# UMASS AMHERST MATH 300 SP '05, F. HAJIR

HOMEWORK 5: COUNTING AND UNCOUNTABILITY

## 1. The classification of sets according to size

Here are some notes on measuring the size of infinite sets. For doing the problems below, you will need to read over the notes from HW 4 as well as pages 4-10 of the Meeks notes.

Let us begin by reviewing some facts about finite sets. Two main facts are as follows.

**Lemma 1.1** (Main Lemma of Finite Sets)**.** *Suppose $f : X \hookrightarrow Y$ is an injective map of sets. Then,*

*(a) If $x_1, \ldots, x_n$ is a non-repeating sequence of length $n \geq 1$ in $X$, then $f(x_1), \ldots, f(x_n)$ is a non-repeating sequence of length $n$ in $Y$.*

*(b) If $Y$ is finite, then so is $X$ and $|X| \leq |Y|$.*

**Theorem 1.2** (Main Theorem of Finite Sets)**.** *Suppose $X$ and $Y$ are finite sets. Then, $X \sim Y$ if and only if $|X| = |Y|$.*

*Proof.* If $X \sim Y$, then there exists a bijective map $f : X \to Y$. Since $f$ is injective, $|X| \leq |Y|$ by the Main Lemma of Finite Sets. Since $f$ is bijective, we also have $f^{-1} : Y \to X$ which is bijective hence injective, so we also get $|Y| \leq |X|$. Since $|X| \leq |Y| \leq |X|$, we must have $|X| = |Y|$. The other direction is also easy, it is left as a homework problem. □

A set $X$ is infinite if and only if there exists an injective map $f : \mathbb{N} \hookrightarrow X$. For, if such a map exists, then $f(1), f(2), f(3), \ldots$ is a non-repeating infinite sequence in $X$, and in the other direction, if $x_1, x_2, \ldots$ is an infinite non-repeating sequence in $X$, then by putting $f(n) = x_n$ for $n \geq 1$, we see immediately that $f$ gives an injection $\mathbb{N} \hookrightarrow X$.

How should we measure the "size" of infinite sets? Our first thought might be (as it was for a period lasting at least 4000 years!) that all infinite sets are of the same size. It was only in the 19th century that Georg Cantor cured us of this naïveté singlehandedly. Here is an analogy. Suppose in all your life, you have never seen or heard of a car. The fastest method of travel you know about is riding a horse. Then one day you discover cars. In the beginning of your discovery, you will not be able to tell the difference between a Yugo and a Ferrari. To you, these new "machine horses" are so extraordinarily fast and superior to your prior experiences that the relative differences between them are irrelevant. But after driving in a Yugo for a while, then moving on to a Chevrolet and a Mercedes and a Ferrari, you come to distinguish and understand the differences between them. Then one day you see a Cessna plane cruising down a runway and you think well, here is another machine horse, and then you see it take off and fly, and you realize "Whoa! Now that thing belongs to a whole higher class of machine horses by itself!!"

So it was for our mathematical understanding of infinity. At first, you are familiar with finite sets. You realize early on that some finite sets are "alike" (they have the same size) and are in that sense "equivalent." Then one fine day, you realize "Hey, there is NO largest counting number!" i.e. "Whoa! the set of counting numbers (which 'classify' finite sets) is not a finite set itself!" In other words, you have just discovered the existence of infinite sets. This is such a monumental discovery, infinity is such a remote concept, that to you just assume that all infinite sets are of the same size. This relative homogeneity of infinite sets is expressed in your language: you have a name for sets of size 1, size 2, size 3, etc. but you have only one name for sets which are not finite (infinite) reinforcing the notion that all infinites sets are of the same size.

Let us look more closely to see whether all infinite sets really are of the same size. Well, what does it mean for two sets to be of the same size? For finite sets, it means that there is a one-to-one correspondence between the two sets: we can line up the elements of one set against the elements of the other set in an exact match-up. Cantor had the important insight that this definition should be used for infinite sets as well. Recall that we write $X \sim Y$ if there is a bijection $X \to Y$.

**Definition 1.3.** For any pair of arbitrary sets, $X$ and $Y$, we say that $X$ and $Y$ have the same size and write $|X| = |Y|$ if and only if $X \sim Y$.

Having accepted this definition, we now must dare ask, with Cantor, a fundamental question.

Let's imagine a conversation Cantor could have had with a fictitious colleague (and a very old friend) who has just come for a visit from out of town, let us call him Professor Groeg Rotnac, as they stroll along on Ulmestrasse one summer evening in the early 1870s.

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## Professors Cantor and Rotnac Take a Walk

*Prof Rotnac:* What work is occupying you these days, esteemed colleague?

*Prof Cantor:* I have been fascinated and considerably puzzled by a fundamental question which on the surface appears quite simple but whose complexities, in my opinion, run very deep. This question has been troubling me for years and only recently have I been able to make any progress. Given the simplicity of the question, I hesitate to mention it, even, to such a dear and old friend as yourself.

*Prof. Rotnac:* Come now, we have known each other practically from infanthood! We have no need of reticence! Please, I am curious, I will treat your question with respect.

*Prof. Cantor:* Very well, I would expect no less from you. You would agree that two sets, be they finite or infinite, should be said to be of the same size if there is a one-to-one correspondence, i.e. a bijection, between them.

*Prof. Rotnac:* On that, we are in complete agreement. You might say, on that there is a correspondence between our thoughts. Ha Ha Ha.

*Prof. Cantor:* Yes, that is very witty of you. Very good. Now, here is my innocent-sounding question: If $X, Y$ are two infinite sets, are they necessarily of the same size? In other words, if $X, Y$ are infinite sets, then is there necessarily a bijection from $X$ to $Y$?

*Prof. Rotnac:* Mit Verlaub, with all due respect, I believe you are teasing me.

*Prof. Cantor:* Need I remind you that you promised you would take my question at face value? Here is my question again, please treat it seriously: If $X$ and $Y$ are infinite sets, can one always find a bijection from $X$ to $Y$?

*Prof. Rotnac:* JAWOHL, NATÜRLICH, of course one can, my dear Georg! DAS IST EIN KINDERSPIEL: It is child's play not worthy of a Herr Doctor Professor of your standing!

*Prof. Cantor:* Indulge me and present your KINDER-proof, bitte.

*Prof. Ratnoc:* Very well, list the elments of $X$ and $Y$ as $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$, respectively. Then map $x_i$ to $y_i$ for each $i \geq 1$.

*Prof. Cantor:* Your proof rests on the notion that if $X$ is any infinite set, then one can simply list all its elements in a single infinite list, i.e. that there is a bijection from the natural numbers $\mathbb{N}$[1] to $X$. Then, since there is a bijection from $X$ to $\mathbb{N}$ and another from $\mathbb{N}$ to $Y$, composing them one gets a bijection from $X$ to $Y$. Is that about it?

*Prof. Ratnoc:* You have expressed my thoughts perfectly, as you always do.

*Prof. Cantor:* Yes, but this brings us to the crux of the matter. I find your claim that one can always list all the elements of an arbitrary infinite set in a single infinite list LÄSTIG, troubling at best, indeed downright false as we shall hopefully see in a moment. Pray discuss how you would justify your claim that if $X$ is any infinite set, then its elements can be listed in one infinite list.

*Prof. Rotnac:* Well, simply choose one element at random, call it $x_1$, then a different one, call it $x_2$, then an $x_3$ different form $x_1$ and $x_2$ and so on. Since the set is infinite, one will never run out of elements to choose, and one will eventually list every single element.

*Prof. Cantor:* I am afraid that will not work.

*Prof. Rotnac:* Of course it will.

*Prof. Cantor:* NEIN.

*Prof. Rotnac:* I see that you are quite sure of yourself. Very well, why not, pray?

*Prof Cantor:* The trouble with your procedure is your claim that that "one will eventually list every single element." Let me illustrate with a very simple example. Suppose we attempt to use your procedure to list all the elements of the set $\mathbb{N}$. You said I can choose the elements as I please. Suppose I happen to choose 2, then 4, then 6, then 8, and so on, always skipping the odd numbers. In this way, I will never stop and will never list all the elements since I miss every odd number.

*Prof. Rotnac:* Very well, the procedure does not work if you apply it in a deliberately obtuse manner, but surely there must be a way of listing the elements correctly so that none gets left out. I grudgingly admit that I have not yet given you a correct procedure that will always list all the elements of a given infinite list, but I do not give up on the claim that such a procedure exists. For instance, of course one *can* list all the elements of $\mathbb{N}$, simply ask a child to do it and she will do it in the right order!

*Prof. Cantor:* For the set $\mathbb{N}$, yes, of course, but recall that we are asking if the elements of an arbitrary infinite set $X$ can always be given in one infinite list (let us call that simply "listing the elements" and a set for which we can list the elements in one infinite list I will call "listable"). So far, I was simply showing you that your procedure is not guaranteed to work. Let us continue our investigation for various infinite sets and attempt to find the means of listing the elements, which you so adamantly maintain must exist.

*Prof. Rotnac:* Very well, how about Herr Dr Professor Kronecker's favored set, $\mathbb{Z}$?

*Prof. Cantor:* (With barely disguised disgust) BITTE do not speak of that distinguished yet pompous gentleman.

[1]For the sake of convenience, we allow several anachronisms, such as the use of words like "bijection" in this discourse. Another anachronism is the nuse of $\mathbb{N}$ to stand for the set of natural numbers. The notation $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and so on was not standardized until many years later when Bourbaki came along

*Prof. Rotnac:* Please forgive me; I did not know that mentioning his name would upset you so. But do let us take up the case of the set of all integers, $\mathbb{Z}$, positive, negative and 0. This is a good test case to start with, because it is usually thought of as a "two-sided" infinite list. and you would like a one-sided infinite list? Hmm, just a moment. I have it! We start at 0, then go to 1, then circle around to $-1$, then circle around to 2, then circle around to $-2$, and so on, getting: $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$. Every element is now listed in one infinite list.

*Prof. Cantor:* Excellent work, MEINE FREUND! You have discovered a method I call "interleaving." Here is what I mean: Given two sets $X$ and $Y$ which are listable (say as $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$), then $X \cup Y$ is also listable by interleaving the two sets, as follows: $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$, just as you interleaved the positives and negatives. Moreover, suppose given a finite collection of sets $X_1, X_2, \ldots, X_n$, each of which is listable, then you can imagine ....

*Prof. Rotnac:* Tut tut, just a moment, I know what you are about to say: by listing the first element of the first, the first element of the second, etc. up the the first element of the last set, then going to the second element of the first set and so on, this shows that $X_1 \cup \cdots \cup X_n$ is listable!

*Prof. Cantor:* Sharp as the axe of the mightiest lumberjack of the Black Forest is your mind, Herr Dr Professor.

*Prof. Rotnac:* Very funny. Well, that takes care of quite a few infinite sets already!

*Prof. Cantor:* Yes, you can well imagine the next step: what if one has an infinite list of listable sets, is their union listable? Again, there is a simple way to show that this is so (I should say it's simple after one sees how to do it, but I can confess to my esteemed colleage because he is also an intimate friend that it took me many months to come up with this answer!). Namely, let us write $x_{11}, x_{12}, x_{13}, \cdots$ for the first list, then $x_{21}, x_{22}, x_{23}, \cdots$ for the second list and so on. So that $x_{ij}$ is the $j$th element in the $i$th list. Do you see the lists in your mind's eye?

*Prof. Rotnac:* Yes, yes, you have put them there beautifully. I see them as a 2-dimensional grid with the first list at the bottom going off to the right, the second list just above it and so on.

*Prof. Cantor:* You flatter me, but it's true that you imagine them just as I do. Well, now, let us say that the element $x_{ij}$ has "weight" $i + j$. The lightest element, so to speak, is $x_{11}$ and it has weight 2, the next two lightest are $x_{12}$ and $x_{21}$ of weight 3, then next three lightest are of weight 4, they are $x_{31}, x_{22}, x_{13}$. The next four lightest are of weight 5, they are $x_{14}, x_{23}, x_{32}, x_{41}$. And then ...

*Prof. Rotnac:* ICH HABE DEIN SPIEL DURCHSCHAUT, I can see your game! You list the elements by increasing weight, and since there are only finitely elements of each weight, you simply order those finitely many elements however you please as you go along. In fact, I think you have been zigzagging them!

*Prof. Cantor:* Exactly! I have always admired your quick uptake of ideas.

*Prof. Rotnac:* Well, as I said before, it appears you are well on your way to proving that every infinite set is listable!

*Prof. Cantor:* That is what I thought as well. Notably, with the "filtering by weight" idea I have just described, one sees that the set of rational numbers $\mathbb{Q}$ is listable. But alas, my success stalled there for quite some time, for I have been utterly unsuccessful in listing all the elements of the set of real numbers $\mathbb{R}$. So much so, that I now believe that this set is not listable!

*Prof. Rotnca:* Now now, do not be hasty. I understand why you are discouraged, but do not give up hope. No one has yet found the holy grail, but that does not prove that it does not exist. Patience, I am sure your fertile mind will soon devise a method for listing all the real numbers as well. After all, each real number can be expressed as infinite list of finitely many digits. Will not some refinement of your zigzagging on diagonals provide the answer?

*Prof. Cantor:* Ah, I have indeed come to the conclusion that it does provide the answer, but the answer is NOT the one that you expect my dear colleague.

*Prof. Rotnac:* You mean to say that you have found a proof that the real numbers are not listable?!

*Prof. Cantor:* Precisely.

*Prof. Rotnac:* (Unbelieving, but excited all the same) Come now, be serious. How can such a monstrous thing be true? You have piqued my curiosity. I am sure you are mistaken, however. Come, let us have your so-called "proof," and if it is not overly elaborate, within minutes we will have found a shortcoming in your argument or my name is not Groeg Rotnac.

*Prof. Cantor:* Very well, I must admit I would be relieved to find an error in the argument, which is indeed quite short, but I am afraid there is none. In fact we will prove that the set of real numbers bounded by 0 and 1, the so-called unit interval, already has so many elements that they cannot all be given in one infinite list. We will proceed by Widerspruch, contradiction. So, suppose you offer me a list $x_1, x_2, x_3, \ldots$ of real numbers between 0 and 1 and claim that **all** real numbers between 0 and 1 figure among your list. I will now show you that, with all due respect, you cannot have done so. Namely, I will produce a real number between 0 and 1 which cannot be on your list. You see what I wish to do?

*Prof. Rotnac:* Yes, I see what you wish to do, but cannot see how you will possibly achieve it!

*Prof. Cantor:* Neither could I for a long time. You will agree, readily, I believe, that by definition, each real number larger than 0 but smaller than 1 can be represented in a unique way with an infinite decimal expansion $c_1c_2c_3\ldots$ where the digits $c_i$ belong to the set $\{0,1,2,3,4,5,6,7,8,9\}$. Let us say we write the first number on your list as $x_1 = 0.a_{11}a_{12}a_{13}\ldots$ the second number on your list as $x_2 = 0.a_{21}a_{22}a_{23},\ldots$ so that the $j$th digit of the $i$th number on the list is $a_{ij}$. Here of course the digits $a_{ij}$ belong to the set $\{0,1,2,3,4,5,6,7,8,9\}$. Are you with me?

*Prof. Rotnac:* Yes, I am. Go on, please.

*Prof. Cantor:* Recall that my goal is to show you that some number in the unit interval must have been left out of the list. Here is how to do it. I will construct such a number $y = b_1b_2b_3\ldots$ by making sure that $y$ differs from each number in the list for at least one digit. In fact, I will choose it one digit at a time so that for each $j \geq 1$, $b_j$ is different from $a_{jj}$. For instance, if $a_{11}$ is not 5, I will choose $b_1$ to be 5, whereas if $a_{11}$ is 5 then I will choose $b_1$ to be some other digit, say 6 for the sake of definiteness. If $a_{22}$ is not 5, I will choose $b_2$ to be 5 and otherwise I will choose it to be 6. And so on. Thus, $y$ cannot be $x_1$ because they differ in the first digit, and $y$ cannot be $x_2$ because they differ in the second digit and so on. For any $j \geq 1$, $y$ cannot be $x_j$ because their $j$th digits do not match. Thus, $y$ (which will be some real number in the interval $[0.5555\ldots, 0.6666\ldots] = [5/9, 6/9]$) does not appear on the list $x_1, x_2, \ldots$ no matter how this list was constructed. This proves that the real numbers bounded by 0 and 1 are already too numerous to be put into one single list.

*Prof. Rotnac:* I am stunned! You have just shown that the set of real numbers is **more infinite** somehow than the set of natural numbers. You have opened my mind to a universe whose existence I did not even suspect! Infinite sets come in different sizes? Infinite sets come in different sizes.

*Prof. Cantor:* You believe the proof then?

*Prof. Rotnac:* Indeed I do. You are travelling down the diagonal modifying the digits as you go along. This "diagonal argument" of yours is simplicity itself. I congratulate you! Have you found any other sets that are not listable? Have you found sets that are even bigger than the set of real numbers?

*Prof. Cantor:* Actually, yes to both questions. Let $\mathcal{P}(\mathbb{N})$ be the set of all subset of $\mathbb{N}$. If you begin to list its elements, you will see that they are quite numerous. Other than the empty set and the complete set $\mathbb{N}$ itself, there is one infinite list of subsets of size 1, a 2-dimensinonal infinite grid of subsets of size 2, a 3-dimensional infinite grid of subsets of size 3 and so on. I have found a rather roundabout way of showing that $\mathcal{P}(\mathbb{N})$ is not listable, on the one hand, and also of showing that $\mathcal{P}(\mathbb{N})$ is of the same size as the set of real numbers on the other. Let me first explain why $\mathcal{P}(\mathbb{N})$ is not listable, the term I prefer, incidentally is "countable" in which category I also include the finite sets. So, here is why $\mathcal{P}(\mathbb{N})$ is not countable. If if were, then there would be a bijection, which is in particular a surjection, form $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. But I claim that for any set $X$, and any map $f : X \to \mathcal{P}(X)$, $f$ is not surjective.

*Prof. Rotnac:* Let me see now, you are saying that if $f : X \to \mathcal{P}(X)$ is any map, then there is some subset $Y$ of $X$ such that $Y \neq f(x)$ for all $x \in X$?

*Prof Cantor:* Admirably put, yes. Here is a set $Y$ which is always missed by $f$, so to speak. Consider the subset $Y_f$ of $X$ which consists of those elements $x \in X$ such that $x$ does not belong to $f(x)$.

*Prof. Rotnac:* That is confusing. Let me see, to each $x$ in $X$ we have an associated subset $f(x)$ of $X$ and you are saying that you admit $x$ to the set $Y_f$ if and only if $x$ itself does **not** belong to the subset $f(x)$ associated to it by $f$. Yes, I see your definition of $Y_f$ is rather bizarre but gives a pefectly valid subset of $X$.

*Prof Cantor:* Very good. Now I claim that $Y_f$ is not in the image of $f$, i.e. is not of the form $f(y)$ with $y \in X$. Why, you ask? Recall that for any given $x \in X$, $x$ belongs to $Y_f$ if and only if $x$ does not belong to $f(x)$. Recall that $y$ belongs to $Y_f$ if and only if $y$ does not belong to $f(y)$. So if we had $f(y) = Y_f$, then this last statement would become "$y$ belongs to $Y_f$ if and only if $y$ does not belong to $Y_f$." This is absurd! Thus, for all $y \in X$, $f(y) \neq Y_f$ and so $f$ is not surjective.

*Prof. Rotnac:* Just when I think you have ceased surprising me, you astound me with an even more devious proof! That is brilliant. Moreover, with this new proof, not only do you have **two** infinite sets of different size, but you can make as many infinite sets of different size as you want.

*Prof. Cantor:* I can? How can I?

*Prof. Rotnac:* Well, say you start with $\mathbb{N}$, then $\mathcal{P}(\mathbb{N})$ is more numerous, then $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ is more numerous still, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))$ is more numerous still and so on!

*Prof. Cantor:* Yes, of course, that is truly marvelous! Thank you my friend for that wonderful observation.

*Prof. Rotnac:* You had already thought of that yourself, hadn't you?

*Prof. Cantor:* To be truthful, yes, but after much thinking. Whereas you made the leap immediately, and I wanted to share in the excitement of your discovery.

*Prof. Rotnac:* One thing puzzles me still. How does $\mathcal{P}(\mathbb{N})$ compare with the set of real numbers? Which one is bigger?

*Prof. Cantor:* Actually, they are both of the same size!

*Prof. Rotnac:* Is that so? Ah yes, you already said that they were. But I don't see why.

*Prof. Cantor:* Well, first of all, there is a natural one-to-one correspondence between $\mathcal{P}(\mathbb{N})$ and $\text{Maps}(\mathbb{N}, \{0, 1\})$, that is to say the set of all maps from $\mathbb{N}$ to the binary set $\{0, 1\}$. Here is how to ....

*Prof. Rotnac:* Just a moment, forgive me for interrupting but I beg you do not tell me! I want to figure this out on my own. Let's see now. You are saying that to give a subset of $\mathbb{N}$ is the same as giving a map from $\mathbb{N}$ to $\{0, 1\}$, eh? Hmm... to give a subset $X$ of $\mathbb{N}$ means to admit certain natural numbers to $X$ and exclude others. And to give a map from $\mathbb{N}$ to $\{0, 1\}$ means that certain natural numbers will be assigned the number 1 and others will be assigned the number 0. I have got it! Being assigned a "1" can mean admit to $X$ and being assigned a "0" means you are to be excluded from $X$. To be perfectly formal, here is a bijective map $F : \mathcal{P}(\mathbb{N}) \to \text{Maps}(\mathbb{N}, \{0, 1\})$. If $X \subseteq \mathbb{N}$, we have to describe a map $F(X) \in \text{Maps}(\mathbb{N}, \{0, 1\})$. We define $F(X) : \mathbb{N} \to \{0, 1\}$ by putting $F(X)(n) = 1$ if $n \in X$ and $F(X)(n) = 0$ if $n \notin X$. The map $F$ is bijective because, for example, it has an inverse $G : \text{Maps}(\mathbb{N}, \{0, 1\})$ defined in a simple way. Given a map $f : \mathbb{N} \to \{0, 1\}$, we let $G(f) = \{n \in \mathbb{N} \mid f(n) = 1\}$. Then $F(G(f))$ is clearly the map $f$ and $G(F(X))$ is just the set $X$.

*Prof. Cantor:* Masterfully done, I am sure. So, as I was saying, $\mathcal{P}(\mathbb{N})$ is in one-to-one correspondence with $\text{Maps}(\mathbb{N}, \{0, 1\})$ and the set of maps from $\mathbb{N}$ to $\{0, 1\}$ is in one-to-one correspondence with the interval $[0, 1]$.

*Prof. Rotnac:* There you go confusing me again, just fresh from my triumph. Give me a moment to think...You have a string of 0's and 1's and you wish to assign to it a number between 0 and 1 in such a way that every real number is covered once. Hmm... If you had a string of digits between 0 and 9 then I would just say write out the decimal expansion and that would do the trick, but you are giving me only 0's and 1's.

*Prof. Cantor:* Pray remind me, what is the decimal expansion?

*Prof. Rotnac:* I assume you are playing dumb for my benefit. Very well, I will play along. If $a_1, a_2, a_3, \ldots$ is a sequence of digits from the set $\{0, 1, \ldots, 9\}$, then we get a corresponding decimal expansion $0.a_1a_2a_3 \cdots$ which is defined to be the real number $a_1/10 + a_2/100 + a_3/1000 + \cdots$ and this converges to a number in the interval $[0, 1]$. Every number in $[0, 1]$ can be represented as a decimal expansion in exactly one way, with $0 = 0.000000 \cdots$ and $1 = 0.99999 \cdots$.

*Prof. Cantor:* Excellent. You are nearly there, meine Kollege. I am only giving you two symbols instead of ten, so instead of representing numbers in base ten (decimal expansion), you should represent them ...

*Prof. Rotnac:* In Base Two, natürlich! That's it. Allow me describe it in complete detail. There is a bijective map $\alpha$ from $\text{Maps}(\mathbb{N}, \{0, 1\})$ to $[0, 1]$ defined as follows: Given a map $f : \mathbb{N} \to \{0, 1\}$, we associate a number $\alpha_f \in [0, 1]$ to it by putting $\alpha_f = f(1)/2 + f(2)/4 + f(3)/8 + \cdots + f(n)/2^n + \cdots$. This series converges to a number in $[0, 1]$. Note that the smallest possible $\alpha_f$ occurs for the function $f_0$ with constant value 0 giving $\alpha_{f_0} = 0$ and the greatest possible value occurs for the function $f_1$ with constant value 1 giving $\alpha_{f_1} = 1/2 + 1/4 + \cdots = 1$.

To show that every real number in between 0 and 1 is in the image of $\alpha$, i.e. has a binary expansion, we must construct a map $\beta : [0,1] \to \text{Maps}(\mathbb{N}, \{0, \})$ which just gives the binary (instead of the decimal) expansion of a real number. Namely, if $x \in [0,1]$, we define a map $\beta_x : \mathbb{N} \to \{0,1\}$ as follows. We know that $0 \leq x \leq 1$ so we try to determine which half of this range contains $x$. Namely, if $x < 1/2$, we put $x_1 = 0$ and if $x \geq 1/2$, we put $x_1 = 1$. Now we have $0 \leq x - x_1/2 \leq 1/2$, so we ask which half of this interval $x - x_1/2$ belongs to, i.e. if $x - x_1/2 < 1/4$, we put $x_2 = 0$, otherwise we put $x_2 = 1$. Then if $x - x_1/2 - x_2/4 < 1/8$ we put $x_3 = 0$, otherwise $x_3 = 1$. In general, having found $x_i$ for $1 \leq i \leq n$, we define $x_{n+1}$ by putting $x_{n+1} = 0$ if $x - x_1/2 - x_2/4 - \ldots - x_n/2^n < 1/2^{n+1}$ and $x_{n+1} = 1$ otherwise. It is easy to see that $x - x_1/2 - x_2/4 - \cdots - x_n/2^n \leq 1/2^n$ and $1/2^n$ goes to 0 of course as $n$ becomes large, hence the series $x_1/2 + x_2/4 + \cdots$ converges to $x$ by construction. If we define $f_x : \mathbb{N} \to \{0,1\}$ by $f_x(n) = x_n$, then $x = f(1)/2 + f(2)/4 + f(3)/8 + \cdots + f(n)/2^n + \cdots$, so that with $\beta(x) = f_x$, we have $\beta$ is the inverse of $\alpha$, and so $\alpha$ is bijective.

*Prof. Cantor:* Dotting the i's and crossing the t's perfectly as usual. So now we see that $\mathcal{P}(\mathbb{N})$ is in one-to-one correspondence with the closed interval $[0,1]$ of real numbers. Now that we have seen infinite sets come in different sizes, I have begun to give names to sizes which infinite sets can take, I call these sizes "cardinal numbers" or "cardinalities." Sets which are in one-to-one correspondence with $\mathbb{N}$ I call of size $\aleph_0$. Those which are in one-to-one correspondence with $\mathcal{P}(\mathbb{N})$ I call of size $\aleph_1$. Those which are in one-to-one correspondence with $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ I call of size $\aleph_2$ and so on.

*Prof. Rotnac:* So the set of real numbers is of which cardinality?

*Prof. Cantor:* Of cardinality $\aleph_1$. For it is easy to see that the set of all real numbers is in one-to-one correspondence with those in the interval $(0,1)$ and the latter is in one-to-one correspondence with $[0,1]$. I have two different proofs that $[0,1]$ is more numerous than $\mathbb{N}$, one via what you call the diagonal argument, and the other based on the fact that $[0,1]$ is equivalent to $\mathcal{P}(\mathbb{N})$ and no surjective map $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ can exist. The two proofs are in fact related in a simple way.

*Prof. Rotnac:* Which one is your favorite?

*Prof. Cantor:* You might imagine that I am more fond of the second, because it allows me to see that there is a whole infinite sequence of infinite cardinals, but the diagonal argument allowed me for the first time to show that infinite sets come in different sizes, so it is particularly dear to me.

*Prof. Rotnac:* My dear friend, you have uncovered eine grundsätzliche Wahrheit, a fundamental truth of nature, which has eluded our science for thousands of years! This is heady stuff, Georg! Have you written to our friend Richard about your findings?

*Prof. Cantor:* I am so pleased that you are in agreement with my results and methods. I have written to Herr Dedekind some of my discoveries, but now that I have your reassurance, I will write to him an expanded version of my results. Speaking of heady stuff, here we are in front of our favorite pub, Wirsthaus Mozart. Shall we go in for a celebratory pint of something cold?

*Prof. Rotnac:* We shall do more than that, meine freund! We will go on a Bierreise machen, a pub crawl to celebrate your results. And I will pay!

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**Definition 1.4.** A set $X$ is called *countable* if it is either finite or equivalent to $\mathbb{N}$. A set is called uncountable if it is not countable. An infinite countable set, i.e. one which is equivalent to $\mathbb{N}$, is also sometimes called countably infinite and is said to have size $\alpha_0$ (read "aleph nought" or "aleph null.") A set which is equivalent to $\mathcal{P}(\mathbb{N})$ is said to have size $\aleph_1$.

**Lemma 1.5.** *A set $X$ is countable if there exists a single infinite list (with or without repetition) which contains all the elements of $X$.*

*Proof.* First suppose $X$ is countable. If it is finite, then certainly its elements can be listed. If it is infinite, then by definition, there is a bijection $f : \mathbb{N} \to X$ hence $f(1), f(2), \dots$ is a single infinite list containing all the elements of $X$ (with no repetition in fact). In the other direction, suppose $x_1, x_2, x_3, \dots$ is an infinite list (possibly with repetition) with the property that for all $x \in X$, there exists $n \in \mathbb{N}$ such that $x = x_n$. If $X$ is finite, then it is certainly countable. Now suppose $X$ is infinite. From the list $x_1, x_2, \dots$, we easily produce a list *without* repetition that contains all the elements of $X$. Namely, let $n_1 = 1$ and put $y_1 = x_1$. Let $y_2 = x_{n_2}$ where $n_2 = \min\{n > n_1 | x_n \notin \{y_1\}\}$. Let $y_3 = x_{n_3}$ where $n_3 = \min\{n > n_2 | x_n \notin \{y_1, y_2\}\}$. Similarly, having defined $n_1, n_2, \dots, n_k$ and $y_k$ as above, we define $n_{k+1} = \min\{n > n_k | x_n \notin \{y_1, y_2, \dots, y_k\}\}$ and put $y_{k+1} = x_{n_{k+1}}$. Then the map $f : \mathbb{N} \to X$ defined by $f(m) = y_m$ is a bijection, so $X$ is countable. □

**Theorem 1.6.** *(a) If $X_1, \dots, X_N$ is a finite collection of countable sets, then $X_1 \cup \dots \cup X_N$ is countable.*

*(b) If $X_1, X_2, \dots$ is a countably infinite collection of countable sets, then $\cup_{n=1}^{\infty} X_n$ is countable.*

*The two parts can be summarized by saying that "a countable union of countable sets is countable."*

*Proof.* The proof of both assertions was given in the discussion between Cantor and his friend above. □

In practice, mathematicians encounter finite sets, countable sets, and sets of size $\aleph_1$ on an everyday basis. Sometimes when we want to compare two infinite sets to see if there could be some kind of relationship between them, it's a good first question to ask if they are both countable.

Recall that for infinite sets, we have given a meaning to the equality $|X| = |Y|$, namely this means that $X \sim Y$, i.e. there is a bijection from $X$ to $Y$. We will also define a partial ordering on the size of sets, as follows.

**Definition 1.7.** If $X$ and $Y$ are sets, we write $|X| \leq |Y|$ if there exists an injective map $X \hookrightarrow Y$. We write $|X| < |Y|$ if there exists an injective map $X \hookrightarrow Y$ but no bijective map $X \xrightarrow{\sim} Y$ exists.

Note that the partial ordering we have defined on sizes of sets is transitive. Namely, suppose $|X| \leq |Y|$ and $|Y| \leq |Z|$. Then does it follows that $|X| \leq |Z|$? Yes, for suppose $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow Z$ are injections. Then $g \circ f$ injects $X$ into $Z$. But now suppose $|X| \leq |Y|$ and $|Y| \leq |X|$. We would expect in this case that $|X| = |Y|$. In other words, if $X$ injects into $Y$ and $Y$ injects into $X$, then there is a bijection from $X$ to $Y$. Cantor was not able to show this, but in 1898, Schroeder and Bernstein succeeded in demonstrating this property. Here is their theorem.

**Theorem 1.8** (Schroeder-Bernstein). *(a) Suppose $B \subseteq A$ and there is an injection $f : A \to B$. Then there is a bijection $h : A \xrightarrow{\sim} B$.*

*(b) Suppose there is an injective map $f : X \hookrightarrow Y$ and an injective map $g : Y \hookrightarrow X$. Then there is a bijection $h : X \xrightarrow{\sim} Y$.*

*(c) Suppose $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$.*

*Proof.* The proof that follows was adapted from `http://planetmath.org`

(a) Inductively define a sequence $(C_n)$ of subsets of $A$ by $C_0 = A \setminus B$ and $C_{n+1} = f(C_n)$. Then the $C_n$ are pairwise disjoint. We will prove this by contradiction. Suppose the countable collection of sets $C_0, C_1, C_2, \ldots$ is not pairwise disjoint. Let

$$Z = \{j \geq 0 | C_j \cap C_k \neq \emptyset \text{ for some } k > j\}.$$

We have assumed that $Z$ is not empty and are seeking a contradiction. By the Well-Ordering Principle, since $Z$ is non-empty, it has a least element, call it $m$. Let $k > m$ be the least integer larger than $m$ such that $C_m \cap C_k \neq \emptyset$; it exists because $m \in Z$. Since $C_0$ is disjoint with any following $C_n$, we have $0 < m$ and thus that $C_m = f(C_{m-1})$ and $C_k = f(C_{k-1})$. But this implies that $f(C_{m-1} \cap C_{m-1}) \neq \emptyset$ and so $C_{m-1} \cap C_{k-1}$ cannot be empty, hence $m-1 \in Z$, contradicting the minimality of $m$. This contradiction shows that the collection $C_0, C_1, \ldots$ is pairwise disjoint.

Now let $C = \bigcup_{k=0}^{\infty} C_k$, and define $h : A \to B$ by

$$h(z) = \begin{cases} f(z), & z \in C \\ z, & z \notin C \end{cases}.$$

If $z \in C$, then $h(z) = f(z) \in B$. But if $z \notin C$, then $z \in B$, and so $h(z) \in B$. Hence $h$ is well-defined; $h$ is injective by construction. Let $b \in B$. If $b \notin C$, then $h(b) = b$. Otherwise, $b \in C_k = f(C_{k-1})$ for some $k \geq 0$, and so there is some $a \in C_{k-1}$ such that $h(a) = f(a) = b$. Thus $b$ is bijective; in particular, if $B = A$, then $h$ is simply the identity map on $A$.

Now (b) is a simple consequence of (a). Suppose $f : X \to Y$ and $g : Y \to X$ are injective. Then the composition $g \circ f : X \to g(Y)$ is also injective. By the lemma, there is a bijection $h' : X \to g(Y)$. The injectivity of $g$ implies that $g^{-1} : g(Y) \to Y$ exists and is bijective. Define $h : X \to Y$ by $h(z) = g^{-1} \circ h'(z)$; this map is a bijection.

For (c), we simply apply (b) to conclude that if $X$ injects into $Y$ and $Y$ injects into $X$, then $X$ and $Y$ have the same cardinality. □

Note that if $X$ is an infinite set, then $\mathbb{N}$ injects into $X$, hence for all infinite sets, $|\mathbb{N}| \leq |X|$. The picture that emerged from Cantor's research is the following schematic of sets classified according to their size.

| $\{\}$ | $\{1\}$ | $\{1,2\}$ | $\{1,2,3\}$ | $\cdots$ | $\mathbb{N}$ | $\mathcal{P}(\mathbb{N})$ | $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ | $\cdots\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | $\cdots$ | $\aleph_0$ | $\aleph_1$ | $\aleph_2$ | $\cdots\cdots$ |

Cantor had established that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are of size $\aleph_0$ and that $\mathcal{P}(\mathbb{N}), \mathbb{R}$ and $[0, 1]$ are of size $\aleph_1$. Cantor believed that there were no sets of size strictly between $\aleph_0$ and $\aleph_1$, in other words, he believed that if $X$ is an uncountable subset satisfying $|\mathbb{N}| < |X| \leq |\mathbb{R}|$, then $X$ is equivalent to $\mathbb{R}$. Equivalently, he believed that every uncountable subset of $\mathbb{R}$ is equivalent to $\mathbb{R}$. This statement became known as the Continuum Hypothesis. Following fundamental work of Curt Gödel in 1940 and that of Paul Cohen in 1962, the Continuum Hypothesis was found to be in a most mysterious class of problems, those which are "undecidable."

What this means in practice, is that believing the Continuum Hypothesis to be true does not lead to any contradictions within the totality of all logical consequences of the standard axioms of set theory, and the same can be said of believing the Continuum Hypothesis to be false! In other words, there are two perfectly consistent paradigms in which to do set theory: one in which we accept the standard axioms of set theory plus the Continuum Hypothesis and another in which we accept the standard axioms of set theory plus the negation of the Continuum Hypothesis! In practice, most problems that mathematicians work on are not affected one way or another by which of these these two set-theoretical paradigms we work in, because the sets we usually deal with are, so to speak, too small. However, in the field of mathematical logic, one of the objects of study to consider ramifications of adopting one set of logic axioms versus another. Please note, with my apologies!, that I grossly misrepresented the status of knowledge concerning the Continuum Hypothesis in class, although I stated the Hypothesis itself correctly.

## 2. Problems

1. Recall that $\mathrm{Maps}(X,Y)$ is the set consisting of all maps from $X$ to $Y$.

(a) With $X = \{a,b,c\}$ and $Y = \{0,1\}$, list all elements of $\mathrm{Maps}(X,Y)$ explicitly. What is $|\mathrm{Maps}(X,Y)|$?

(b) Suppose $X, Y$ are finite sets. Use the "Multiplication Counting Principle" to show that

$$|\mathrm{Maps}(X,Y)| = |Y|^{|X|}.$$

(c) If $Y = \{0,1\}$, then $|\mathrm{Maps}(X,Y)| = 2^{|X|}$.

(d) By thinking of "0" as "does not belong to the subset" and "1" as "belongs to the subset," show that the sets $\mathcal{P}(X)$ and $\mathrm{Maps}(X,\{0,1\})$ are equivalent.

(e) Use (c) and (d) to conclude that $|\mathcal{P}(X)| = 2^{|X|}$. Now perhaps you understand why the power set is called the power set.

2. (a) Suppose $X$ and $Y$ are two finite sets and $|X| = |Y|$. Show that $X \sim Y$, i.e. show that there is a bijection from $X$ to $Y$.

(b Suppose $X$ and $Y$ are two infinite countable (also called "countably infinite") sets. Prove that $X \sim Y$.

3. Aaron and Garret are going to a big city of 500,000 inhabitants for spring break. On the way there, they have the following conversation.

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*Garret:* Hey Aaron, what do you think the chances are that in this city, you can find two people that have exactly the same number of hairs on their heads?

*Aaron:* One Hundred Percent, my friend.

*Garret:* How can you be so sure?

*Aaron:* Because I happen to know that no human being has more than 250,000 hairs on their head.

*Garret:* Oh, well in that case you are right. But can we be sure to find **three** people all with the exact same number of hairs on their heads?

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(a) Explain why Aaron is right.

(b) Answer Garret's final question.

4. For each of the following subsets, state whether the set has a least element or not. If it does, determine the least element.
(a) $\{x \in \mathbb{R} | x > 0 \text{ and } 5x \in \mathbb{Z}\}$
(b) $\{x \in \mathbb{R} | 5x \in \mathbb{Z}\}$
(c) $\{x \in \mathbb{R} | x = 1.012\square\square\square\cdots\}$ (In other words the set consists of those real numbers having a base 10 expansion that begins with 1.012).
(d) $\bigcup_{n\in\mathbb{N}}\{x \in \mathbb{R} | x > 0 \text{ and } nx \in \mathbb{Z}\}$

5. Suppose $S \subseteq \mathbb{N}$ is a subset of the set of the set of natural numbers with the following two properties: i) $1 \in S$ and ii) If $n \in S$ is any element of $S$, then $n + 1 \in S$.
Use the Well-ordering Principle to give a rigorous proof of the fact that $S = \mathbb{N}$.
Hint: Let $X = \mathbb{N} \setminus S$. You want to prove that $X = \{\}$. Use proof by contradiction.

6. Prove that $\mathbb{N} \times \mathbb{N}$ is countable by explicitly describing a bijective map $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

7. Show that if $X$ is a countable set, and $Y \subseteq X$, then $Y$ is countable.

8. Give a bijection from $(0, 1) = \{x \in \mathbb{R} | 0 < x < 1\}$ to $\mathbb{R}$, thereby showing that $|(0, 1)| = |\mathbb{R}|$. Hint: think about a function that has an asymptote going to $-\infty$ near 0 and one going to $+\infty$ near 1.

9. (a) Show that if $X$ and $Y$ are countable sets, then $X \cup Y$ is a countable set. (Hint: if $X$ and $Y$ are both countably infinite, say $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$, then *interleave the two sequences* (the way the odds and evens are interleaved)).
(b) Let $I = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Prove that $I$ is uncountable. (Hint: Proof by contradiction is your friend).

10. (a) Use interleaving to give a bijection $(0, 1) \times (0, 1) \to (0, 1)$.
(b) Show (a) and 8 to prove that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$, another great achievement of Cantor.

11. Suppose $X$ is finite and $Y$ is countably infinite. Show that $X \times Y$ is countably infinite.

12. Suppose $X$ is a non-empty set and $f : X \to \mathcal{P}(X)$ is defined $f(x) = X \setminus x$. Consider the subset $Y_f = \{x \in X | x \notin f(x)\}$ of $X$ (which plays a prominent role in Cantor's theorem). Determine $Y_f$ for the particular $f$ we have just defined.

13. (a) Convert the rational number 147.05 (written in base ten) to base 4.
(b) Convert the base 3 rational number $(120.\overline{21})_3 = (120.21212121\cdots)_3$ to base ten.