UMASS AMHERST MATH 300 SP '05, F. HAJIR

FINAL EXAM REVIEW

Mathematics is cumulative, so concepts we have learned throughout the course will appear on the final. However, there will be a decided imbalance toward having more questions from the last few weeks of the course, specifically from HW 7 (Number Theory) HW 8 (Relations) and HW 9 (Complex Numbers). Correspondingly, on this sample exam, I have included mostly questions from HW 7,8,9, but you should review exams 1 and 2 as well as the sample exams to recall the material we discussed earlier in the term.

As with Exams 1 and 2, the final will have three parts: 1. Defintions, 2. Short Answer, and 3. Problems. The problems will usually but not always require you to write a cogent, concise and correct proof. Some of the problems will be statements that were already proved in class or are taken directly from homework. But at least some of the problems will require you to prove a statement that has not been presented to you before, or to find a counterexample to a statement.

For the definitions, it is important to be extremely precise. For instance if I ask you define what it means for $f : X \to Y$ to be surjective, the response "f is surjective means that for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$" receives full credit, and "everyting in Y gets hit by somebody in X" receives only partial credit because "gets hit by" is not sufficiently precise.

Be sure to memorize the definitions well enough so that you can just rattle them off. Many of you lost a large number of points on the Definitions on Exam 1. Also, many of you spent too much time on the Short Answers, which do not count as much as the Problems, then ran out of time.

Here begineth the sample exam.

The points are distributed approximately as follows: 25% Definitions, 25% Short Answer, and 50% Problems.

You may wish to give yourself 2 hours and take this exam in a quiet room without notes under the time constraint (or not, this is just a suggestion; it may be a good suggestion for some students and not so good for others). The actual exam will be somewhat similar but not identical to this one in length and in the variety of the problems.

Sample Final Exam

1. Definitions

[All the definitions I asked you on exams 1 and 2 as well as on the sample exams]

a. If a and b are integers, $\gcd(a, b)$ is

b. Two integers a, b are relatively prime if

c. We say $a|b$ if (don't just rephrase, give the actual mathematical property defining this concept)

d. We say $a \equiv b \bmod n$ if
e. State Bézout's theorem.
f. If X is a set equipped with an equivalence relation $\sim$, then the set $\widetilde{X}$ is defined to be
g. A relation $\sim$ from a set X to itself is said to be transitive (h. reflexive; i. symmetric, j. an equivalence relation) if
k. If $\sim$ is a relation from X to Y, and $x \in X$, then the fiber above x is defined by $R_{x,\bullet} =$
l. If $\sim$ is an equivalence relation on X, then a $\sim$-equivalence class is
m. If $x, y \in \mathbb{R}$, the real part (n. imaginary part; o. modulus; p. complex conjugate) of $z = x + iy$ is
q. The triangle inequality states that if $w, z \in \mathbb{C}$, then

2. Short Answers

a. The quantity $\min\{x \in \mathbb{Z} \mid a|x \wedge b|x\}$ is called
b. In the above sentence, this minimum is guaranteed to exist by the .... Principle because ....
c. Use the Euclidean algorithm to compute $\gcd(432, 168)$ as well as $\operatorname{lcm}(432, 168)$.
d. Sketch and shade the region in the complex plane defined by

$$R = \{z \in \mathbb{C} \mid |z - i| < |z + i|\}.$$

e. For $w \in \mathbb{C}$, define a map $\delta_w : \mathbb{C} \to \mathbb{C}$ via $\delta_w(z) = wz$ for all $z \in \mathbb{C}$. If $w = 1 + i\sqrt{3}$, the map δ_w can be represented by a radial dilation by a factor .... followed by a counterclockwise rotation around the origin of measure ....
f. Suppose z, w, v are three complex numbers such that $|z - w| = |z - v| + |v - w|$. Draw a picture of what this means geometrically. What can you conclude about the geometric configuration of the complex numbers z, w, v?
g. Write the complex number $z = (7 + 4i)/(3 - 2i)$in polar form, i.e. find real numbers r, θ such that $z = re^{i\theta}$.
h. TRUE or FALSE: If $z_0 \in \mathbb{C}$, then the equation $w^5 = z_0$ has 5 distinct solutions in $\mathbb{C}$.
i. TRUE or FALSE: If X is a countable set, and Δ is a partition of X, then Δ is also a countable set.
j. TRUE or FALSE: If $x, y \in \mathbb{Z}$, and $3x + 17y = 2$, then $\gcd(x, y)$ is either 1 or 2.
k. TRUE or FALSE: The equation $3x + 18y = 1$ has no solution with $x, y \in \mathbb{Q}$.

3. Problems

A. On the set $X = \mathbb{R}$ of real numbers, we define a relation $\sim$ as follows: for $x, y \in \mathbb{R}$ we have $x \sim y$ if and only if $x - y \in \mathbb{Z}$.
(i) Prove that $\sim$ is an equivalence relation.
(ii) Prove that $[0, 1) \to \widetilde{X}$ given by $t \mapsto \widetilde{t} = \mathbf{cl}(t)$ is a bijection.

B. Suppose $r, s, m, n \in \mathbb{Z}$ and $\gcd(m, n) = 1$.
(i) Show that the set

$$\{x \in \mathbb{Z} \mid x \equiv r \bmod m \wedge x \equiv s \bmod n \wedge 1 \leq x \leq mn\}$$

is a singleton. In other words, there exists a unique integer in the interval $[1, mn]$ that gives remainder r when divided by m and remainder s when divided by n.

Hint: By Bézout, we can find a, b such that $am + bn = 1$. Now try $x = ams + bnr$. It satisfies two of the three needed conditions. How do you "fix" it to get the third condition?

(ii) How many integers in the interval $[1, 3000]$ give remainder 73 when divided by 100 and remainder 1 when divided by 3? What is the least such integer?

C. Let p be a prime number. Let $X = \mathbb{Z}$ be the set of integers, and for $x, y \in \mathbb{Z}$, write $x \sim y$ if and only if $x \equiv y \bmod p$, i.e. if and only if $p|(x-y)$. Thus, the set $\widetilde{X}$ has p elements, namely $\widetilde{0}, \widetilde{1}, \ldots, \widetilde{p-1}$. On the set $\widetilde{X}$, let us define two operations, $+, \times$ as follows:

$$\widetilde{a} + \widetilde{b} := \widetilde{a+b}, \qquad \widetilde{a}\widetilde{b} := \widetilde{ab}.$$

Show that if $\widetilde{a} \neq \widetilde{0}$, then there exists $b \in \mathbb{Z}$ such that $\widetilde{a}\widetilde{b} = \widetilde{1}$.

D. Suppose $\sim$ is an equivalence relation on X. Suppose S, T are $\sim$-equivalence classes. Prove that either $S = T$ or $S \cap T = \emptyset$.

E. State the triangle inequality, then use it to prove that for $u, v \in \mathbb{C}$,

$$|u| - |v| \leq |u - v|.$$

F. Find six complex roots of the equation $z^6 + z^3 + 1 = 0$. Hint: let $w = z^3$ so that $w^2 + w + 1 = 0$. Solve for w and put the solutions w_1, w_2 in $re^{i\theta}$ form, then solve $z^3 = w_1$ and $z^3 = w_2$.

G. (i) Suppose $z_0, z_1 \in \mathbb{C}$ and $z_0 \neq z_1$. Consider a map $z : [0, 1] \to \mathbb{C}$ defined by $z(t) = tz_1 + (1-t)z_0$ for $0 \leq t \leq 1$. Note that $z(0) = z_0$ and $z(1) = z_1$. Thinking of this map as a path in the complex plane, describe (geometrically) what this path is.

(ii) Let $B = \{z \in \mathbb{C} \mid |z| < 1\}$ be the inside of the unit circle; it's called the unit disc. Use the triangle inequality to show that, given two distinct points $z_0, z_1 \in B$, every point of the line segment joining z_0 to z_1 is inside B also. (This is clear geometrically, I am asking for an "algebraic" proof). You have just shown that the unit disc is *convex*.

4. Extra Credit

A. Suppose $n \geq 2$ is an integer. Write down an explicit formula for a non-identity 2 by 2 matrix M with real entries such that $M^n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.