroup of G.

5. A subgroup H of a group G is called a *normal subgroup* of G if for all $g \in G$ and all $h \in H$, $ghg^{-1} \in H$. Show that H is a normal subgroup of G if and only if every right coset of H coincides with the corresponding left coset of H, i.e. for all $g \in G$, $gH = Hg$.

6. Suppose $f : G_1 \to G_2$ is a group homomorphism. Prove that $\ker(f)$ is a normal subgroup of G_1. [We already know it's a subgroup; now show that it's normal].

7. Suppose $f : G_1 \to G_2$ is a group homomorphism. Suppose $x \in G_1$ has order k and $f(x)$ has order ℓ in G_2. Show that $\ell | k$, i.e. k is a multiple of ℓ.

8. Suppose $f : G_1 \to G_2$ is a group homomorphism, where G_1 and G_2 are two finite groups satisfying $gcd(|G_1|, |G_2|) = 1$. Show that $f(x) = e_2$ for all $x \in G_1$, i.e. f is the trivial homomorphism. Hint: use the previous problem.

HERE'S AN EXTRA PROBLEM FOR THOSE WHO WANT TO DO IT BUT IT'S NOT PART OF THE REGULAR ASSIGNEMENT: Suppose H, K are subgroups of G and that $H \subseteq K$. Show that H is a subgroup of K. Is it true that if K is a normal subgroup of G and H is a normal subgroup of K, then H is a normal subgroup of G? Either give a proof, or find a counterexample. Hint: take $G = S_3$ and list **all** its subgroups. Now play with different $H \subset K \subset G$ to get a feeling for the problem.