UMASS AMHERST MATH 411 SECTION 2, FALL 2009, F. HAJIR

HOMEWORK 1

1. Some Definitions

Suppose S and T are sets. A *map* (or a *function*) F from S to T is a rule which assigns to each element of S a unique element of T. We say that S is the *source* of F and T is the *target* of F. Thus, if $s \in S$ is an element of s, the notation $F(s)$ carries no ambiguity: there is a well-defined rule which tells us what $F(s)$ is; its value does not depend on the time of day or the person who is evaluating it. For example, let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_{>0}$ the set of positive real numbers. If we say that for $x \in \mathbb{R}_{>0}$, $F(x) = \sqrt{x}$, where $\sqrt{x}$ means the unique positive square root of x, then $F : \mathbb{R}_{>0} \to \mathbb{R}$ is a well-defined function. However, if we say that for $x \in \mathbb{R}_{>0}$, $F(x)$ is defined by the equation $F(x)^2 = x$, then F is not a function, because $F(x) = x^2$ does not identify $F(x)$ uniquely: we would have $F(x) = \sqrt{x}$ or $F(x) = -\sqrt{x}$ for any given x.

Now let us go back to arbitrary sets S and T and a function $F : S \to T$. We say F is *surjective* (or *onto*) if every element of T is an "F-value" i.e. for every $t \in T$, there exists $s \in S$ such that $F(s) = t$. We say F is *injective* (or *one-to-one*) if whenever $s, s' \in S$ and $s \neq s'$, then $F(s) \neq F(s')$. In other words, F is injective means that no two distinct elements of S are sent to the same element of T by F. For any set S, we define the identity function $\mathrm{Id}_S : S \to S$ by $\mathrm{Id}_S(s) = s$ for all $s \in S$.

We say a map F is *bijective* if F is surjective as well as injective. A bijective map is also sometimes called a *set equivalence* or a *one-to-one correspondence*. We say that $F : S \to T$ has an inverse, if there exists a function $G : T \to S$ such that $F \circ G = \mathrm{Id}_T$ and $G \circ F = \mathrm{Id}_S$. If F has an inverse G, then G is unique. We denote by $\mathrm{Maps}(S, T)$ the set of all maps from S to T. The subset of $\mathrm{Maps}(S, S)$ consisting of *bijective* maps from S to itself is denoted $\mathrm{Perm}(S)$ or $\mathrm{Sym}(S)$. In class, we discussed the fact that $(\mathrm{Sym}(S), \circ, \mathrm{Id}_S)$ where $\circ$ is composition of functions and Id_S is the identity map of S, is always a group. If $S = \{1, 2, 3, \ldots, n\}$ is the set of the first n positive integers, then $\mathrm{Sym}(S)$ has a special name S_n and is called the nth *symmetric group*.

2. Some Problems

PROBLEM 1. Define $F : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by

$$F(a/b, c/d) = (a + c)/(b + d).$$

Is F a well-defined function? (Hint: the point here is that an element of $\mathbb{Q}$ is a rational number and as such can be expressed as a/b in infinitely many ways. For example, the rational number $1/3$ is also expressible as $2/6, 27/81$ etc.)

PROBLEM 2. Suppose $F : S \to T$ has an inverse G. Show that G is a bijection.

PROBLEM 3. Suppose $G : S \to T$ and $F : T \to U$ are functions. Prove the following:

i) If F, G are both injective, then so is $F \circ G$.
ii) If F, G are both surjective, then so is $F \circ G$.
iii) If F, G are both bijective, then so is $F \circ G$.

3. Some more definitions

Definition 3.1. Suppose G is a set. A *composition law (binary operation)* on G is a rule which assigns to each ordered pair of elments of G, say a and b, an elment of G, call it $a * b$. In other words, a composition law is a map

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (a,b) & \mapsto & a * b \end{array}$$

which sends the ordered pair $(a, b) \in G \times G$ to the element $a * b \in G$.

Whenever G is a set and $*$ is a binary operation on G, we say $(G, *)$ is a *magma*.

We say $*$ is *associative* if $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$. We say $*$ is *commutative* if $a * b = b * a$ for all $a, b \in G$. In this course, we will always work with associative laws of composition, but we will frequently work with non-commutative composition laws.

4. Some More Problems

PROBLEM 4. Perhaps the most natural example of a composition law is given by composition of functions (that's where the name comes from). So, let S be a set, and let $G = \mathrm{Maps}(S, S)$ be the set of all maps from S to itself. In other words, an element of $\mathrm{Maps}(S, S)$ is just a function $f : S \to S$ whose source and target are both S. Then composition of functions defines a law of composition on G: if $f, g \in \mathrm{Maps}(S, S)$, then we put $f * g = f \circ g$ where $f \circ g$ is the "composite" map defined by $(f \circ g)(s) = f(g(s))$ for $s \in S$.

PROBLEM 4A. Explain why the operation of composition defines an associative law of composition on $\mathrm{Maps}(S, S)$. (Start with three functions $f, g, h \in \mathrm{Maps}(S, S)$, then ....).

Here is an example. Let $S = \{\mathrm{Burger}, \mathrm{Fries}, \mathrm{Coke}\}$. Suppose we define two function $f, g : S \to S$ as follows:

$$f(\mathrm{Burger}) = \mathrm{Coke}, \qquad f(\mathrm{Fries}) = \mathrm{Fries}, \qquad f(\mathrm{Coke}) = \mathrm{Coke}$$

and

$$g(\mathrm{Burger}) = \mathrm{Coke}, \qquad g(\mathrm{Fries}) = \mathrm{Fries}, \qquad g(\mathrm{Coke}) = \mathrm{Burger}.$$

Now let $h = f \circ g$. For example have $h(\mathrm{Burger}) = f(g(\mathrm{Burger})) = f(\mathrm{Coke}) = \mathrm{Coke}$.

PROBLEM 4B. Complete the table below:

	Burger	Fries	Coke
f			
g			
$h = f \circ g$			
$i = g \circ f$			
$j = f \circ f$			
$k = g \circ g$			
$f \circ h$			
$j \circ g$			

Can you explain why $f \circ h$ and $j \circ g$ are the same function (without doing any calculation of what functions they actually are)?

PROBELM 5. In this problem, we will define the concept of a "lefty group" which is very similar to the definition of group we gave in class. The goal is to show that lefty groups and groups actually are the same thing.

Definition 4.1. Suppose $(G, *)$ is a magma and $e \in G$ is an element of G. We say that e is a *lefty identity* for $*$ if $e * a = a$ for all $a \in G$. We say that e is *righty identity* for $*$ if $a * e = a$ for all $a \in G$.

A *lefty group* is a triple $(G, *, e)$ where G is a set, $e \in G$ is an element of G and $*$ is a composition law on G satisfying the following properties:

i) $*$ is associative
ii) e is *lefty identity* for $*$, i.e. $e * a = a$ for all $a \in G$.
iii) every $a \in G$,

has a *lefty inverse*, i.e. for every $a \in G$, there exists $b \in G$ such that $b * a = e$.

Now suppose $(G, *, e)$ is a lefty group

Probelm 5A. Suppose $a \in G$ and let b be its lefty inverse, i.e. $b * a = e$. Show that b is also a righty inverse of a, i.e. $a * b = e$.

Hint: Let c be the lefty inverse of b and write $a * b = e * (a * b) = (c * b) * (a * b)$ and now use associativity.

Problem 5B. By assumption, e is a lefty identity. Prove that e is also a righty identity.

Hint: for $a \in G$, we must show $a * e = a$. Start with $a * e = a * (b * a)$ where b is the left inverse of a and use associativity together with A.

Problem 5C. Show that if $e' \in G$ satisfies $e' * a = a$ for all $a \in G$, (in other words, e' is a lefty identity for $*$), then $e' = e$.

Problem 5D. Show that if $b, b' \in G$ are both lefty inverses of $a \in G$, then $b = b'$, i.e. the lefty inverse of any element is unique.

Problem 5E. It is clear that if $(G, *, e)$ is a group, then it is also a lefty group. Using A and B, show that every lefty group is a group. Therefore, the concepts of "lefty group" and "group" actually coincide.

PROBLEM 6. Suppose $(G, *, e)$ is a group and $g \in G$. Show that if $g * g = g$, then $g = e$.

PROBLEM 7. Let $S_3 = \mathrm{Perm}(\{1, 2, 3\})$ be the set of all permutations of the set $\{1, 2, 3\}$. Its six elements are

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$
$$c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Complete the following 6×6 "multiplication" table of $\alpha \circ \beta$ for this group.

$\alpha \setminus \beta$	e	a	b	c	d	f
e						
a						
b						
c						
d						
f						

Verify that each line has a unique "e". How can you "see" at a glance whether this group is commutative or not from the table?

PROBLEM 8 (Extra Credit). On the real interval $(0, 1)$, we define the following six functions:

$$E(x) = x, A(x) = \frac{1}{x}, B(x) = 1 - x, C(x) = \frac{1}{1-x}, D(x) = \frac{x}{x-1}, F(x) = \frac{x-1}{x}.$$

Let $G = \{A, B, C, D, E, F\}$. Verify that composition of functions makes $(G, \circ, E)$ into a group by completing the following "multiplication" table.

$\alpha \setminus \beta$	E	A	B	C	D	F
E						
A						
B						
C						
D						
F						

Can you give a bijection from $\{a, b, c, d, e, f\}$ to $\{A, B, C, D, E, F\}$ which will carry the table of PROBLEM 7 exactly onto the table of this problem? In what sense(s) would you say the group $(G, \circ, E)$ of this problem is or is not the "same" as the group S_3 of PROBLEM 7?