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HOMEWORK 8: (EQUIVALENCE) RELATIONS AND PARTITIONS

1. Relations

Recall the concept of a function f from a source set X to a target set Y. It is a rule for mapping each element x of the source to a single, well-defined, element of the target, which we call $f(x)$. A function from X to Y gives a very neat relationship between these two sets. Not all relationships between two sets are so "neat," and we will now consider a more general notion, that of a *relation* between X and Y.

Definition 1.1. Suppose X and Y are sets and $R \subseteq X \times Y$ is an arbitrary subset of the Cartesian product of X and Y. We say that R determines a *relation* from X to Y in the following way: If $x \in X$ and $y \in Y$ we say that x is related to y (and write $x \sim_R y$) if $(x, y) \in R$, and we say that x is not related to y ($x \not\sim_R y$) if $(x, y) \notin R$. If R is a relation from X to Y, and $x \in X$, we say that the fiber above x is the set $R_{x,\bullet} = \{y \in Y \mid (x, y) \in R\}$. Similarly, for $y \in Y$ $R_{\bullet,y} = \{x \in X \mid (x, y) \in R\}$ is the fiber below y. The Graph of the relation R is simply the set R itself. If X and Y are subsets of $\mathbb{R}$ with X, Y lying along the x-axis, y-axis as usual, then $R_{x,\bullet}$ is simply the intersection of the vertical line passing through x with the graph of R and $R_{\bullet,y}$ is the intersection of the horizontal line through y with this graph. The relation R determines a function $X \to Y$ determines a function if and only if for each $x \in X$, the fiber above it, $R_{x,\bullet}$ is a singleton, i.e. contains a single element, which is then the value of the function at x. Thus, a relation can be thought of as a function when its graph passes the "vertical line test."

Here is an informal summary of the above formal definition. First, informally speaking, a relation R between X and Y is a rule which, given $x \in X$ and $y \in Y$ determines whether x is "related" to y or not. If x is related to y, we write $x \sim_R y$ and otherwise we write $x \not\sim_R y$. The graph of R is defined by $\{(x, y) \in X \times Y \mid x \sim_R y\}$. If for each $x \in X$, there is a unique $y \in Y$ such that $x \sim_R y$, then R is a special kind of relation called a function; in that case, we write y is the value of this function at x if $x \sim_R y$.

Let us consider some examples.

Example 1.2. Let $X = Y = \mathbb{R}$, and let

$$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

In other words, x is related to y if and only if (x, y) is a point on the unit circle. The graph of the relation is just the unit circle. For instance, $1 \sim_R 0$ because $1^2 + 0^2 = 1$ and $0.5 \not\sim_R 0.5$ because $0.5^2 + 0.5^2 = 0.5 \neq 1$. Now let's talk about fibers. We have $R_{0,\bullet} = \{1, -1\}$ because $0^2 + y^2 = 1$ has two solutions $y = \pm 1$. On the other hand, the fiber below 1 has only one element, because $R_{\bullet,1} = \{x \in X \mid x^2 + 1 = 1\} = \{0\}$. By looking at the picture, one can see immediately that the fiber above x has $0, 1$ or 2 elements depending on whether $|x|$ is greater than 1, equal to 1, or less than one, respectively. In particular, this relation is *not* a function.

Example 1.3. Let $X = [0, 2]$ and $Y = [0, 12]$. Let $R = \{(x, y) \in \mathbb{R}^2 \mid y - 3x^2 = 0\}$. The graph of this relation is a piece of the parabola $y = 3x^2$. Since there is exactly one y-value in Y for each x value in X, this relation actually determines a function $X \to Y$, namely $\mathcal{R}(x) = 3x^2$.

Example 1.4. A relation can also be determined by just listing which elements of X are related to which of Y. For example, let $X = \{a, b, c, d\}$ and $Y = \{0, 1\}$. Then, we define a relation R by defining the related pairs to be are $a \sim 0$, $b \sim 1$, $b \sim 0$, $c \sim 1$. This means that these are ALL the related pairs. In other words, the subset R of $X \times Y$ is simply $R = \{(a, 0), (b, 1), (b, 0), (c, 1)\}$. This relation is not a function: here are two reasons; the fiber above b is too big (it is $\{0,1\}$) and the fiber above d is too little (it's empty!).

Example 1.5. Let $X = Y = [-1, 1]$ and define $x \sim_R y$ if and only if $xy \geq 0$. Thus, $R = \{(x, y) \in [0, 1] \times [0, 1] \mid xy \geq 0\}$. What does the graph of this relation look like? Think about it on your own for a minute. No Peekie until one minute's worth of thinking has occurred.

What did you get? The union of the unit squares in the first and third quadrants (including the axes), I hope. Verify that $R_{x,\bullet} = [0, 1]$ if $x > 0$ and that $R_{x,\bullet} = [-1, 0]$ if $x < 0$. What is $R_{0,\bullet}$?

In some of the examples above, the sets X and Y were the same, so the relation from X to Y can be called a self-relation on X; we call this a "relation on X" for short. Certain relations on X (those which satisfy three very nice properties we'll describe in a moment) are alled *equivalence relations*; they play a very important role in mathematics! We now define them.

Definition 1.6. A relation $\sim$ from a set X to itself is called an *equivalence relation* if it is **reflexive** (i.e. $x \sim x$ for all $x \in X$), **symmetric** (i.e. for all $x, y \in X$, $x \sim y \Rightarrow y \sim x$), and **transitive** (i.e. for all $x, y, z \in X$, $(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z$). These three conditions can be stated as follows: for each $x \in X$, x is always in the fiber above itself (reflexive), for each $x, y \in X$, if x is in the fiber above y, then y is in the fiber above x (symmetric), and finally for $x, y, z \in X$, if y is in the fiber above x and z is in the fiber above y, then z is in the fiber above x (transitive).

Given our philosophy of the importance of non-examples, let us start with two relations which are not equivalence relations.

Non-Example 1.7. Let us look at the first example of a relation we gave, namely $X = Y = \mathbb{R}$ and $x \sim y$ if and only if $x^2 + y^2 = 1$. This relation fails two of the three tests needed for being an equivalence relation. First, it is not reflexive: for $x \in X$, we do not always have $x \sim x$. For starters, for an equivalence relation, there are no empty fibers (why?) but this relation has plenty of empty fibers (just consider numbers of absolute value greater than 1). In fact, there are just 2 real numbers x such that $x \sim x$! What are they? This relation also fails transitivity. (Give an example). This relation is, however, symmetric.

Example 1.8. Let $X = Y = \mathbb{Z}$ with the divisibility relation, i.e. $x|y$ if and only if the equation $y = dx$ has a unique solution $d \in \mathbb{Z}$. Note that this is an asymmetric relation: $x|y$ does not imply $y|x$. But $x|y$ and $y|z$ implies $x|z$ and $x|x$ of course. We see that this relation is reflexive and transitive, but not symmetric. It plays a very important role in the study of the algebraic properties of the set $\mathbb{Z}$.

Non-Example 1.9. Now consider $X = Y = [-1, 1]$ with $x \sim y$ if and only if $xy \geq 0$. We have xRx because $x^2 \geq 0$ for all $x \in X$. If $xy \geq 0$, then clearly $yx \geq 0$ also. But this relation fails the last property of transitivity: for example, $1 \sim 0$ and $0 \sim -1$ are true but $1 \sim -1$ is false!

Example 1.10. We can tweak the previous relation just a little bit and make it into an equivalence relation. Namely, if we take $X_0 = [-1, 0) \cup (0, 1]$, i.e. $X_0 = [-1, 1] \setminus \{0\}$, and define $x \sim y$ if and only if $xy \geq 0$, then $\sim$ defines an equivalence relation on X_0. Or, if we keep $X = [-1, 1]$ as before but now define $x \sim y$ to mean that either $xy > 0$ or $x = y = 0$. You should check that in either of these cases, we get an equivalence relation.

Many relations from a set X to itself that occur "naturally" are equivalence relations. When that happens, it is a signal to us that the property defining the relation is a useful way of understanding the set. For example, if you consider the set of students at some elementary school, and consider the relation $\sim_1$ of "being in the same grade," then YOU CAN EASILY CONVINCE YOURSELF [**This is a shorthand way of saying : It's very important for you to do this exercise!**] that this is an equivalence relation. What that means is that breaking up the students according to their grade is a meaningful and interesting way to "organize" the set of elementary school students. Another relation $\sim_2$ of "being in the same class" is also an equivalence relation. They are not necessarily the same relation (because a school might have so many students that it has three different Kindergarten classes, for example). Another relation might be the classify the students according to age.

Now let us consider the fibers of an equivalence relation R on X. For $x \in X$, since $R_{x,\bullet} = R_{\bullet,x}$, we write simply R_x for either of these sets. This may appear an innocent definition, but the reader is hereby alreted to the fact that the fibers of an equivalence relation play a tremendously important role in all of mathematics! Often we annotate an equivalence relation with just a $\sim$ instead of giving a name (such as R) to its graph, so it's useful to have some other kind of notation for these fibers.[1] Consequently, we make the following very important definition.

Definition 1.11. If $\sim$ is an equivalence relation on a set X with graph $R = \{(x, y) \in X \times X \mid x \sim y\}$, then for each $x \in X$ the $\sim$-*equivalence class* of x, or simply the *equivalence class of* x, is the fiber above x, namely

$$R_x = \mathbf{cl}(x) = \widetilde{x} = [x] = \{y \in X \mid x \sim y\}.$$

It is the set of all elements of X equivalent to x. The set of all equivalence classes, i.e. the set $\widetilde{X} = \{\widetilde{x} \mid x \in X\}$ is called "X modulo $\sim$," and sometimes denoted by $X/\sim$ and called a "quotient of X." Note that $\widetilde{X} \subseteq \mathcal{P}(X)$ since its elements are subsets of X. The (surjective) map $X \twoheadrightarrow \widetilde{X}$ defined by $x \mapsto \mathbf{cl}(x) = \widetilde{x}$ is called the "quotient map," or the "natural map" from X to $\widetilde{X}$.

Example 1.12. Consider the set $\mathbb{Z}$ of all integers. If $a \in \mathbb{Z}$ we say that the *parity* of a is *even* if $a = 2b$ for some $b \in \mathbb{Z}$ and that its parity is *odd* otherwise. Let us write $a \sim b$ whenever a and b have the same parity. In other words, $a \sim b$ means that a and b are both odd or both even. Another way to say this is that $a - b$ is even. Let us check that this defines an equivalence relation on $\mathbb{Z}$: first, $a - a$ is always $0 = 2 \cdot 0$ hence always even. If

[1] In general, whenever a mathematical concept is given multiple names or multiple notations, this should serve as a clear indication to the student that a concept of great import has just been encountered

$a-b$ is even, then $b-a=-(a-b)$ is also even. If $a-b$ and $b-c$ are even, then $a-c$ is even, because $a-c=(a-b)+(b-c)$ is the sum of two evens. Thus, parity defines an equivalence relation on $\mathbb{Z}$ with two equivalence classes (the odds O and the evens E). The quotient of $\mathbb{Z}$ by this equivalence relation is the set $\{O, E\}$.

Part of the Fundamental Theorem of Equivalence Relations says the following: Suppose X is a set and $\sim$ is an equivalence relation on X; then $\widetilde{X}$ is a partition of X, i.e. $\sim$ *partitions* the set into non-overlapping non-empty subsets that cover the whole set. You might say, it polarizes the set into non-overlapping "clans" of equivalent elements. It might help you to think of equivalent elements (i.e. those which are related to each other by the given equivalence relation) as "relatives" and the set of all relatives of a given element x is the *clan* of x, a less technical term for "equivalence class" of x. The fact that "clan" and "class" both start with "cla" is linguistically useful; is that an accident?

To justify the remarks of the previous paragraph, let us make some important observations about $\mathbf{cl}(x)=\widetilde{x}$ where x is an arbitrary element of X (relative to a fixed equivalence relation $\sim$ on X).

- $\mathbf{cl}(x)=\widetilde{x}$ is never empty (because $x \in \mathbf{cl}(x)$!). "Every clan has at least one member."
- For the same reason, every element of X belongs to some clan. (If $x \in X$, then $x \in \mathbf{cl}(x)$).
- On the other hand, if x and z and two elements of X, then their equivalence classes $\mathbf{cl}(x)$ and $\mathbf{cl}(z)$ are either identical or disjoint (recall that two sets are called disjoint if their intersection is the empty set $\emptyset$). In other words, if $x, y \in X$, then either $\widetilde{x}=\widetilde{y}$ or $\widetilde{x} \cap \widetilde{y}=\emptyset$. The proof of this is left as an important homework exercise.

Now let us recall the definition of a partition of a set.

Definition 1.13. Suppose X is a set, A is another set (an "auxiliary" or "indexing" set) and $\Delta=\{X_\alpha | \alpha \in A\}$ is a collection of subsets of X. We say that $\{X_\alpha\}$ is a *partition of X with indexing set A* if (1) for all $\alpha \in A$, $X_\alpha \neq \emptyset$; (2) for all $\alpha, \beta \in A$, $X_\alpha \cap X_\beta=\emptyset$; (2) $\cup_{\alpha \in A} X_\alpha=X$. In other words, a partition of X is a collection of non-empty pairwise disjoint (non-overlapping) subsets of X whose collective union is X. the subests X_α are non-overlapping and together entirely cover X.

Remark. As you will show in one of your homework problems, a collection $\{X_\alpha | \alpha \in A\}$ of non-empty subsets of X is a partition of X with indexing set A if and only if for every $x \in X$ there exists a unique $\alpha \in A$ such that $x \in X_\alpha$.

Remark. If $\Delta=\{X_\alpha \mid \alpha \in A\}$ is a partition of X with indexing set A, then the map $A \to \Delta$ defined by $\alpha \mapsto X_\alpha$ is a one-to-one correspondence from A to Δ. Thus, the point of the indexing set A is just to have a convenient way to refer to the elements of Δ.

Example 1.14. let $X=\mathbb{Z}$, let $A=\{0,1\}$, let X_0 bet the set of even integers, and let X_1 be the set of odd integers. Then $\{X_0, X_1\}$ is a partition of $\mathbb{Z}$ because every integer is either odd or even (and no integer is both odd and even). You may note that this partition is mandated by the parity equivalence relation we discussed earlier. If we impose the parity relation on the integers and then order all the integers to band together into the corresponding clans, we will have exactly two clans, the evens and the odds, i.e. X_0 and X_1. The partition $\Delta=\{X_0, X_1\}$ is the set of clans under the parity equivalence relation.

Thus, if X is a set and $\sim$ is an equivalence relation on X, then X breaks up (is partitioned into) non-overlapping spanning equivalence classes. The set of these equivalence classes, i.e.

$\widetilde{X}$ or $X/\sim$ called "X modulo $\sim$," is also called *the partition of X associated to $\sim$*. The elements of X are, on the one hand, subsets of X, on the other hand, they should be thought of as the "clans" which together make up X. Thus, if $x \in X$ is a "point" in X, then $\mathbf{cl}(x) = \widetilde{x}$ does double duty: as $\mathbf{cl}(x) \subset X$ it is a subset of X, and as $\widetilde{x}$, it is "a point" in the set $\widetilde{X}$.

For example, for $\mathbb{Z}$ under the parity equivalance, the set $\mathbb{Z}/\sim$ is the set $\{X_0, X_1\}$ consisting of two elements. Note that the elements of $\mathbb{Z}/\sim$ are themselves sets: X_0 is the set of even integers and X_1 is the set of odd integers, but as elements of the set $\mathbb{Z}/\sim$, we just think of them as "the even equivalence class" and the "the odd equivalence class." One psychological technique is to say that the equivalence relation gloms all odds together into one object and all the evens together into another object (it "forgets" or erases the distinguishing features of the integers and remembers only their parity). Any single member of an equivalence class is then a "representative" of that class, just as any member of a clan is a representative of his or her clan.

The main fact that one should understand about partitions also happens to be the main fact one should understand about equivalence relations, namely that **To specify a partition of X is "the same as" specifiying an equivalence relation on X and vice versa.** The process of going back and forth between the two concepts, the first half of which we have already outlined above, is as follows.

From an equivalence relation to a partition: If $(X, \sim)$ is a set together with an equivalence relation $\sim$ on it, then the set $\widetilde{X} = \{\mathbf{cl}(x) \mid x \in X\}$ of the clans of X, also called $X/\sim$ read "X modulo $\sim$", is a partition of X.

From a partition to an equivalence relation: On the other hand, if (X, Δ) is a set together with a partition of it, then this partition induces an equivalence relation on X via the rule $x \sim y$ if and only if there exists $S \in \Delta$ such that $x \in S$ and $y \in S$, i.e. if and only if x and y belong to the same piece of the partition.

Note that the equivalence relation we have attached to Δ is the unique equivalence relation under which the partition formed by the equivalence classes is just the partition we started with. Likewise, the reader should check that if you start with $(X, \sim)$ then pass to (X, Δ) and then attach an equivalence relation to Δ, then you get back the original $\sim$.

Theorem 1.15 (The Fundamental Theorem of Equivalence Relations). *Suppose X is a set.*

(a) If $\sim$ is an equivalence relation on X, then the set of $\sim$-equivalence classes, $\widetilde{X} = X/\sim$, is a partition of X.

(b) If Δ is a partition of X, then Δ induces an equivalence relation $\sim_\Delta$ by the rule $x \sim y$ if and only if $x \in S$ and $y \in S$ for some $S \in \Delta$.

(c) If Δ is a partition of X, then $X/\sim_\Delta = \Delta$, and if $\Delta = \widetilde{X}$, then $\sim_\Delta = \sim$.

The point of the above theorem is that equivalence relations and partitions are two ways of looking at the same thing. Sometimes it is more convenient to use the partition language, and other times it is more useful to think in terms of relations. It is a frequent theme in a mathematician's experience that two objects that had been under separate study are revealed to offer different perspectives on the same underlying idea. Whenever this happens, the cohabitation, by the two seemingly different concepts, of the same "idea-landscape," serves to illuminate both concepts and to elevate the latter to a higher category of importance in the consciousness of the mathematician. On this issue, consult the wonderful book[2] by Barry Mazur, one of the most distinguished, not to mention eloquent, mathematicians of our times.

2. The set of rational numbers

Earlier we introduced the set of rational numbers in a practical way by defining:

$$\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+, \gcd(m,n) = 1\}.$$

The problem, if you want to call it that, with this definition is that as written, $\frac{2}{-4}$ is not a rational number. We have to make the added stipulation that $\frac{2}{-4} = \frac{-1}{2}$ by putting the fraction in reduced form. The problem here is to be able to "forget" the fact that the fraction $\frac{2}{-4}$ determine by the ordered pair of numbers 2 and -4 looks different from the fraction $-1/2$. Our machinery of equivalence classes gives a neat solution to this little dilemma. Namely, to construct the set of rational numbers, let $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$ be the set of non-zero integers. We start with the set $X = \mathbb{Z} \times \mathbb{Z}_0$. On this set, we define the equivalence relation $(a, b) \equiv (c, d)$ if and only if $ad - bc = 0$.[3] [Don't take my word for it: you must actually check that this really is an equivalence relation!] Then we define the set $\mathbb{Q}$ of rational numbers to be $\mathbb{Q} = X/\sim$! In other words, a rational number is really an equivalence class of (infinitely) many ordered pairs of integers (the second of which is non-zero). For instance,

$$\{(n, -2n) \mid n \in \mathbb{Z}_0\} = \{\dots, (3, -6), (2, -4), (1, -2), (-1, 2), (-2, 4), (-3, 6), \dots\}$$

is a rational number. You might object that this is a horrendously cumbersome way of "dealing" with what is after all a pretty basic object, and you would be right. But you cannot argue witht the fact that our construction is very rigorous and "correct" somehow. If we are ever in doubt about the truth of some subtle point about rational numbers, we can fall back on this very rigorous understanding. On some level, our mind keeps track of the fact that rational numbers can be represented in infinitely many different ways. For instance, in wishing to add $1/2$ to $1/3$, we prefer to think of these as $3/6$ and $2/6$ respectively. What allowed us to give a definition of $\mathbb{Q}$ without all this fuss was that we are able to give in each equivalence class in X a well-chosen representative, namely the one with positive second coordinate whose first coordinate is least in absolute value. This is a different way of saying (a, b) where $b > 0$ and $\gcd(a, b) = 1$. By the way, we have a natural injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ given by $a \mapsto \mathbf{cl}(a, 1)$, which you should think of as $a \mapsto a/1$.

Now we will express our previous proof that $\mathbb{Q}$ is a countable set in a slightly different way. Recall that a set S is countable if its elements can be listed, with or without repetition, as $s_1, s_2, \dots$. In other words, S is countable if and only if there exists a surjection $\mathbb{N} \twoheadrightarrow S$. Thus, if S is a countable set and $\sim$ is an equivalence relation on S, then $\widetilde{S}$ is also countable. (see the homework exercises). Since $\mathbb{N}$ and $\mathbb{Z}_0$ are countable, and since the direct product of two countable sets is countable, we have $\mathbb{N} \times \mathbb{Z}_0$ is countable, hence so is $\mathbb{Q} = \widetilde{\mathbb{N} \times \mathbb{Z}_0}$ where this is the quotient under the equivalence relation $(a, b) \sim (c, d) \Leftrightarrow ad - bc = 0$.

[2] Barry Mazur, Imagining Numbers (especially the square root of minus fifteen) FSG 2002.

[3] Where else have you seend that expression $ad - bc$ before?!

3. Problems

1. Consider the following relation on $\mathbb{Z}$: if $a, b \in \mathbb{Z}$, then $a \sim b$ if and only if $a \cdot b$ is even. Prove or Disprove: $\sim$ defines an equivalence relation on $\mathbb{Z}$.

2. Suppose X is a set and $\sim$ is an equivalence relation on X. Suppose $x, z \in X$. Prove that either $\mathbf{cl}(x) = \mathbf{cl}(z)$ or else $\mathbf{cl}(x) \cap \mathbf{cl}(z) = \emptyset$. [Hint: In other words, show that $\mathbf{cl}(x) \neq \mathbf{cl}(z) \Rightarrow \mathbf{cl}(x) \cap \mathbf{cl}(z) = \emptyset$.

3. Suppose $\sim$ is an equivalence relation on a set X with graph R. For $x \in X$, show that $R_{x,\bullet} = R_{\bullet,x}$.

4. If X is a set and $\sim$ is an equivalence relation on it, then we have a map $X \to X/\sim$ defined by $x \mapsto \mathbf{cl}(x)$. Show that this map is surjective. (Hint: this is a very easy problem; it requires only a careful examination of the definitions involved).

5. Show that every quotient of a countable set is countable, i.e. show that if S is a countable set and $\sim$ is an equivalence relation on S, then $\widetilde{S}$ is also countable.

6. Suppose $\Delta \subseteq \mathcal{P}(X) \setminus \emptyset$ is a collection of non-empty subsets of X. Show that Δ is a partition of X if and only if for every $x \in X$ there exists a unique $S \in \Delta$ such that $x \in S$.

7. Suppose $X = \mathbb{Z}$ is the set of integers, and let n be a positive integer. Define an equivalence relation on X as follows. For $a, b \in \mathbb{Z}$, we write $a \sim_n b$ if and only if n divides $a - b$, i.e. if and only if $a - b = nk$ for some $k \in \mathbb{Z}$. This equivalence relation is called congruence modulo n. The more common notation is $a \equiv b \bmod n$.

(i) Show that this really is an equivalence relation.

(ii) Show that if $n = 1$, then all integers are equivalent to each other.

(iii) Show that if $n = 2$, then the resulting equivalence relation is the parity equivalence relation discussed above.

(iv) Show that under $\sim_n$, $\mathbb{Z}$ breaks up into n equivalence relations corresponding to the n possible remainders $0, 1, 2, \cdots, n-1$ for division by n. Thus, the set $\mathrm{Rem}(n) = \{0, 1, 2, \ldots, n-1\}$ is a natural indexing set for the partition of $\mathbb{Z}$ corresponding to congruence modulo n.

(v) Use (iv) to show that we can write $\mathbb{Z}/\sim_n = \{X_0, X_1, \cdots, X_{n-1}\}$ where

$$X_j = \{a \in \mathbb{Z} | \text{ the remainder of } a \text{ divided by } n \text{ is } j\}.$$

8. In this problem, you will show that the two concepts of "equivalence relation on a set X" and "partition of X" are really the same concept, i.e. you will prove the fundamental theorem of equivalence relations.

(a) Suppose X is a set equipped with an equivalence relation $\sim$. Show the the set of equivalence classes of X under $\sim$ is a partition of X.

(b) Conversely, suppose $\Delta = \{X_\alpha | \alpha \in A\}$ is a partition of a set X. Now define a relation $\sim_\Delta$ on X as follows: if $x, y \in X$, then $x \sim_\Delta y$ if and only if there exists $\alpha \in A$ such that $x, y \in X_\alpha$. Prove that $\sim_\Delta$ is an equivalence relation on X.

(c) Prove that if $\sim$ is an equivalence relation on X, then $\sim_{\widetilde{X}} = \sim$.

(d) Prove that if Δ is a partition of X, then $X/\sim_\Delta$ is just Δ.

9. With the fundamental theorem of equivalence relations we established that equivalence relations on X and partitions on X are basically the same object and give rise to a map $X \to \widetilde{X}$. In this problem, you will how a map $X \to Y$ induces an equivalence relation on X.

Suppose X, Y are sets and $f : X \to Y$ is an arbitrary map. For $y \in Y$, the fiber of f at y (or above y) is defined to be the set $\mathbf{f}^{-1}(y) = \{x \in X \mid f(x) = y\}$. The notation $\mathbf{f}^{-1}(y)$ should not be confused with the inverse function f^{-1}. Note that we are not assuming that f is bijective. Thus, the set $\mathbf{f}^{-1}(y)$ could be empty or it could have more than one element. **If** f **is** bijective, however, **then** for each $y \in Y$, $\mathbf{f}^{-1}(y)$ is a singleton set whose only element is $f^{-1}(z)$.

Let Δ be the set of **non-empty** fibers of f, i.e. $\Delta = \{\mathbf{f}^{-1}(y) \mid y \in \text{Image}(f)\}$.

(a) Show that Δ is a partition of X.

(b) Define a relation $\sim$ on X by the rule $x \sim x'$ if and only if $f(x) = f(x')$ for $x, x' \in X$. Prove that $\sim$ defines an equivalence relation on X.

(c) For the relation $\sim$ defined in (b), prove that the equivalence classes of $\sim$ coincide with the elements of Δ, in other words, the non-empty fibers of f are precisely the equivalence classes of the equivalence relation $\sim$.

(d) For the equivalence relation $\sim$ defined in (b), define a simple and natural bijection $\varphi : \text{Image}(f) \to \widetilde{X}$ explicitly. Also define explicity the inverse map $\varphi^{-1} : \widetilde{X} \to \text{Image}(f)$ making sure to show that this map is well-defined.

(e) Suppose $n \geq 1$ is a positive integer, and recall that $\text{Rem}(n) = \{0, 1, 2, \ldots, n-1\}$. Let $X = \mathbb{Z}$ and $Y = \text{Rem}(n)$. Define the *reduction map modulo* n by $f : \mathbb{Z} \to \text{Rem}(n)$ where $f(x)$ is the remainder when x is divided by n, i.e. $f(x) = r$ where $x = nq + r$ for some $q \in \mathbb{Z}$ and $0 \leq r \leq n-1$. Show that the fibers of the map f are precisely the equivalence classes of congruence modulo n, and thus the equivalence relation one obtains on $\mathbb{Z}$ by the method of (b) is just congruence modulo n.

(f) Suppose $X = C^\infty(\mathbb{R})$ is the set consisting of all infinitely-differentiable functions on $\mathbb{R}$, i.e. functions $g : \mathbb{R} \to \mathbb{R}$ such that $g', g'', \ldots, g^{(n)}$ are well-defined functions from $\mathbb{R}$ to $\mathbb{R}$ for all $n \geq 1$. Define a relation $\sim$ on X as follows: for $g, h \in X$, $g \sim h$ if and only if $g - h$ is a constant function i.e. if and only if there exists $c \in R$ such that $g(x) - h(x) = c$ for all $x \in \mathbb{R}$. There is a very familiar map $D : X \Rightarrow X$ such that the fibers of D are exactly the $\sim$-equivalence classes of $\sim$. What is D?! Explain. Letting $z : \mathbb{R} \to \mathbb{R}$ be the zero map, i.e. $z(x) = 0$ for all $x \in \mathbb{R}$, what is the fiber $\mathbf{D}^{-1}(z)$ above z?