UMASS AMHERST MATH 300 SP '05, F. HAJIR

EXAM 2 REVIEW

Mathematics is cumulative, so concepts we have learned throughout the course will appear on Exam 2. However, the primary material for Exam 2 is as follows: the counting principles from HW4, all of HW 5, and all of HW 6. You should also know the definition of divisibility, which already came up much earlier in the course. However, more sophisticated topics from number theory, such as primes, gcd, Euclidean algorithm etc. will not appear on this exam. Thus, the main topics covered are: Well-ordering principle, the Pigeonhole Principle, the Principle of Induction (including complete induction), Cantor's theory of cardinality of sets, and how to write numbers in different bases. We also discussed ever so briefly the multiplication counting principle but it is unlikely that I would ask any questions regarding that topic this time.

As with Exam 1, Exam 2 will have three parts: 1. Defintions, 2. Short Answer, and 3. Problems. The problems will usually but not always require you to write a cogent, concise and correct proof. Some of the problems will be statements that were already proved in class or are taken directly from homework. But at least some of the problems will require you to prove a statement that has not been presented to you before, or to find a counterexample to a statement.

For the definitions, it is important to be extremely precise. For instance if I ask you define what it means for $f : X \to Y$ to be surjective, the response "f is surjective means that for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$ receives full credit, and "everyting in Y gets hit by somebody in X" receives only partial credit because "gets hit by" is not sufficiently precise.

Be sure to memorize the definitions well enough so that you can just rattle them off. Many of you lost a large number of points on the Definition on Exam 1. Also, many of you spent too much time on the Short Answers, which do not count as much as the Problems, then ran out of time.

Here begineth the sample exam.

The points are distributed approximately as follows: 25% Definitions, 25% Short Answer, and 50% Problems.

You may wish to give yourself 1.5 hours and take this exam in a quiet room without notes under the time constraint (or not, this is just a suggestion; it may be a good suggestion for some students and not so good for others). The actual exam will be somewhat similar but not identical to this one in length and in the variety of the problems. Make sure you attend the Review Session run by Molly on Monday April 18 if you have any questions!

Sample Exam 2

1. Definitions

The Well-ordering Principle states that
The Principle of Mathematical Induction states that
The Pigeon-hole Principle states that
Cantor's theorem states that
The Schroeder-Bernstein Theorem states that
The set of rational numbers $\mathbb{Q}$ is defined by
A set X is countable means that
A set X is uncountable means that
A set X is finite means that
If X is a set, $|X| = \aleph_0$ means that
If X is a set, $|X| = \aleph_2$ means that
If X and Y are arbitrary sets, $|X| = |Y|$ means that
If X and Y are arbitrary sets, $|X| \leq |Y|$ means that

2. Short Answer

An office building has ten floors, numbered 1 through 10. Ten people walk into the elevator on floor 1. Can you conclude that at least two of the ten people are headed to the same floor? Explain.

There are 25 pennies in a jar. Twelve of them are dated 1984. Nine of them are dated 2000, and Four of them are dated 2005. What is the least number of pennies you have to draw from the jar in order to guarantee that you will have at least four pennies with the same date?

Write down two **uncountable** sets, X and Y, which are not equivalent to each other.

Give a bijection $(0,1) \to \mathbb{R}$.

Consider the set $X = \{1,2,3,4,5\}$ and the map $f : X \to \mathcal{P}(X)$ defined by $f(1) = \{4\}$, $f(2) = \{3,4\}$, $f(3) = \{2,3,4\}$, and $f(4) = \{1,2,3,4\}$. Calculate $Y_f = \{x \in X \mid x \notin f(x)\}$. Is Y_f in the image of f? Are you surprised by this? Why or why not?

Write the number 0.123 in base 5.

TRUE OR FALSE: For each statement below, Indicate whether it is True or False.
Every infinite subset of an uncountable set is uncountable.
For *arbitrary* sets X, Y, Z, if $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.
If X is an infinite set, then there is a injection $X \to \mathbb{N}$.
If $|X| = \aleph_0$, then $|X \times X| = \aleph_0$.
If X is equivalent to Y, then there is an injection $X \hookrightarrow Y$ as well as an injection $Y \hookrightarrow X$.
If X is a countable set, then every infinite subset of X is equivalent to X.

3. Problems

Define a sequence of integers g_n for $n \geq 1$ by the following recursion rule:

$$g_1 = 1, \qquad g_{n+1} = g_n + n - 1 \qquad \text{for all } n \geq 1.$$

Use the principle of mathematical induction to prove that $g_n = n(n+1)/2$ for all integers $n \geq 1$.

Use the principle of mathematical induction to prove that $3 | 2^{2n} - 1$ for all $n \geq 1$.

Use the principle of mathematical induction to prove that for all integers $n \geq 1$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Prove Cantor's theorem: If X is an arbitary set, and $f : X \to \mathcal{P}(X)$ is a map from X to the set of all subsets of X, then f is not surjective. Hint: Proof by contradiction.

Describe a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, thereby proving that $\mathbb{N}$ is countable. (Drawing a picture is a good idea, but it should be accompanied by a careful description of the map you are constructing).

Using Cantor's diagonal argument, prove that the interval $(0, 1)$ is not countable.

4. Extra Credit

A. Trivial Credit

A correct response to any one of the following trivia questions is worth 1 point. The total number of points you may gain in part A is 1 point.

A1. What is Garret's middle name?

A2. In what state(s) has Molly taught high school mathematics?

A3. What is the color of the nameplate on Farshid's office door?

A4. What is Anna's middle name?

A5. In what semester did Laura take Math 300?

A6. How many tattoos does Aaron have?

B. Non-trivial Credit

Prove by induction that a regular n-gon (with $n \geq 3$) has $n(n-3)/2$ diagonals. (A regular n-gon is a figure in the plane with n edges of equal length and n angles of equal measure). A regular 3-gon is simply an equilateral triangle, and everyone knows the expression "Be there or be regular 4-gon." A diagonal is a line segment that joins to non-adjacent corners of the n-gon (i.e. a line joining two corners is a diagonal as long as it is not an edge).

C. More Non-trivial Credit

Prove or Disprove: If X is a subset of Y, Y is a subset of Z, and X is equivalent to Z, then X is equivalent to Y. (Here these sets need not be finite).