UMASS AMHERST MATH 300 SP '05, F. HAJIR

HOMEWORK 6: INDUCTION

1. Remembrances of Things Past

Let's take a moment to recall some material which will hopefully be familiar to you from your study of infinite series and matrices, etc.

A finite sum $a_1 + a_2 + \cdots + a_n$ can be expressed compactly as $\sum_{j=1}^{n} a_j$. Note that it could also be written as $\sum_{umass=1}^{n} a_{umass}$. If $a_1, a_2, \ldots$ is an infinite sequence of real numbers, then for the infinite sum $a_1 + a_2 + \cdots$ we write

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j,$$

i.e. the sum of the series is defined to be the limit of the partial sums, should this limit exist. If it does not, we say that the series diverges.

A product of terms $a_1 a_2 \cdots a_n$ is written as $\prod_{j=1}^{n} a_j$.

You have probably encountered the factorial notation (when you studied the Taylor series of e^x for example): for an integer $n \geq 0$, we define $n! = \prod_{j=1}^{n} j$. Note that $0! = 1$ because by definition, $0!$ is an empty product: an empty product should be interpreted as 1 always, just as an empty sum should be interpreted as 0. A very useful fact is that $n! \approx e^{-n} n^{n-0.5}\sqrt{2\Pi}$; this rather good approximation (for large n) is known as *Stirling's formula.*

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are two matrices with real coefficients, then the product of the two matrices AB is defined to be the matrix

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Matrices represent linear transformations of a plane (equipped with a basis) and the matrix product defined above represents the composite of the two linear transformations corresponding to A and B, respectively under a fixed basis. The determinant of the matrix A is defined to be $ad - bc$; one can check directly that $\det(AB) = \det(A)\det(B)$, i.e. the determinant map is multiplicative. For a more conceptual explanation, you should have attended an Undergraduate Colloquium on March 24, 2005 [notes on my web page might pop up one day ...]

2. Mathematical Induction

One of the most powerful methods for proving certain kinds of propositions is mathematical induction. Keep in mind that inductive reasoning is quite distinct from mathematical induction (from now on I will just say "induction" for short). Induction is based on the well-ordering principle. There are several different "flavors" of induction; we will talk about three: ordinary induction, complete induction, and two-variable induction.

The basic idea is as follows. Imagine a typical guy. Let's call him Larry. Larry is a snacker. His favorite snack is Potato Chips. Larry has a peculiar habit. If he eats a chip from a bag, then he simply must eat another chip from that bag (as long as another chip exists in the bag). So, what can we conclude from this? That if you give Larry a bag of chips **and** he eats a first chip, then he will completely eat the whole bag no matter how many chips are in the bag. That's it, that's Mathematical Induction. Note the main "engine" of the conclusion we reached about Larry is the "inductive step" that each consummed chip necessarily will entail the consumption of another chip. The other necessary hypothesis (for Larry to eat the whole bag) is of course is that he take that first plunge and actually eat the first chip (the "base case"). Here is another analogy. Imagine you set up a bunch of dominoes in your room and you space them so closely that you know for sure that should any domino ever topple forward, then its succeeding neighbor will also fall forward. Then, if any domino along the chain falls forward, **all** of the succeeding ones will as well.

Okay, now you are psychologically prepared for a more formal statement of the Principle of Mathematical Induction.

Theorem 2.1 (The (Ordinary) Principle of Mathematical Induction)**.** *Suppose for each integer $n \geq 1$, $P(n)$ is a statement. Assume moreover that*

- *$P(1)$ is true, and*
- *Whenever $P(k)$ happens to be true for some $k \geq 1$, then $P(k+1)$ is also true, i.e.*

$$P(k) \Rightarrow P(k+1) \text{ for all } k \geq 1.$$

Then $P(n)$ is true for all $n \geq 1$.

Proof. We will give a proof by contradiction. Suppose it is not the cae that $P(n)$ holds for all n, i.e. the set $S = \{n \geq 1 \mid P(n) \text{ is false }\}$ is non-empty. By the well-ordering principle, S has a least element, let us call it m. We know that $m > 1$ since $P(1)$ is true by assumption. Since m is the smallest element of S, we know that $m - 1 \notin S$. By definition of S, therefore, $P(m-1)$ is true. Since $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$, we conclude that $P(m)$ is true, but $m \in S$ so $P(m)$ is false. This is a contradiction. So S must be empty after all, i.e. $P(n)$ holds for all $n \geq 1$. □

Remark. Problem 5 on HW5 was just a different way of stating the above Principle.

We will now use mathematical induction to prove some formulas that we guessed using inductive reasoning but have not yet verified, as well as some other kinds of statements.

Example 2.2. For each integer $n \geq 1$, prove that $P(n) : 1 + 2 + \cdots + n = n(n+1)/2$. This is crying out for proof by induction.

We first check $P(1)$: $1 = 1(1+1)/2$ CHECK.

Now we check the "induction step" i.e. $P(k) \Rightarrow P(k+1)$ for $k \geq 1$. So, suppose $P(k)$ holds for some $k \geq 1$. Then $1 + 2 + \cdots + k = k(k+1)/2$. We have to show that this implies $P(k+1)$. Okay, let's try to calculate $1+2+\cdots+k+(k+1)$ using what we have been allowed to assume, namely that the sum of the first k integers has a nifty closed-form formula. Okay

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\ &= k(k+1)/2 + (k+1) \\ &= (k+1)(1 + k/2) \\ &= (k+1)(k+2)/2. \end{aligned}$$

Hey, we just showed that, for $k \geq 1$, $P(k+1)$ follows if $P(k)$ is true. That completes the "induction step." So, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

Example 2.3. Show that if we order the odd positive integers in increasing size (as usual), for $n \geq 1$, the nth one is $2n-1$. In other words, $P(n)$: if O_n is the nth odd integer, then $O_n = 2n-1$. Let's check the base case: $n = 1$: $O_1 = 1$ and $2(1)-1 = 1$ so it checks out okay. Now let's do the inductive step. If, for some $k \geq 1$, $O_k = 2k-1$, then, since $O_{k+1} = O_k + 2$, we have $O_{k+1} = (2k-1)+2 = (2k+1)$. We might scratch our heads here a little and say "Um, like, are we done now?" or "What the heck are we trying to do here?!" The answers are "Not quite dude" and "Like you have to show $P(k) \Rightarrow P(k+1)$ dude," respectively. Let's just work backwards just a tiny bit: what is $P(k+1)$? It says that $O_{k+1} = 2(k+1)-1$. So far, we have $O_{k+1} = 2k+1$. Oh but wait, $2k+1 = 2k+2-1 = 2(k+1)-1$ so we got it.

Just to be totally clear let's now re-give the induction step. We suppose that $k \geq 1$ is some integer and that $P(n)$ happens to hold for $n = k$, then try to derive that $P(n)$ would follow for $n = k+1$. So, we assume $O_k = 2k-1$, then compute $O_{k+1} = 2k-1+2 = 2k+1 = 2k+2-1 = 2(k+1)-1$, so $P(k) \Rightarrow P(k+1)$.

Example 2.4. Prove that for $n \geq 1$, the sum of the first n odd positive integers is n^2. So, let us define T_n to be the sum of the first n odd integers, i.e. $T_n = 1+3+5+\cdots+(2n-1)$, since we figured out that $2n-1$ is the nth odd number. Let's check the base case. Always do that first. In fact, it pays to do just a few more cases after the base case. So, for $n = 1$, $T_n = T_1 = 1$ and $n^2 = 1^2 = 1$ so they agree. We also check $T_2 = 1+3 = 2^2$, $T_3 = 1+3+5 = 9$ and that's good too. Okay, let's move to the heart of the matter, then inductive step. We must show that $P(k)$ implies $P(k+1)$ for all $k \geq 1$. Thus, if $P(k)$ holds, then $T_k = k^2$ which implies that

$$T_{k+1} = T_k + (2k+1) = k^2 + 2k + 1 = (k+1)^2,$$

so $P(k+1)$ holds, just as easy as that. The equality $T_n = n^2$ has now been proved for all $n \geq 1$ by the principle of mathematical induction.

Example 2.5. Show that for $n \geq 5$, $2^n > n^2$. So we have $P(n) : 2^n > n^2$. You mean out "statement" is not an equality, it's an *in*equality? And we don't start with $n = 1$? Dude, can we still apply induction?! Of course *we* can, Dude! Base Case and Inductive Step that's what it's all about! Base Case $n = 5$: $2^5 = 32 > 25 = 5^2$ no sweat. Inductive Step: must show, for $k \geq 5$, that $2^k > k^2$ implies $2^{k+1} > (k+1)^2$. So suppose for some integer $k \geq 5$, $2^k > k^2$. Then $2^{k+1} = 2 \cdot 2^k > 2k^2$. Looking ahead (or working backwards) it sure would be nice if $2k^2$ would be gracious enough to dominate $(k+1)^2$ for $k \geq 5$. So, we boldly claim $2k^2 > (k+1)^2$ for $k \geq 5$ which solves the problem assuming we can prove our claim. Now, we write down $2k^2 \overset{?}{>} (k+1)^2$ on a Blue Wall napkin and expand and bring stuff from one side to the other until we see the following argument work itself out in reverse order of how we will now present it:[1] Since $k \geq 5$, we have $k > k-2 > 1$ so $k(k-2) > 1$ so $k^2 - 2k > 1$ so $k^2 > 2k+1$ so $2k^2 = k^2 + k^2 > k^2 + 2k + 1$ so $2k^2 > (k+1)^2$.

Example 2.6. Prove by induction that $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9 for all $n \geq 1$. The base case: $n = 1$, $1+8+27 = 36 = 4*9$ CHECK. Now we must show $P(k) \Rightarrow P(k+1)$

[1] One problem with reading mathematics is that the little scratchwork is never kept, even off in the margins, the proofs are all presented neat and crisp. I'm not saying this should change, but if that stuff isn't there in the text you are reading, then you need to be supplying it as you read!

for all $k \geq 1$. So we assume the Induction Hypothesis: $k^3 + (k+1)^3 + (k+2)^3 = 9t$ for some integer t. We must SHOW: $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9s$ for some integer s. We recognize the first two terms of this sum as the last two terms of the sum in the induction hypothesis, so we calculate

$$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= 9t + (k+3)^3 - k^3 \\ &= 9t + k^3 + 9k^2 + 27k + 27 - k^3 \\ &= 9t + 9k^2 + 27k + 27 \\ &= 9s \qquad s = (t + k^2 + 3k + 3) \in \mathbb{Z}.\end{aligned}$$

By the Principle of Mathematical Induction, we have proved that the sum of three consecutive integer cubes is divisible by 9.

3. Problems

1. Prove by induction that, for all integers $n \geq 1$,

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

2. Prove by induction that for $n \geq 1$, the sum of the cubes of the first n positive integers is the square of the sum of the first n positive integers, i.e.

$$\sum_{j=1}^{n} j^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Feel free to use the formula for Bowling Numbers that we proved in class and in these notes.

3. Prove by induction that, for all integers $n \geq 1$,

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

4. Prove, using induction, that if X is a finite set of size $n \geq 0$, then X has 2^n subsets, i.e. that $|\mathcal{P}(X)| = 2^{|X|}$. You gave a proof of this on HW5. Your proof here must use induction on the size of X.

5. Prove by induction that for $n \geq 2$,

$$\prod_{j=2}^{n} (1 - \frac{1}{j^2}) = \frac{n+1}{2n},$$

6. Prove by induction that if $x > 1$ is a real number then for all integers $n \geq 2$, $(1+x)^n > 1 + nx$.

7. Here is a "proof," using induction, of the statement that all horses are have the same color.

We must prove that for $n \geq 1$,
$H(n)$: If C is a collection of n horses, all the n horses in C have the same color.

We will use induction on n to prove $H(n)$. Base case: $n = 1$; since there is only one horse, it has the same color as all the horses present in the collection, i.e. itself. Inductive Step: we must show that $H(k)$ implies $H(k + 1)$. So we assume known that in any collection of k horses, they all have the same color [the "induction hypothesis"] and must show this to be the case for any collection of $k + 1$ horses. Let C be a collection of $k + 1$ horses. Take any subcollection $S = C \setminus \{h\}$ of size k, i.e. one which leaves out one horse, lets us call it h. We know that all the horses in S have the same color by the induction hypothesis; it remains only to show that h has the same color as all the horses in S. Now let $S' = C \setminus \{h'\}$ be a different collection of k horses in C, i.e. $h' \neq h$. Then all the horses in S' have the same color, again by the induction hypothesis. Since $h \in S'$, h must have the same color as all the other horses too, so all the horses in C have the same color. By the Principle of Mathematical Induction, all horses have the same color.

Write a carefully written discussion of this proof, describing where and how it goes wrong. [For you agree that the statement is false, right?!]

8. Define the famed *Fibonacci Sequence* $F_0, F_1, F_2, \ldots$ as follows. Let $F_0 = 0$, $F_1 = 1$ and define the rest *recursively* by letting $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. Thus, $F_2 = F_1 + F_0 = 1$, and $F_3 = F_2 + F_1 = 1 + 1 = 2$ and so on, each new term being the sum of the two preceding ones.

Let M be the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) Verify for a few small n, say $1 \leq n \leq 5$, that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. Try to prove this holds for all n, either using induction, or some other trick. [*Do Try it*, it's good for you, but if it gets hairy, don't sweat it too much, you'll see why in a minute.]

(b) Using induction on n, prove that, for $n \geq 1$,

$$M^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

[For a reminder about matrix multiplication, see Remembrances of Things Past].

(c) Now Use (b) to prove the formula in (a).

9. Suppose $m \geq 2$ and $X_1, \ldots, X_m$ are countable sets. Use induction on m to prove that $X_1 \cup X_2 \cup \cdots \cup X_m$ is countable.

10. Use induction to prove that for any integer $n \geq 1$, the number $a_n = 5^n + 2(3^{n-1}) + 1$ is divisible by 8, i.e. $a_n/8$ is an integer.

4. Extra Credit Problems

Suppose L is the set of all straight lines in the plane $\mathbb{R}^2$.

(a) Show that L is not countable.

(b) Let $C \subsetneq L$ be a countable subset of L. Let $X = \cup_{c \in C} c$ be the union of all the members of C, so X is the union of countably many lines in $\mathbb{R}^2$. Show that X is a *proper* subset of $\mathbb{R}^2$.

Hint: Use proof by contradiction. Look to contradict the fact that each line is uncountable.