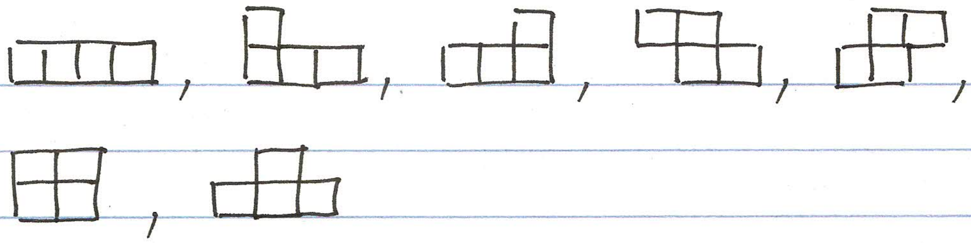
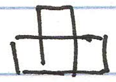
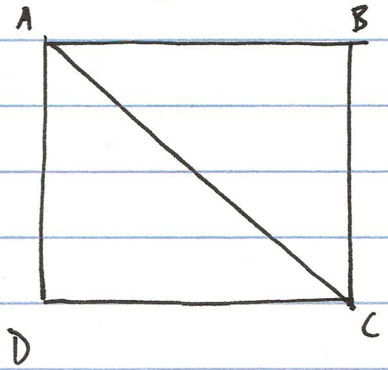
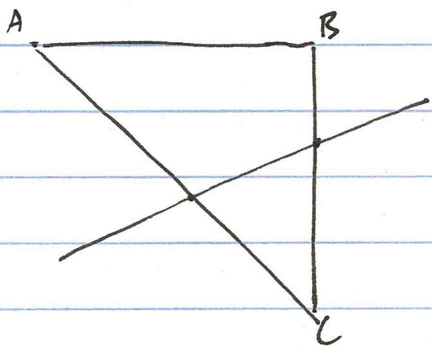
Saturday 11/7/15. Teaching Seminar worksheet solutions.

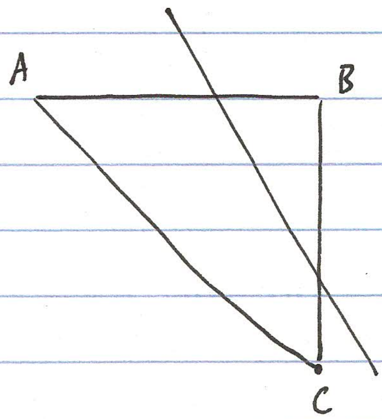
1. a. 

b. Divide the 4×7 rectangle into 1×1 squares, & colour the squares alternately black & white (like a chess board).
Notice that all the tiles except the last one () cover ~~on~~ 2 black squares & 2 white squares, whereas  will cover 3 black & 1 white or vice versa.
But the 4×7 rectangle has the same number of black & white squares (14 of each).
This shows that it is not possible to tile the rectangle using the 7 tetrominos.

2. 

There are two possible types of cut: (up to the symmetry switching A & C)

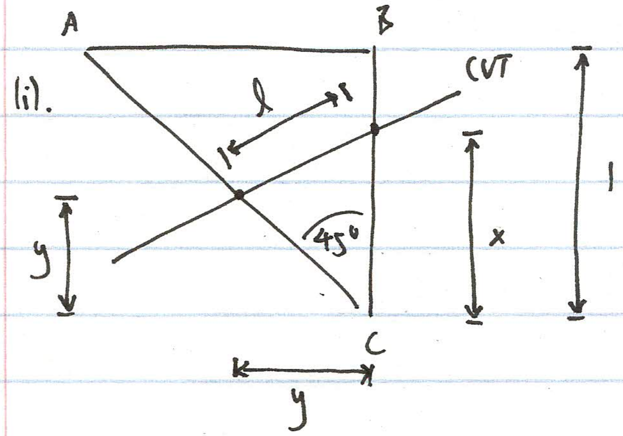
(i) 

(ii) 

Draw a diagram & introduce notation.

We may assume (by scaling) square ABCD has side length 1.

(i).



Condition: Two parts have equal area.

Area of lower part $= \frac{1}{2} x \cdot y = \frac{1}{2}$ area$(\triangle ABC) = 1/4$

i.e., require $xy = 1/2$.

Goal: Minimize length l of cut.

$$l = \sqrt{(y^2 + (x-y)^2)} = \sqrt{(x^2 - 2xy + 2y^2)} \qquad \dagger$$

Equivalently, minimize $z := l^2 = x^2 - 2xy + 2y^2$
subject to $xy = 1/2$.

Eliminate y: $y = 1/2x$ $\dagger\dagger$ $\Rightarrow$ $z = x^2 - 1 + \frac{1}{2x^2}$

Find critical points: $\frac{dz}{dx} = 2x - \frac{2}{2x^3} = 2x - \frac{1}{x^3}$

$\frac{dz}{dx} = 0 \iff x^4 = \frac{1}{2}$, $x = \frac{1}{2^{1/4}}$ $(x > 0)$

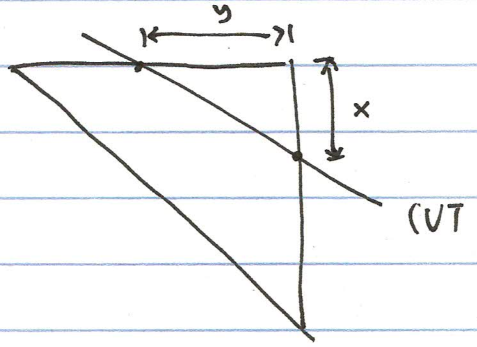
Check this is a MINIMUM: $\frac{d^2z}{dx^2} = 2 + \frac{3}{x^4} > 0$.

Position of cut: $x = \frac{1}{2^{1/4}} = 0.841$, $y \overset{\#}{=} \frac{1}{2x} = \frac{1}{2^{3/4}} = 0.595$

Length $l \overset{+}{=} \sqrt{(x^2 - 2xy + 2y^2)} = \sqrt{\left(\frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right)}$

$= \sqrt{(\sqrt{2}-1)}$. $(= 0.644)$

Case (ii) is similar (and easier)



Condition : $xy = 1/2$.

Minimize $z = l^2 = x^2 + y^2$

Eliminate y : $z = x^2 + \frac{1}{4x^2}$ $\left(y = \frac{1}{2x}\right)$

$\frac{dz}{dx} = 2x - \frac{1}{2x^3} = 0$

$\Leftrightarrow x^4 = 1/4$, $x = \frac{1}{\sqrt{2}}$

$\frac{d^2z}{dx^2} = 2 + \frac{3}{2x^4} > 0 \Rightarrow$ Min.

$\therefore$ Position of cut is

$x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$

Length $l = \sqrt{x^2+y^2} = 1$.

So case (i) gives shortest cut.

Notes on Q 1b.

It's natural to first try to construct a tiling by trial & error. After some time, you may get the impression it's impossible. Now, need to <u>prove</u> it's impossible. Some people had seen a <u>related problem</u>: if we remove two opposite corners from a chessboard (8×8 square) it's impossible to cover the remaining squares by 2×1 rectangular tiles. The idea of the two solutions is the same (colour the squares alternately black & white as usual, & consider the colours of squares a tile can cover).