UMASS AMHERST MATH 411 SECTION 2, FALL 2009, F. HAJIR

SAMPLE QUESTIONS FOR EXAM 2

1. Define the following terms completely and accurately.
A map $f : G \to K$ is a homomorphism if ...
If $f : G \to K$ is a homomorphism, then $\ker(f)$ is ...
If G is a group and H is a subgroup of H, then a left coset of H in G is ...
If G is a group and H is a subgroup of H, then G/H is ...
If G is a group and H is a subgroup of G, then the index of H in G (denoted by $[G : H]$) is ...
A subgroup H of G is called normal if ...
The "natural map" from G to G/H is defined by ...
If H is a normal subgroup of G, we give G/H a natural group structure by defining $aH * bH = ...$
The first isomorphism theorem states that ....
Lagrange's theorem states that ....
A group is simple if ....

2. Suppose H and K are subgroups of a finite group G and $\gcd(|H|, |K|) = 1$. Show that $H \cap K = \{e\}$.

3. For each element of $\mathbb{Z}/12\mathbb{Z}$, determine its order.

4. Suppose $\psi : G \to J$ is a group homomorphism and $Q = \mathrm{im}(\psi) \subseteq J$ is the image of ψ. Let $H = \ker(\psi)$.
(a) Prove that H is a normal sugbroup of G.
(b) Prove that the map $\Psi : G/H \to Q$ defined by $\Psi(gH) = \psi(g)$ is a well-defined map, i.e. if $aH = bH$ for $a, b \in H$, then $\psi(a) = \psi(b)$ (giving $\Psi(aH) = \Psi(bH)$.
(c) Prove that the map Ψ from (b) is a homomorphism.
(d) Prove that $\Psi : G/H \to Q$ is an isomorphism.

5. Suppose $m \geq 1$ is a positive integer. Prove that the subgroup $H = m\mathbb{Z}$ of the group $\mathbb{Z}$ (under addition) has index m.

6. Suppose H and K are subgroups of a group G. Recall that $HK = \{hk | h \in H, k \in K\}$ and $KH = \{kh | k \in K and h \in H\}$. Show that HK is a sugbroup of G if and only if $HK = KH$.

7. In the group S_4, let $V = \{(1), (12)(34), (13)(24), (14)(23)\}$.
(a) Show that V is a subgroup of S_4.
(b) What is $[S_4 : V]$? List the cosets of V explicitly.
(c) Show that V is a normal subgroup of S_4.
(d) Since V is a normal subgroup of S_4, S_4/V has a natural group structure; calculate the coset $(12)V * (123)V$.

(e) Show that $W = \{(1), (12)(34)\}$ is a normal subgroup of V.
(f) Show that W is *not* a normal subgroup of S_4.

Remark. Note that W is normal in V and V is normal in S_4, but W is not normal in S_4. Thus, normality is not transitive.

8. If G is a group and H is a subgroup of G of index 2, then H is a normal subgroup of G.

9. Prove that if G is a finite group, and Q is a homomorphic image of G, then $|Q|$ divides $|G|$.

10. Prove that if G is a group of prime order p, then G is cyclic.

11. Suppose G is a group of order 36 and K is a group of order 48, and that $f : G \to K$ is a homomorphism. Let $H = \ker(f)$. We obviously have $1 \leq |H| \leq 48$. Which of these numbers cannot occur as $|H|$?

12. Suppose G is a group, $x, y \in G$ are conjugate elements in G, i.e. there exists $g \in G$ such that $y = gxg^{-1}$. Prove that x, y have the same order in G.