UMASS AMHERST MATH 411 SECTION 2, FALL 2009, F. HAJIR

SAMPLE EXAM 1

These questions are similar to questions you will find on the exam. The actual exam will be shorter than this sample. The sample has more questions in order to illustrate a wider range of possible questions you can expect. Any concept introduced in HW 1, HW 2, or HW 3 may appear on the Exam, even if it does not appear on this sample exam. But there will be a much heavier emphasis on concepts from the first two homeworks than on the third.

1. Definitions

Give the complete definition of the following mathematical objects.

• If S an T are sets, then their Cartesian product $S \times T$ is ...

• A binary operation $*$ on a set S is ...

• A group is .....

• A map $f : S \to T$ is surjective if ....

• If $f : S \to T$ is a map, the inverse of f is a map ....

• For an integer $n \geq 1$, the group S_n is ... [make sure you give the underlying set as well as the binary operation on that set]

• If $(G, *, e)$ is a group and $H \subseteq G$ is a subset of G, we say that H is a subgroup of G if ...

• If $(G_1, *_1, e_1)$ and $(G_2, *_2, e_2)$ are groups, and $f : G_1 \to G_2$ is a map, then f is a group homomorphism if ...

• With notation as in the previous item, if $f : G_1 \to G_2$ is a homomorphism of groups, then the kernel $\ker(f)$ is defined to be the set ...

2. Problems

1. Suppose $(G, *, e)$ is a group and for all $g \in G$, $g^{-1} = g$. Prove that G is a commutative group.

2. Suppose we are given maps $f : S \to T$ and $g : T \to U$ and know that f is injective and that g is surjective. Does it necessarily follow that $g \circ f : S \to U$ is bijective? If so, give a proof. If not, give a counterexample.

3. Let X be a set and let $G = \mathrm{Perm}(X)$ be the set consiting of all bijective maps from X to X. Prove that under the operation of composition of functions, (i.e. $(f \circ g)(x) = f(g(x))$)
(a) G admits an identity element
(b) every element has an inverse
Conclude that G is a group under this operation.

4. Consider the two elements

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

of S_4. Calculate α^{-1}, β^{-1}, $\alpha \circ \beta$ and $\beta \circ \alpha$. Calculate the order of α and the order of β.

5. Suppose a, b are real numbers and $a \neq 0$. Define a map $\alpha_{a,b} : \mathbb{R} \to \mathbb{R}$ by $\alpha_{a,b}(x) = ax + b$ for all $x \in R$. Show that $\alpha_{a,b} \in \mathrm{Perm}(\mathbb{R})$, in other words, show that $\alpha_{a,b}$ is a bijection.

6. Recall that $G = SL_2(\mathbb{R})$, the set of all 2×2 matrices with *real* coefficients having determinant 1, is a group under matrix multiplication. Let $H = SL_2(\mathbb{Z})$ be the set of all 2×2 matrices with *integer* coefficients having determinant 1. Show that H is a subgroup of G.

7. Let $G_1 = \mathbb{Z}$ be the group of integers under addition and let $G_2 = GL_2(\mathbb{R})$ be the group of 2×2 real matrices with non-zero determinant (under matrix multiplication). Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}).$$

Define a map $f : \mathbb{Z} \to GL_2(\mathbb{R})$ by $f(n) = T^n$ for all $n \in \mathbb{Z}$.
(a) Calculate the order of T in $GL_2(\mathbb{R})$.
(b) Show that f is a homomorphism.
(c) Calculate $\ker(f)$, the kernel of f.
(d) Show that f is injective.