583C Lecture notes

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This document contains lecture notes for 583C, a graduate course on compact complex surfaces at the University of Washington in Spring quarter 2008.

The aim of the course is to give an overview of the classification of smooth projective surfaces over $k = \mathbb{C}$. (We also occasionally discuss non-algebraic surfaces.) The emphasis is on understanding the key examples (including rational and ruled surfaces, K3 surfaces, elliptic surfaces, and surfaces of general type).

Contents

1 Topology

Here we describe the topology of compact complex surfaces.

1.1 Homology and cohomology

We recall the construction of the homology groups of a topological space (see [Hatcher, Sec. 2.1, p. 102] for more details). Let X be a topological space. Suppose given a triangulation of X, that is, a simplicial complex Σ and a homeomorphism $|\Sigma| \simeq X$. The group of *i-chains* $C_i(X,\mathbb{Z})$ is the free abelian group generated by oriented i-simplices of Σ, modulo the following relations: for σ an oriented simplex we identify $-\sigma$ and σ with its orientation reversed. The *boundary map* $d\colon C_i(X,\mathbb{Z}) \to C_{i-1}(X,\mathbb{Z})$ is given by $\sigma \mapsto \partial\sigma$, that is, a oriented simplex σ maps to its boundary $\partial\sigma$ (the sum of its faces with the induced orientation). One checks that $d^2 = 0$. The *i-th (integral) homology group* of X is

$$H_i(X,\mathbb{Z}) = \frac{\ker(d\colon C_i(X,\mathbb{Z}) \to C_{i-1}(X,\mathbb{Z}))}{\operatorname{Im}(d\colon C_{i+1}(X,\mathbb{Z}) \to C_i(X,\mathbb{Z}))}.$$

A chain with no boundary is called a *cycle*. So $H_i(X,\mathbb{Z})$ is the group of cycles modulo boundaries in dimension i. The group of *i-cochains* $C^i(X,\mathbb{Z})$ is the dual $\operatorname{Hom}(C_i(X,\mathbb{Z}),\mathbb{Z})$ of $C_i(X,\mathbb{Z})$. The boundary map $d\colon C_i \to C_{i-1}$ induces the *coboundary map* $C^{i-1} \to C^i$ which we also denote by d. The *ith (integral) cohomology group* of X is

$$H^i(X,\mathbb{Z}) = \frac{\ker(d\colon C^i(X,\mathbb{Z}) \to C^{i+1}(X,\mathbb{Z}))}{\operatorname{Im}(d\colon C^{i-1}(X,\mathbb{Z}) \to C^i(X,\mathbb{Z}))}.$$

(Note: there is a more geometric interpretation of the real cohomology groups in terms of smooth differential forms on X (the de Rham approach) which we discuss in Sec. 1.7.)

The homology and cohomology groups of X do not depend on the choice of triangulation. In fact, one can define singular homology and cohomology groups intrinsically (without choosing a triangulation) and show that they are isomorphic to the simplicial homology and cohomology groups described above. Here we define the singular homology and cohomology groups as follows: let $C_i^{\text{sing}}(X,\mathbb{Z})$ denote the free abelian group generated by all continuous maps $\sigma\colon \Delta^i \to X$, where Δ^i is the standard i-simplex, and proceed as above to obtain $H_i^{\text{sing}}(X,\mathbb{Z})$ and $H^i_{\text{sing}}(X,\mathbb{Z})$.

There is an intersection product (or cap product) on the homology of an oriented smooth manifold defined as follows (see [GH, p. 49–53]). Let X be an oriented smooth manifold of real dimension d. Let $\alpha \in H_k(X,\mathbb{Z})$ and $\beta \in H_{d-k}(X,\mathbb{Z})$ be two homology classes. We can find piecewise smooth cycles A and B representing α and β which intersect transversely. For each point $P \in A \cap B$, we define the intersection index $i_P(A,B) = \pm 1$ as follows. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_{d-k}$ be oriented bases of the tangent spaces $T_P A$ and $T_P B$. Then $i_P(A, B) = 1$ if $v_1, \ldots, v_k, w_1, \ldots, w_{d-k}$ is a positively oriented basis of $T_P X$ and $i_P(A, B) = -1$ otherwise. We define the *intersection number*

$$A \cdot B := \sum_{P \in A \cap B} i_P(A, B).$$

One shows that $A \cdot B$ is independent of the choice of representatives A, B of α, β. (Convince yourself that this is true for 1-cycles on the real 2-torus.) So, we obtain a bilinear form

$$\cap \colon H_k(X, \mathbb{Z}) \times H_{d-k}(X, \mathbb{Z}) \to \mathbb{Z}, \quad ([A], [B]) \mapsto A \cdot B,$$

the *intersection* or *cap product*. Note that $\alpha \cap \beta = (-1)^{k(d-k)} \beta \cap \alpha$.

Remark 1.1. One can also define an intersection product

$$\cap \colon H_i(X, \mathbb{Z}) \times H_j(X, \mathbb{Z}) \to H_{i+j-d}(X, \mathbb{Z}),$$

but we will not need this.

1.2 Curves

We review the topological classification of complex curves. Let X be a smooth complex projective curve (or, equivalently, a compact Riemann surface). Then X is a compact oriented smooth manifold of (real) dimension 2 together with a complex structure. As a smooth manifold, X is diffeomorphic to a sphere with g handles, where g is the *genus* of X (the number of holes). In particular, there is a unique topological invariant, the genus g.

Remark 1.2. The moduli space M_g of curves of genus g is a space whose points correspond to isomorphism types of complex projective curves of genus g. For $g \geq 2$ it is an (irreducible) quasiprojective variety of dimension $3g - 3$ with quotient singularities. The space M_g can be understood as the space of complex structures on the smooth surface of genus g (the Teichmüller approach).

We describe the integral homology of X. We have $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H_2(X, \mathbb{Z}) = \mathbb{Z}$. (More generally, recall that if X is a connected topological space then $H_0(X, \mathbb{Z}) = \mathbb{Z}$, generated by the class of a point, and if X is an oriented smooth manifold of dimension d then $H_d(X, \mathbb{Z}) = \mathbb{Z}$, generated by the so called fundamental class of X.) The first homology group $H_1(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$, generated by the 1-cycles $a_1, b_1, \ldots, a_g, b_g$, where a_i,

b_i encircle the ith hole in two independent ways and meet transversely in a single point, and are disjoint from a_j, b_j for $j \neq i$. In particular the intersection product

$$\cap\colon H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \to \mathbb{Z}$$

is the skew bilinear form with matrix

$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

with respect to the basis $a_1, b_1, \ldots, a_g, b_g$ (for an appropriate choice of orientations of the a_i, b_i).

We describe the fundamental group of X (see [Fulton, p. 242]). Recall that the smooth manifold X can be obtained from a $4g$-gon as follows. Going anticlockwise around the boundary of the $4g$-gon, we label the edges $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$, and then glue the edges in pairs according to the labels (here a^{-1} corresponds to a with the orientation reversed). All the vertices of the polygon are identified to a single point $x \in X$ and the edges of the polygon become loops $a_1, b_1, \ldots, a_g, b_g$ based at $x \in X$. Using the van Kampen theorem, one deduces that

$$\pi_1(X,x) = \frac{\langle a_1, b_1, \ldots, a_g, b_g \rangle}{([a_1,b_1]\cdots[a_g,b_g])},$$

where $[a,b] = aba^{-1}b^{-1}$ denotes the commutator of a and b. That is, $\pi_1(X,x)$ is the free group generated by $a_1, b_1, \ldots, a_g, b_g$ modulo the single relation given by the product of the commutators of a_i, b_i.

If X is a connected topological space then the abelianisation $\pi_1(X,x)^{\mathrm{ab}}$ of the fundamental group $\pi(X,x)$ is identified with the first homology group $H_1(X,\mathbb{Z})$ ([Hatcher, p. 166, Thm. 2.A.1]). (The abelianisation G^{ab} of a group G is the largest abelian quotient of G. Explicitly, it is the quotient of G by the normal subgroup generated by all commutators.) This gives another way to compute $H_1(X,\mathbb{Z})$.

1.3 Poincaré duality and the universal coefficient theorem

Theorem 1.3. *(Poincaré duality I) [GH, p. 53] Let X be a compact oriented smooth manifold of dimension d. Then the intersection product*

$$\cap\colon H_i(X,\mathbb{Z}) \times H_{d-i}(X,\mathbb{Z}) \to \mathbb{Z}$$

is unimodular, that is, the induced map

$$H_i(X,\mathbb{Z})/\operatorname{Tors} \to H_{d-i}(X,\mathbb{Z})^* \quad \alpha \mapsto (\alpha \cap \cdot)$$

is an isomorphism.

Recall that, if L, M are abelian groups and

$$b\colon L \times M \to \mathbb{Z}$$

is a bilinear pairing, we say b is *unimodular* if the induced map

$$L/\operatorname{Tors} \to M^* := \operatorname{Hom}(M,\mathbb{Z}), \quad l \mapsto b(l,\cdot)$$

is an isomorphism. Equivalently, if we pick bases for L/Tors and M/Tors, the matrix of b with respect to these bases has determinant ± 1.

Theorem 1.4. *(Universal coefficient theorem)[Hatcher, p. 195, Thm. 3.2] Let X be a topological space. Then there are natural exact sequences*

$$0 \to \operatorname{Ext}^1(H_{i-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^i(X,\mathbb{Z}) \to H_i(X,\mathbb{Z})^* \to 0.$$

In particular

$$H^i(X,\mathbb{Z})/\operatorname{Tors} \simeq H_i(X,\mathbb{Z})^*$$

and

$$\operatorname{Tors} H^i \simeq \operatorname{Tors} H_{i-1}$$

(this last isomorphism is not canonical).

Note: If you do not know what Ext^1 is, you can ignore the first statement in the theorem.

Proof. (Sketch) Recall that the homology $H_i(X,\mathbb{Z})$ is the homology of the complex of chains

$$\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$$

for some triangulation of X, and the cohomology $H^i(X,\mathbb{Z})$ is the cohomology of the dual complex

$$\cdots \leftarrow C_{i+1}^* \leftarrow C_i^* \leftarrow C_{i-1}^* \leftarrow \cdots$$

Observe that there is a natural map $H^i(X,\mathbb{Z}) \to H_i(X,\mathbb{Z})^*$. Recall that each C_i is a free abelian group (generated by the simplices of dimension

i). One shows that the complex $(C_., d)$ splits as a direct sum of shifts of complexes of the following types:

$$0 \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0, \quad 1 \mapsto n$$

Dualising these complexes, we deduce that $H^i(X, \mathbb{Z})/\operatorname{Tors} \simeq H_i(X, \mathbb{Z})^*$ and $\operatorname{Tors} H^i(X, \mathbb{Z}) \simeq \operatorname{Tors} H_{i-1}(X, \mathbb{Z})$. The exact sequence in the statement is obtained by being careful about naturality. □

We can now give a slightly stronger form of Poincaré duality.

Theorem 1.5. *(Poincaré duality II) [GH, p. 53] Notation as in 1.3. There is a natural isomorphism*

$$H_i(X, \mathbb{Z}) \simeq H^{d-i}(X, \mathbb{Z})$$

which induces the isomorphism $H_i(X, \mathbb{Z})/\operatorname{Tors} \simeq H_{d-i}(X, \mathbb{Z})^*$ *given by the intersection product. In particular,* $\operatorname{Tors} H_i(X, \mathbb{Z}) \simeq \operatorname{Tors} H^{d-i}(X, \mathbb{Z})$.

Proof of 1.3, 1.5. (Sketch) Fix a triangulation $X \simeq |\Sigma|$. Consider the dual complex Σ' of Σ. This is a cell complex (not necessarily a simplicial complex) with support $|\Sigma'| = |\Sigma|$, and cells of dimension $d - i$ in bijection with simplices of Σ of dimension i. It is constructed as follows: consider the barycentric subdivision $\hat{\Sigma}$ of $|\Sigma|$. For $v \in \Sigma$ a vertex, the corresponding d-cell $v' \in \Sigma'$ is the union of all the simplices in $\hat{\Sigma}$ which contain v. For $\sigma \in \Sigma$ an i-simplex, the corresponding $(d-i)$-cell $\sigma' \in \Sigma'$ is the intersection of the d-cells v' corresponding to the vertices v of σ. (Please draw a picture for $d = 2$). We observe that, for an i-simplex $\sigma \in \Sigma$, the corresponding cell $\sigma' \in \Sigma'$ is the unique $(d - i)$-cell of Σ' meeting σ, and intersects it transversely in one point. Now let $(C_.(\Sigma, \mathbb{Z}), d)$ be the chain complex for the triangulation $X \simeq |\Sigma|$ and $(C_.(\Sigma', \mathbb{Z}), d)$ the chain complex for the cellular subdivision $X \simeq |\Sigma'|$. (Note that we can use cellular subdivisions to compute homology exactly as for triangulations.) One shows that the maps

$$C_i(\Sigma, \mathbb{Z}) \xrightarrow{\sim} C^{d-i}(\Sigma', \mathbb{Z}) = C_{d-i}(\Sigma', \mathbb{Z})^*, \quad \sigma \mapsto (\sigma')^*.$$

commute (up to sign) with the differentials d. Here $C_j(\Sigma', \mathbb{Z})$ is the free abelian group with basis given by the j-cells τ of Σ' and for such a τ we write τ^* for the corresponding element of the dual basis of $C^j(\Sigma', \mathbb{Z}) = C_j(\Sigma', \mathbb{Z})^*$. Passing to homology we obtain the isomorphism $H_i(X, \mathbb{Z}) \simeq H^{d-i}(X, \mathbb{Z})$. □

1.4 Topological invariants of surfaces

Let X be a compact oriented smooth 4-manifold. Then $H_0(X, \mathbb{Z}) = \mathbb{Z}$, $H_4(X, \mathbb{Z}) = \mathbb{Z}$, and $H_1(X, \mathbb{Z}) = \pi_1(X, x)^{\mathrm{ab}}$. The intersection form

$$Q := \cap \colon H_2(X, \mathbb{Z}) / \operatorname{Tors} \times H_2(X, \mathbb{Z}) / \operatorname{Tors} \to \mathbb{Z}$$

is symmetric and unimodular. We also have $\operatorname{Tors} H_2 \simeq \operatorname{Tors} H_1$, $\operatorname{Tors} H_3 = 0$, and $H_3(X, \mathbb{Z}) \simeq H_1(X, \mathbb{Z})^*$ by Poincaré duality and the universal coefficient theorem.

So, to recap, the topological invariants are the fundamental group $\pi_1(X, x)$ and the intersection form Q on $H_2(X, \mathbb{Z}) / \operatorname{Tors}$.

Remark 1.6. Assume X is a smooth projective complex surface. We can usually reduce to the simply connected case ($\pi_1(X, x) = 0$) as follows. If $\pi_1(X, x)$ is finite, let $p \colon \tilde{X} \to X$ be the universal cover of X. Then $\tilde{X}$ inherits the structure of a smooth complex surface from X, and $\tilde{X}$ is projective because p is a finite morphism. So X is the quotient of the smooth projective surface $\tilde{X}$ by the free action of the finite group $\pi_1(X, x)$. If $H_1(X, \mathbb{Z}) = \pi_1(X, x)^{\mathrm{ab}}$ is infinite, then the Albanese morphism is a non-trivial morphism from X to a complex torus, and we can use this to study X. See Sec. 11.2.

1.5 Results of Freedman and Donaldson

We state without proof two results about the classification of smooth 4 manifolds. See [BHPV, Ch. IX] for more details (note: unfortunately, this material is only contained in the 2nd edition).

Theorem 1.7. *(Freedman '82) A simply connected compact oriented* 4*-manifold is determined up to (oriented) homeomorphism by its intersection form Q.*

Theorem 1.8. *(Donaldson '83) There exist infinitely many smooth complex projective surfaces which are homeomorphic but not diffeomorphic.*

1.6 Classification of quadratic forms

Let L be a free abelian group of finite rank. Let $Q \colon L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form.

We can pick a basis of the $\mathbb{R}$-vector space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ such that the matrix of Q with respect to this basis is diagonal with diagonal entries n_+ 1's and n_- (-1)'s. The pair (n_+, n_-) is the *signature* of Q.

We say Q is *positive definite* if $Q(x,x) > 0$ for all $x \neq 0$, *negative definite* if $Q(x,x) < 0$ for all $x \neq 0$, and *indefinite* otherwise. In terms of the signature, Q is positive definite if $n_- = 0$, negative definite if $n_+ = 0$, and indefinite otherwise.

We say Q is *even* if $Q(x,x)$ is even for all $x \in L$, and Q is *odd* otherwise.

Theorem 1.9. *[Serre70, Ch. V] An indefinite unimodular quadratic form is determined up to isomorphism by its signature and parity. (The same holds for definite forms if the rank is ≤ 8).*

The quadratic forms as in the theorem can be described explicitly as follows. If Q is odd, Q is of type

$$(1)^{n_+} \oplus (-1)^{n_-}.$$

(That is, with respect to some basis of L, the quadratic form Q has matrix the diagonal matrix with diagonal entries n_+ 1's and n_- (-1)'s.) If Q is even, Q is of type

$$H^a \oplus (\pm E_8)^b$$

for some $a > 0$ and $b \geq 0$, where H is the *hyperbolic plane* with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8 is the positive definite quadratic form of rank 8 with matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Remark 1.10. The classification of definite quadratic forms is much more involved. However, if X is a smooth projective complex surface with definite quadratic form Q, then $H_2(X,\mathbb{Z})$ has rank 1 and $Q = (1)$.

Remark 1.11. A free abelian group L together with a symmetric bilinear form $Q\colon L \times L \to \mathbb{Z}$ is sometimes called a *lattice*.

Example 1.12. Let $\pi\colon X \to \mathbb{P}^2$ be the blowup of n distinct points $P_1, \dots, P_n$ in the complex projective plane $\mathbb{P}^2$. Let $H \subset \mathbb{P}^2$ be a hyperplane not containing any of the P_i and $E_1, \dots, E_n$ the exceptional curves. Then $H_2(X, \mathbb{Z})$ is free with basis $\pi^{-1}H, E_1, \dots, E_n$, and the intersection product Q has matrix $(1) \oplus (-1)^n$ with respect to this basis,

Example 1.13. Let X be a K3 surface (that is, a compact complex surface X such that the canonical divisor K_X is trivial and $H^1(X, \mathbb{Z}) = 0$) Then $H_2(X, \mathbb{Z})$ is free of rank 22 and the intersection form Q has type $H^3 \oplus (-E_8)^2$.

1.7 de Rham cohomology

Here we describe the de Rham approach to cohomology via differential forms.

We first note some elementary facts about homology and cohomology groups. Let X be a topological space. Then we can define (simplicial) homology and cohomology groups $H_i(X, A)$, $H^i(X, A)$ with coefficients in an abelian group A as before. If $\mathbb{F}$ is a field of characteristic 0 (for example $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) then $H_i(X, \mathbb{F}) = H_i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$. Recall that $H^i(X, \mathbb{Z}) / \operatorname{Tors} = H_i(X, \mathbb{Z})^*$. If $\mathbb{F}$ is a field then $H^i(X, \mathbb{F}) = H_i(X, \mathbb{F})^*$.

Now let X be a compact smooth manifold of (real) dimension d. Let $C^k_{\mathrm{dR}}(X, \mathbb{R})$ denote the $\mathbb{R}$-vector space of smooth $\mathbb{R}$-valued k-forms ω on X. That is, locally on X,

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots dx_{i_k} = \sum_{|I| = k} f_I dx_I$$

where $x_1, \dots, x_d$ are local coordinates on X and the f_I are smooth $\mathbb{R}$-valued functions on X. Let

$$d\colon C^k_{\mathrm{dR}}(X, \mathbb{R}) \to C^{k+1}_{\mathrm{dR}}(X, \mathbb{R})$$

be the *exterior derivative*, that is (working locally),

$$d\omega = d(\sum f_I dx_I) = \sum df_I \wedge dx_I$$

where

$$df = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dx_i$$

Then $d^2 = 0$, and we define the *(real) de Rham cohomology groups* $H^i_{\mathrm{dR}}(X, \mathbb{R})$ by

$$H^i_{\mathrm{dR}}(X, \mathbb{R}) = \frac{\ker(d\colon C^i(X, \mathbb{R}) \to C^{i+1}(X, \mathbb{R}))}{\operatorname{Im}(d\colon C^{i-1}(X, \mathbb{R}) \to C^i(X, \mathbb{R}))}.$$

We say a smooth differential form ω is *closed* if $d\omega = 0$, and we say ω is *exact* if $\omega = d\eta$ for some η. So, $H^i_{\mathrm{dR}}(X, \mathbb{R})$ is the $\mathbb{R}$-vector space of closed i-forms modulo exact forms.

There is a natural $\mathbb{R}$-bilinear pairing

$$H^i_{\mathrm{dR}}(X, \mathbb{R}) \times H_i(X, \mathbb{R}) \to \mathbb{R}, \quad (\omega, \gamma) \mapsto \int_\gamma \omega. \tag{1}$$

Note that this is well defined by Stokes' theorem: if $\omega = d\eta$ then $\int_\gamma d\eta = \int_{d\gamma} \omega = 0$ because $d\gamma = 0$, and if $\gamma = d\beta$ then $\int_{d\beta} \omega = \int_\beta d\omega = 0$ because $d\omega = 0$.

Theorem 1.14. *(de Rham's theorem) [GH, p. 44] The map*

$$H^i_{\mathrm{dR}}(X, \mathbb{R}) \to H_i(X, \mathbb{R})^* = H^i(X, \mathbb{R})$$

induced by (1) is an isomorphism.

The *wedge product* or *exterior product* on de Rham cohomology is the product

$$H^i_{\mathrm{dR}}(X, \mathbb{R}) \times H^j_{\mathrm{dR}}(X, \mathbb{R}) \to H^{i+j}_{\mathrm{dR}}(X, \mathbb{R}), \quad (\omega, \eta) \mapsto \omega \wedge \eta$$

induced by wedge product of forms. Note that this is well defined because

$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^{\deg \omega} \omega \wedge d\xi$$

so, if ω is closed and $\eta = d\xi$ is exact, then

$$\omega \wedge d\xi = (-1)^{\deg \omega} d(\omega \wedge \xi)$$

is exact.

Now we discuss Poincaré duality from the de Rham point of view. Recall that the intersection product defines a non-degenerate bilinear form

$$\cap \colon H_i(X, \mathbb{R}) \times H_{d-i}(X, \mathbb{R}) \to \mathbb{R}.$$

Consider the identification

$$H_i(X, \mathbb{R}) \xrightarrow{\sim} H_{d-i}(X, \mathbb{R})^* = H^{d-i}(X, \mathbb{R}) = H^{d-i}_{\mathrm{dR}}(X, \mathbb{R})$$

induced by $\cap$ and the de Rham isomorphism. Here an i-cycle α maps to a $(d-i)$ form ω (determined up to an exact form) such that for any $(d-i)$-cycle $\beta \in H_{d-i}(X, \mathbb{R})$,

$$\alpha \cap \beta = \int_\beta \omega.$$

We identify the form ω explicitly. Assume for simplicity that the homology class α is represented by a smooth submanifold $A \subset X$ of dimension i. There exists a "tubular neighbourhood" N of A in X isomorphic to a neighbourhood of the zero section in a vector bundle over A (the normal bundle of A in X). We construct a form ω such that the support of ω is contained in N, ω is closed, and $\int_{N_a} \omega = 1$ for all $a \in A$, where N_a denotes the fibre of the bundle $N \to A$ over a. Explicitly, locally on X write $A = (x_1 = \cdots = x_{d-i} = 0) \subset X$, where $x_1, \ldots, x_d$ are local coordinates on X, and let $\omega = f dx_1 \wedge \cdots \wedge dx_{d-i}$ where $f = f(x_1, \ldots, x_{d-i})$ is a smooth bump function on $\mathbb{R}^{d-i}$ supported in a small neighbourhood of the origin, with integral 1. Globally, we can use a partition of unity to patch the local forms together. Now suppose that $\beta \in H_{d-i}(X, \mathbb{Z})$, and represent β by a piecewise smooth $(d-i)$-cycle B intersecting A transversely in a finite number of points. Then $\int_B \omega = A \cap B$ — at each intersection point, we get the integral of the bump function f (which equals 1 by construction), with a sign given by the orientations.

We show that the pairing

$$\wedge \colon H^i_{\mathrm{dR}}(X, \mathbb{R}) \times H^{d-i}_{\mathrm{dR}}(X, \mathbb{R}) \to \mathbb{R}, \quad (\omega, \eta) \mapsto \int_X \omega \wedge \eta$$

given by the wedge product is identified with the pairing

$$\cap \colon H_{d-i}(X, \mathbb{R}) \times H_i(X, \mathbb{R}) \to \mathbb{R}$$

given by the intersection product via the isomorphisms

$$H_i(X, \mathbb{R}) \simeq H^{d-i}_{\mathrm{dR}}(X, \mathbb{R}), \quad H_{d-i}(X, \mathbb{R}) \simeq H^i_{\mathrm{dR}}(X, \mathbb{R})$$

described above. In particular, the wedge pairing is nondegenerate. Assume for simplicity that $\alpha \in H_i(X, \mathbb{R})$, $\beta \in H_{d-i}(X, \mathbb{R})$ are represented by closed submanifolds $A, B \subset X$ of dimensions $i, d-i$. Let ω, η be representatives of the corresponding deRham cohomology classes which are supported in a small tubular neighbourhood of A, B. We may assume that A, B intersect transversely in a finite number of points. Let P be an intersection point of A, B, then we can choose local coordinates $x_1, \ldots, x_d$ at P such that

$$A = (x_1 = \cdots = x_{d-i} = 0) \subset X, \quad B = (x_{d-i+1} = \cdots = x_d = 0) \subset X.$$

As above we may assume that, working locally at $P \in X$,

$$\omega = f(x_1, \ldots, x_{d-i}) dx_1 \wedge \cdots \wedge dx_{d-i}, \quad \eta = g(x_{d-i+1}, \ldots, x_d) dx_{d-i+1} \wedge \cdots \wedge dx_d,$$

where f, g are bump functions at $0 \in \mathbb{R}^{d-i}$ and $0 \in \mathbb{R}^i$ with integral 1. We find that the contribution to $\int_X \omega \wedge \eta$ from this chart is $i_P(A, B) = \pm 1$, the intersection number of A, B at P (where the sign comes from the orientations — see the description of the intersection product). Adding together the local contributions we deduce $\int_X \omega \wedge \eta = \alpha \cap \beta$, as required. (See [GH, p. 58–59] for an alternative argument.)

Remark 1.15. The cup product

$$\cup\colon H^i(X, \mathbb{R}) \times H^j(X, \mathbb{R}) \to H^{i+j}(X, \mathbb{R})$$

on (simplicial) cohomology corresponds to the wedge product on de Rham cohomology via the de Rham isomorphism [GH, p. 60]. (The cup product can be defined as follows: Let $\langle v_0, \ldots, v_n \rangle$ denote the simplex with vertices $v_0, \ldots, v_n$. For $\phi \in C^i(X, \mathbb{Z}), \psi \in C^j(X, \mathbb{Z})$ simplicial cochains, define $\phi \cup \psi \in C^{i+j}(X, \mathbb{Z})$ by

$$\phi \cup \psi(\langle v_0 \cdots v_{i+j} \rangle) = \phi(\langle v_0 \cdots v_i \rangle)\psi(\langle v_i \cdots v_{i+j} \rangle).$$

This induces a well defined product on cohomology. See [Hatcher, p. 206, Sec. 3.2] for more details.)

2 Hodge theory

Let X be a complex manifold of complex dimension n. Consider $\mathbb{C}$-valued k-forms ω on X. That is, locally $\omega = \sum_{|I|=k} f_I dx_I$ where $x_1, \ldots, x_{2n}$ are local real coordinates on X, and f_I is a smooth $\mathbb{C}$-valued function on X. We define the complex de Rham cohomology group $H^k_{\mathrm{dR}}(X, \mathbb{C})$ as the space of closed $\mathbb{C}$-valued k-forms modulo exact forms (as in the real case). Note of course that $H^k_{\mathrm{dR}}(X, \mathbb{C}) = H^k_{\mathrm{dR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Now let $z_1, \ldots, z_n$ be local complex coordinates on X, and write $z_i = x_i + iy_i$ for each $i = 1, \ldots, n$. So $x_1, y_1, \ldots, x_n, y_n$ are local real coordinates on X. We can write a $\mathbb{C}$-valued k-form in terms of the $dz_i = dx_i + idy_i$, $d\bar{z}_i = dx_i - idy_i$ (instead of the dx_i, dy_i). A (p, q)-*form* is a $\mathbb{C}$-valued $(p+q)$-form which is locally of the form

$$\sum_{|I|=p, |J|=q} f_{I,J} dz_I \wedge d\bar{z}_J$$

for some $\mathbb{C}$-valued functions $f_{I,J}$ on X. Let

$$H^{p,q}(X) \subset H^{p+q}_{\mathrm{dR}}(X, \mathbb{C})$$

denote the complex subspace of de Rham cohomology classes represented by a closed (p,q)-form. Note immediately that $H^{q,p} = \bar{H}^{p,q}$, that is, $H^{q,p} \subset H^k(X,\mathbb{C})$ is the complex conjugate of the subspace $H^{p,q} \subset H^k(X,\mathbb{C})$.

Theorem 2.1. *(Hodge decomposition) [GH, p. 116] Let X be a smooth complex projective variety (or, more generally, a compact complex Kähler manifold). Then*

$$H^k_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover, there is a natural isomorphism $H^{p,q}(X) \simeq H^q(X,\Omega^p_X)$. That is, $H^{p,q}(X)$ is isomorphic to the qth cohomology group of the sheaf Ω^p_X of holomorphic p-forms on X.

A *holomorphic p-form* ω on X is a $\mathbb{C}$-valued form which is locally of the form $\omega = \sum_{|I|=p} f_I dz_I$ where $z_1,\ldots,z_n$ are local complex coordinates on X and f_I is a holomorphic $\mathbb{C}$-valued function on X. The sheaf Ω^p_X of holomorphic p-forms is the data of the spaces of holomorphic p-forms $\Omega_X(U)$ on U for each open set $U \subset X$ together with the restriction maps $\Omega^p_X(U) \to \Omega^p_X(V)$ for $V \subset U$. (We will discuss sheaves and cohomology of sheaves in more detail in Sec. 3).

Remark 2.2. Recall that the wedge product defines a nondegenerate pairing

$$H^k_{\mathrm{dR}}(X,\mathbb{C}) \times H^{2n-k}_{\mathrm{dR}}(X,\mathbb{C}) \to H^n(X,\mathbb{R}) \simeq \mathbb{C}, \quad (\omega,\eta) \mapsto \int_X \omega \wedge \eta.$$

By the Hodge decomposition, this pairing decomposes into a direct sum of nondegenerate pairings

$$H^{p,q} \times H^{n-p,n-q} \to H^{n,n} \simeq \mathbb{C}.$$

(Note that the wedge product of a (p,q)-form and an (r,s)-form is a $(p+r,q+s)$-form, so can only be non-zero if $p+r, q+s \leq n$.) In terms of sheaf cohomology, we have a nondegenerate pairing

$$H^q(\Omega^p_X) \times H^{n-q}(\Omega^{n-p}_X) \to H^n(\Omega^n_X) \simeq \mathbb{C}.$$

We observe that this is an instance of Serre duality [Hartshorne, p. 244, III.7.7]. Indeed, the pairing of sheaves

$$\wedge \colon \Omega^p_X \times \Omega^{n-p}_X \to \Omega^n_X =: \omega_X$$

determines an identification

$$\Omega^{n-p}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^p_X, \omega_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^p_X, \mathcal{O}_X) \otimes \omega_X = (\Omega^p_X)^\vee \otimes \omega_X.$$

So the pairing above can be rewritten as

$$H^q(\Omega_X^p) \times H^{n-q}((\Omega_X^p)^\vee \otimes \omega_X) \to H^n(\omega_X) \simeq \mathbb{C}.$$

This is the Serre duality pairing for the sheaf Ω_X^p.

We define some notation. Let $b_i(X) = \dim_{\mathbb{R}} H^i(X, \mathbb{R}) = \dim_{\mathbb{C}} H^i(X, \mathbb{C})$, the *ith Betti number*, and $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$, the *Hodge numbers.* For $\mathcal{F}$ a coherent sheaf on X (for example, $\mathcal{O}_X$, Ω_X^p), let $h^i(X, \mathcal{F}) = \dim_{\mathbb{C}} H^i(X, \mathcal{F})$.

Now let X be a smooth complex projective surface. Then

$$H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1} = H^0(\Omega_X) \oplus H^1(\mathcal{O}_X)$$

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} = H^0(\omega_X) \oplus H^1(\Omega_X) \oplus H^2(\mathcal{O}_X)$$

The *irregularity* of X is $q := h^0(\Omega_X) = h^1(\mathcal{O}_X)$. The *geometric genus* of X is $p_g := h^0(\omega_X) = h^2(\mathcal{O}_X)$. We have

$$b_1 = b_3 = 2q, \quad b_2 = 2p_g + h^{1,1}$$

by the Hodge decomposition and Poincaré duality.

3 Sheaves and cohomology

Here we give a quick introduction to sheaves and their cohomology. For more details, see [GH, p. 34–43].

3.1 Sheaves

Let X be a topological space. A *sheaf* $\mathcal{F}$ on X is the following data:

(1) For every open set $U \subset X$, an abelian group $\mathcal{F}(U)$, the *sections of* $\mathcal{F}$ *over* U, and

(2) for every inclusion of open sets $V \subset U$ a homomorphism $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$, the *restriction map*,

such that

(1) $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

(2) For $U \subset X$ open and $U = \bigcup_{i \in I} U_i$ an open covering of U, the sequence

$$\begin{array}{ccccccc} 0 & \to & \mathcal{F}(U) & \to & \bigoplus_{i\in I} \mathcal{F}(U_i) & \to & \bigoplus_{i \neq j \in I} \mathcal{F}(U_i \cap U_j) \\ & & s & \mapsto & (s|_{U_i}) & & \\ & & & & (s_i) & \mapsto & (s_i|_{U_{ij}} - s_j|_{U_{ij}}) \end{array}$$

is exact.

Here for $s \in \mathcal{F}(U)$ and $V \subset U$ we write $s|_V$ for $\rho_{UV}(s)$. We also sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$.

Example 3.1. (1) If X is a complex manifold or smooth algebraic variety of dimension n, the holomorphic or regular functions on X form a sheaf $\mathcal{O}_X$, the *structure sheaf*, and the holomorphic or regular p-forms form a sheaf Ω^p_X for $0 \leq p \leq n$ (by convention $\Omega^0_X = \mathcal{O}_X$). We write $\omega_X = \Omega^n_X$.

(2) Similarly, if X is a smooth manifold of dimension n, the smooth functions form a sheaf $\mathcal{A}_X$, and the smooth p-forms form a sheaf $\mathcal{A}^p_X$, $0 \leq p \leq n$. Note: The use of these sheaves is not essential in the theory of smooth manifolds, because we can always globalise local data using smooth bump functions. They are however sometimes used in an auxiliary role (e.g., proof of de Rham theorem [GH, p. 43–44]).

(3) If X is a topological space and A is an abelian group (for example, $A = \mathbb{Z}$), the locally constant A-valued functions on X form a sheaf $\underline{A}$, the *constant sheaf with stalk A*.

A morphism of sheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a homomorphism $\alpha_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subset X$, compatible with the restriction maps. The *kernel* of α is the subsheaf $\ker \alpha$ of $\mathcal{F}$ defined by $(\ker \alpha)(U) = \ker \alpha_U$. The *image* of α is the subsheaf $\operatorname{Im} \alpha$ of $\mathcal{G}$ defined as follows: $s \in \mathcal{G}(U)$ lies in $(\operatorname{Im} \alpha)(U)$ if there exists a open covering $U = \bigcup U_i$ of U such that $s|_{U_i} \in \operatorname{Im} \alpha_{U_i}$ for all i. (Note: $(\operatorname{Im} \alpha)(U) \neq \operatorname{Im} \alpha_U$ in general.) The *cokernel* of α is the sheaf $\operatorname{coker} \alpha$ defined as follows: an element of $(\operatorname{coker} \alpha)(U)$ is given by an open covering $U = \bigcup U_i$ of U and sections $s_i \in \mathcal{G}(U_i)$ such that

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \operatorname{Im} \alpha_{U_i \cap U_j}$$

for all $i \neq j$. Two such data $(s_i \in \mathcal{F}(U_i))$, $(s'_j \in \mathcal{F}(U'_j))$ define the same element of $(\operatorname{coker} \alpha)(U)$ if for all $P \in U$ there exists an an open neighbourhood

V of P and i, j such that $V \subset U_i \cap U'_j$ and $s_i|_V - s'_j|_V \in \operatorname{im}(\alpha_V)$. (Note: $(\operatorname{coker} \alpha)(U) \neq \operatorname{coker} \alpha_U$ in general.)

The *stalk* of a sheaf $\mathcal{F}$ at a point $P \in X$ is

$$\mathcal{F}_P = \varinjlim_{U \ni P} \mathcal{F}(U),$$

the direct limit over open neighbourhoods U of P of the abelian groups $\mathcal{F}(U)$. That is, an element of $\mathcal{F}_P$ is given by a section $s \in \mathcal{F}(U)$ for some open neighbourhood U of P, and $(s \in \mathcal{F}(U))$, $(t \in \mathcal{F}(V))$ define the same element of $\mathcal{F}_P$ if there exists an open neighbourhood W of P such that $W \subset U \cap V$ and $s|_W = t|_W$.

Example 3.2. (1) Let X be a complex manifold, $P \in X$ a point, and $z_1, \ldots, z_n$ local complex coordinates at P. Then an element of $\mathcal{O}_{X,P}$ is a power series in $z_1, \ldots, z_n$ with positive radius of convergence.

(2) If X is an algebraic variety then $\mathcal{O}_{X,P}$ is the local ring of regular functions at P.

We say a sequence of sheaves

$$\mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G}$$

is *exact* if the induced sequence of stalks at P is exact for all $P \in X$. Equivalently, $\operatorname{im} \alpha = \ker \beta$ (where $\operatorname{im} \alpha$ and $\ker \beta$ are the subsheaves of $\mathcal{F}$ defined above).

Example 3.3. Let X be a complex manifold. The *exponential sequence* is the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \rightarrow & \underline{\mathbb{Z}} & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_X^\times & \rightarrow & 0 \\ & & & & f & \mapsto & \exp(2\pi i f) & & \end{array}$$

Here $\underline{\mathbb{Z}}$ denotes the constant sheaf with stalk $\mathbb{Z}$, and $\mathcal{O}_X^\times$ denotes the sheaf of holomorphic functions on X which are nowhere zero (with group law pointwise multiplication). Note that $\mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ is *not* surjective in general (for example, if $X = U = \mathbb{C} \setminus \{0\}$).

We introduce some more terminology. Let X be a complex manifold or smooth algebraic variety of dimension n. Then the structure sheaf $\mathcal{O}_X$ is a *sheaf of rings*, that is, each $\mathcal{O}_X(U)$ is a ring and the restriction maps are ring homomorphisms. We say a sheaf $\mathcal{F}$ on $\mathcal{O}_X$ is an $\mathcal{O}_X$*-module* if $\mathcal{F}(U)$

is an $\mathcal{O}_X(U)$-module for each U, and this structure is compatible with the restriction maps. An $\mathcal{O}_X$*-module* $\mathcal{F}$ is *locally free of rank* r if there is an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ for each i. For example Ω_X^p is locally free of rank $\binom{n}{p}$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if there is an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X and an exact sequence

$$\mathcal{O}_{U_i}^{\oplus m_i} \to \mathcal{O}_{U_i}^{\oplus n_i} \to \mathcal{F}|_{U_i} \to 0$$

for each i (that is, $\mathcal{F}$ is locally a cokernel of a map of free sheaves of finite rank).

3.2 Cohomology of sheaves

Let $\mathcal{F}$ be a sheaf on a topological space X, and $\mathcal{U} = \{U_i\}_{i\in I}$ a finite open covering of X. We write $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define

$$C^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p})$$

and

$$d\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}), \quad (ds)_{i_0\cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0\cdots \hat{i_j}\cdots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}.$$

One checks that $d^2 = 0$. We define the *Cech cohomology* $H^p(\mathcal{U}, \mathcal{F})$ *of* $\mathcal{F}$ *relative to the open covering* $\mathcal{U}$ to be the cohomology of the complex $(C^{\cdot}(\mathcal{U}, \mathcal{F}), d)$, that is

$$H^p(\mathcal{U}, \mathcal{F}) = \frac{\ker(d\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}))}{\operatorname{Im}(d\colon C^{p-1}(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F}))}$$

Note immediately that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ by the sheaf axioms.

Now let $\mathcal{V} = \{V_j\}_{j\in J}$ be a refinement of the open covering $\mathcal{U}$. That is, for all $j \in J$ there exists $i \in I$ such that $V_j \subset U_i$. Fix a map $\phi\colon J \to I$ such that $V_j \subset U_{\phi(j)}$ for all j. Then ϕ induces maps

$$\rho_\phi\colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F}) \quad (\rho_\phi s)_{j_0\cdots j_p} = s_{\phi(j_0)\cdots\phi(j_p)}|_{V_{j_0\cdots j_p}}$$

which are compatible with the differentials d, and so induce maps

$$\rho_\phi\colon H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{V}, \mathcal{F})$$

on cohomology. One shows that the maps ρ_ϕ on cohomology do not depend on the choice of ϕ (because the maps ρ_ϕ on complexes for different choices of ϕ are chain homotopic). We define the *Cech cohomology* $H^p(X,\mathcal{F})$ of $\mathcal{F}$ by

$$H^p(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F}),$$

the direct limit over open coverings $\mathcal{U}$ of X of the $H^q(\mathcal{U},\mathcal{F})$.

Example 3.4. Let X be a topological space. Then $H^p(X,\underline{\mathbb{Z}}) \simeq H^p(X,\mathbb{Z})$, that is, the Cech cohomology of the constant sheaf with stalk $\mathbb{Z}$ is isomorphic to the (simplicial) integral cohomology of X. To see this, let $X \simeq |\Sigma|$ be a triangulation, with vertices $\{v_i\}_{i\in I}$. Let $U_i = \mathrm{Star}(v_i)$, the union of the interiors of simplices containing v_i. Then $\mathcal{U} = \{U_i\}_{i\in I}$ is an open covering of X, and $U_{i_0\cdots i_p}$ is non-empty iff $\langle v_{i_0}\cdots v_{i_p}\rangle$ is a simplex of Σ and is connected when nonempty. We obtain an isomorphism

$$C^p(\mathcal{U},\underline{\mathbb{Z}}) \xrightarrow{\sim} C^p(X,\mathbb{Z}) = C_p(X,\mathbb{Z})^* \quad s = (s_{i_0\cdots i_p}) \mapsto (\langle v_{i_0}\cdots v_{i_p}\rangle \mapsto s_{i_0\cdots i_p})$$

which is compatible with the differentials, and so an isomorphism

$$H^p(\mathcal{U},\underline{\mathbb{Z}}) \xrightarrow{\sim} H^p(X,\mathbb{Z}).$$

We can make the open covering $\mathcal{U}$ arbitrarily fine by subdividing Σ. So, passing to the limit over all coverings, we deduce

$$H^p(X,\underline{\mathbb{Z}}) \xrightarrow{\sim} H^p(X,\mathbb{Z})$$

as claimed.

Cech cohomology groups can be computing using the following theorem.

Theorem 3.5. *(Leray theorem) Let X be a topological space, $\mathcal{F}$ a sheaf on X, and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of X such that*

$$H^q(U_{i_0\cdots i_p},\mathcal{F}) = 0 \text{ for all } q > 0 \text{ and } i_0,\dots i_p. \tag{2}$$

Then $H^p(X,\mathcal{F}) = H^p(\mathcal{U},\mathcal{F})$.

Example 3.6. The hypothesis (2) is satisfied in the following cases:

(1) X an algebraic variety, $\mathcal{F}$ a coherent sheaf, and $\mathcal{U}$ an open covering of X by affine open sets.

(2) X a complex manifold of dimension n, $\mathcal{F}$ a coherent sheaf, and $\mathcal{U}$ a covering of X by polydiscs (that is, open sets of the form Δ^n for $\Delta = (|z| < r) \subset \mathbb{C}$, some $r > 0$).

(3) X a topological space, $\mathcal{F} = \underline{A}$ a constant sheaf, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X such that $U_{i_0 \cdots i_p}$ is contractible for all $i_0, \ldots, i_p$ (cf. Ex. 3.4).

Here is one of the main applications of sheaf cohomology. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves. Then there is an associated long exact sequence of cohomology

$$\begin{array}{cccccccc} 0 & \to & H^0(X,\mathcal{E}) & \to & H^0(X,\mathcal{F}) & \to & H^0(X,\mathcal{G}) \\ & \xrightarrow{\delta} & H^1(X,\mathcal{E}) & \to & H^1(X,\mathcal{F}) & \to & H^1(X,\mathcal{G}) \\ & \xrightarrow{\delta} & H^2(X,\mathcal{E}) & \to & \cdots & & \end{array}$$

To see this, assume for simplicity that there exist arbitrarily fine open coverings $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that the sequence

$$0 \to \mathcal{E}(U_{i_0 \cdots i_p}) \to \mathcal{F}(U_{i_0 \cdots i_p}) \to \mathcal{G}(U_{i_0 \cdots i_p}) \to 0$$

is exact for all $i_0, \ldots, i_p$. (This is always satisfied in practice.) Then we have an exact sequence of complexes

$$0 \to C^{\cdot}(\mathcal{U},\mathcal{E}) \to C^{\cdot}(\mathcal{U},\mathcal{F}) \to C^{\cdot}(\mathcal{U},\mathcal{G}) \to 0$$

which (as usual, see [Hatcher, p. 116-7]) induces a long exact sequence of cohomology

$$\cdots \to H^i(\mathcal{U},\mathcal{E}) \to H^i(\mathcal{U},\mathcal{F}) \to H^i(\mathcal{U},\mathcal{G}) \xrightarrow{\delta} H^{i+1}(\mathcal{U},\mathcal{E}) \to \cdots$$

Taking the limit over open coverings $\mathcal{U}$ gives the result.

In particular, given a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

we have an exact sequence of abelian groups

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \xrightarrow{\delta} H^1(X,\mathcal{E})$$

So, a section $t \in \mathcal{G}(X)$ is the image of a section $s \in \mathcal{F}(X)$ iff $\delta(t) = 0$ in $H^1(X,\mathcal{E})$.

3.3 Analytic and algebraic approaches

If X is a smooth complex variety then there is an associated complex manifold, denoted X^{an}. Note that the topology on X is the Zariski topology (the open sets are complements of finite unions of closed subvarieties) and the topology on X^{an} is the usual Euclidean topology.

We state some results from [Serre56]. If $\mathcal{F}$ is a coherent sheaf on X there is an naturally associated coherent sheaf $\mathcal{F}^{\mathrm{an}}$ on X^{an}. Now suppose X is projective, so X^{an} is compact. Then, for $\mathcal{F}$ a coherent sheaf on X, there are natural isomorphisms

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

for each i . In particular, $\mathcal{F}(X) \simeq \mathcal{F}^{\mathrm{an}}(X)$. Note that this is *not* true in general if X^{an} is not compact. For example, if $X = \mathbb{A}^1_x$ and $\mathcal{F} = \mathcal{O}_X$, then $\mathcal{F}^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}$, the sheaf of holomorphic functions on X^{an}, and $\exp(x) \in \mathcal{O}_{X^{an}}(X) \setminus \mathcal{O}_X(X)$.

4 Divisors and Line bundles

Let X be a smooth algebraic variety over $k = \mathbb{C}$.

A *divisor* on X is a finite formal $\mathbb{Z}$-linear combination of irreducible codimension one closed subvarieties

$$D = \sum n_i Y_i, \quad n_i \in \mathbb{Z}$$

We say D is *effective* and write $D \geq 0$ if $n_i \geq 0$ for all i. For $0 \neq f \in k(X)$ a nonzero rational function on X, the *principal divisor* associated to f is

$$(f) := \sum_{Y \subset X} \nu_Y(f) \cdot Y.$$

Here the sum is over codimension one subvarieties $Y \subset X$ and $\nu_Y(f)$ is the order of vanishing of f along Y. That is, locally at a general point $P \in Y$ we can write $Y = (g = 0)$ and $f = g^\nu h$, where h is regular at P and not divisible by g, and $\nu = \nu_Y(f) \in \mathbb{Z}$. So, (f) is the divisor of zeroes and poles of f counted with multiplicities.

A line bundle L over X is a morphism $p \colon L \to X$ with each fibre L_x a complex vector space of dimension 1, which is locally trivial in the following sense: there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, and local trivialisations

$$\phi_i \colon L|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}$$

compatible with the vector space structure on the fibres and the projection maps p, pr_1. Then

$$\phi_j \circ \phi_i^{-1} \colon U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}, \quad (x, v) \mapsto (x, g_{ij}(x) \cdot v),$$

where the *transition functions* $g_{ij} \colon U_{ij} \to \mathbb{C}^\times$ are nowhere zero regular functions, that is, $g_{ij} \in \mathcal{O}_X^\times(U_{ij})$. The (g_{ij}) define a Cech cocycle in $C^1(\mathcal{U}, \mathcal{O}_X^\times)$. Indeed, we have $g_{jk}g_{ij} = g_{ik}$ so $(dg)_{ijk} = g_{jk}g_{ik}^{-1}g_{ij} = 1$. If we change the trivialisations ϕ_i by composing with multiplication on fibres by $f_i \in \mathcal{O}_X^\times(U_i)$, then (g_{ij}) is replaced by $(f_j g_{ij} f_i^{-1}) = (g_{ij} f_j f_i^{-1}) = g \cdot df$. Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of line bundles. The set $\mathrm{Pic}(X)$ is an abelian group with group law (fibrewise) tensor product: $(L \otimes M)_x := L_x \otimes_{\mathbb{C}} M_x$. This corresponds to multiplication of transition functions. We deduce that

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

Let $p \colon L \to X$ be a line bundle. We can consider the sheaf $\mathcal{L}$ of regular sections of L, that is, for $U \subset X$ open,

$$\mathcal{L}(U) = \{s \mid s \colon U \to L|_U \text{ regular}, \, p \circ s = \mathrm{id}_U\}.$$

In terms of local trivialisations,

$$\mathcal{L}(X) = \{(s_i) \mid s_i \in \mathcal{O}_X(U_i), \; s_j = g_{ij}s_j\}.$$

Conversely, given $\mathcal{L}$ we can reconstruct $p \colon L \to X$.

A *rational section* s of L is a section over some (Zariski) open subset U. In terms of local trivialisations, s is given by $s_i \in k(X)$ such that $s_j = g_{ij}s_i$, cf. above. For s a nonzero rational section we can define the divisor $D = (s) = \sum_{Y \subset X} \nu_Y(s) \cdot Y$ of zeroes and poles of s exactly as for rational functions, using local trivialisations of L. D is determined by L modulo prinicipal divisors (f) (because if s, t are two nonzero rational sections then $f = t/s$ is a rational function). We say D_1, D_2 are *linearly equivalent* if they differ by a principal divisor, write $\mathrm{Div}(X)$ for the group of divisors, and $\mathrm{Cl}(X)$ for the *divisor class group* of divisors modulo linear equivalence. Then the above construction defines an isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X), \quad L \mapsto (s). \tag{3}$$

The inverse of this isomorphism can be described as follows. Given D a divisor, define a sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X) \mid f = 0 \text{ or } (D + (f))|_U \geq 0\}$$

Then $\mathcal{O}_X(D)$ is the sheaf of sections of a line bundle L, and $L \mapsto D$ under the isomorphism (3).

4.1 The Picard group

Let X be a smooth complex projective variety of dimension n. Let X^{an} denote the associated compact complex manifold. Consider the exact sequence of sheaves (the *exponential sequence*)

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}}^{\times} \to 0,$$

where the second arrow is given by $f \mapsto \exp(2\pi i f)$. The induced long exact sequence of cohomology gives

$$0 \to H^1(X, \mathbb{Z}) \to H^1(\mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X) \to \cdots$$

Here we used the following facts:

(1) The sequence of global sections is

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^{\times} \to 0$$

(a global regular function on a projective variety is constant), in particular it is exact,

(2) $H^i(\mathcal{O}_{X^{\mathrm{an}}}) = H^i(\mathcal{O}_X)$ by GAGA [Serre56],

(3) $H^1(X, \mathcal{O}_{X^{\mathrm{an}}}^{\times}) = \operatorname{Pic} X^{\mathrm{an}}$ (we can define the Picard group for a complex manifold in the same way as above), and

(4) $\operatorname{Pic} X^{\mathrm{an}} = \operatorname{Pic} X$ by GAGA.

The map $\operatorname{Pic} X \to H^2(X, \mathbb{Z})$ in the exact sequence above is called the *first Chern class* and denoted c_1. We will describe it explicitly shortly.

The maps $H^i(X, \mathbb{Z}) \to H^i(\mathcal{O}_X)$ in the long exact sequence are the following composition (see [GH, p. 163]):

$$H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{C}) \simeq H^i_{\mathrm{dR}}(X, \mathbb{C}) \twoheadrightarrow H^{0,i} \simeq H^i(\mathcal{O}_X).$$

Here the first map is given by extension of scalars from $\mathbb{Z}$ to $\mathbb{C}$, the second is the de Rham isomorphism, the third is the projection onto the factor $H^{0,i}$ of the Hodge decomposition of $H^i_{\mathrm{dR}}(X, \mathbb{C})$, and the fourth is the Dolbeault isomorphism of the Hodge summand $H^{0,i}$ with $H^i(\mathcal{O}_X)$ (stated earlier as part of Hodge decomposition theorem). In particular, we obtain an exact sequence

$$0 \to H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \to \operatorname{Pic} X \xrightarrow{c_1} H^{1,1} \cap H^2(X, \mathbb{Z}) \to 0$$

Indeed, by the long exact sequence and the above description of the map $H^2(X,\mathbb{Z}) \to H^2(\mathcal{O}_X)$, the image of $c_1\colon \operatorname{Pic}(X) \to H^2(X,\mathbb{Z})$ is the subgroup of classes $\omega \in H^2(X,\mathbb{Z})$ such that, writing $\omega = \omega^{2,0}+\omega^{1,1}+\omega^{0,2}$ using the de Rham isomorphism and Hodge decomposition, we have $\omega^{0,2}=0$. But since ω is a real class, $\omega^{2,0} = \overline{\omega^{0,2}}$, so this is equivalent to $\omega \in H^{1,1}$. (This is the *Lefschetz theorem on* $(1,1)$*-classes.*) The kernel of c_1 is denoted $\operatorname{Pic}^0(X)$ and called the *Picard variety.* It is a complex torus of dimension $q = h^1(\mathcal{O}_X)$. Indeed, we have $\operatorname{Pic}^0(X) = H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ by the exact sequence, so it remains to show that $H^1(X,\mathbb{Z}) \subset H^1(\mathcal{O}_X)$ is a lattice, that is, the map of $\mathbb{R}$-vector spaces

$$H^1(X,\mathbb{Z}) \otimes \mathbb{R} \to H^1(\mathcal{O}_X)$$

is an isomorphism. (Then the quotient $H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is diffeomorphic as a smooth manifold to the real torus $(S^1)^{2q}$.) This fact follows from the Hodge decomposition and the description of the map $H^1(X,\mathbb{Z}) \to H^1(\mathcal{O}_X)$ above: $H^1(\mathcal{O}_X)$ has complex dimension q, and $H^1(X,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ has complex dimension $2q$, equivalently, $H^1(X,\mathbb{Z})$ has rank $2q$. So $H^1(X,\mathbb{Z})\otimes\mathbb{R}$ and $H^1(\mathcal{O}_X)$ have the same real dimension $2q$, and it suffices to show that the above map is injective. If ω is in the kernel then $\omega = \omega^{1,0}+\omega^{0,1}$ where $\omega^{0,1}=0$, and $\omega^{1,0} = \overline{\omega^{0,1}}$ because ω is a real class, so $\omega = 0$ as required.

We can describe the first Chern class $c_1\colon \operatorname{Pic} X \to H^2(X,\mathbb{Z})$ explicitly as follows. If L is a line bundle on X, let D be an element of the associated divisor class (the locus of zeroes and poles of a rational section of L). Then D defines a cycle $[D] \in H_{2n-2}(X,\mathbb{Z})$ of real codimension 2. Let

$$\mathrm{PD}\colon H_{2n-2}(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

denote the Poincaré duality isomorphism. Then $c_1(L) = \mathrm{PD}([D])$. See [GH, p. 141–143].

To recap, the Picard group $\operatorname{Pic}(X)$ is an extension

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{c_1} H^{1,1} \cap H^2(X,\mathbb{Z}) \to 0$$

of the discrete group $H^{1,1} \cap H^2(X,\mathbb{Z})$ by the continuous group $\operatorname{Pic}^0(X)$. Moreover $\operatorname{Pic}^0(X) = H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is a complex torus of dimension q.

Example 4.1. If $H^1(\mathcal{O}_X) = 0$ then $\operatorname{Pic}(X)$ is discrete, and if in addition $H^2(\mathcal{O}_X) = 0$ then $\operatorname{Pic} X \simeq H^2(X,\mathbb{Z})$. This is the case for rational surfaces.

Example 4.2. For a K3 surface X we have $H^1(\mathcal{O}_X) = 0$ and $H^2(\mathcal{O}_X) \simeq \mathbb{C}$. Thus $\operatorname{Pic} X \simeq H^{1,1} \cap H^2(X,\mathbb{Z})$. The abelian group $H^2(X,\mathbb{Z})$ has rank 22, and $H^{1,1}$ has complex dimension 20. The complex subspace $H^{1,1} \subset$

$H^2(X,\mathbb{C})$ is preserved by complex conjugation, so corresponds to a real subspace $H^{1,1}_{\mathbb{R}} \subset H^2(X,\mathbb{R})$ of the same dimension. The intersection $\operatorname{Pic} X = H^{1,1}_{\mathbb{R}} \cap H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$ has rank $0 \le \rho(X) \le 20$, and all these values occur. (Note however that the examples with $\rho(X) = 0$ are non algebraic complex manifolds.) A general projective K3 surface has $\operatorname{Pic} X \simeq \mathbb{Z}$ generated by an ample line bundle (that is, some multiple of the generator corresponds to the divisor class given by a hyperplane section in an embedding $X \subset \mathbb{P}^N$).

4.2 The intersection product on the Picard group

Let X be a smooth projective surface over $k = \mathbb{C}$. Recall that the first Chern class

$$c_1\colon \operatorname{Pic} X \to H^2(X,\mathbb{Z})$$

is identified via Poincaré duality with the map

$$\operatorname{Cl} X \to H_2(X,\mathbb{Z}), \quad D \mapsto [D]$$

from the divisor class group to homology given by regarding a divisor as a 2-cycle. We have the (topological) intersection product

$$\cap\colon H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Z}$$

and we define the intersection product on $\operatorname{Pic} X = \operatorname{Cl} X$ as the induced product,

$$D_1 \cdot D_2 := [D_1] \cap [D_2].$$

Algebraically, if $C_1, C_2 \subset X$ are irreducible curves,

$$C_1 \cdot C_2 = \sum_{P \in C_1 \cap C_2} (C_1 \cdot C_2)_P$$

where the *intersection multiplicity* $(C_1 \cdot C_2)_P$ of C_1, C_2 at $P \in X$ is defined as follows: locally at $P \in X$ write $C_i = (f_i = 0) \subset X$, then

$$(C_1 \cdot C_2)_P := \dim_k \mathcal{O}_{X,P}/(f_1, f_2) \tag{4}$$

One can check this agrees with the topological intersection product by a C^∞ perturbation argument, see [GH, p. 62].

Remark 4.3. If X is a complex manifold and $Z, W \subset X$ are complex submanifolds meeting transversely, then, since the orientations of Z, W, X are induced by the complex structure, at each point $P \in Z \cap W$ the intersection index $i_P(Z, W) = +1$. That is, there are no signs in the topological intersection product $[Z] \cap [W]$. (This easy observation is extremely important in algebraic combinatorics.)

Remark 4.4. If X is defined over an arbitrary algebraically closed field k one can define the intersection product using the equation (4). One then needs to show that it is well defined modulo linear equivalence.

Suppose $C \subset X$ is an irreducible curve. We describe two ways to compute the self-intersection C^2. First, we can find a rational function f such that the principal divisor (f) contains C with multiplicity 1 (just take f a local equation of C at a point of X). Then $D = C - (f)$ is linearly equivalent to C and does not contain C as a component, so $C^2 = D \cdot C$ and we can compute $D \cdot C$ as above. Alternatively, we have the general formula

$$D \cdot C = \deg \mathcal{L}|_C$$

where $\mathcal{L} = \mathcal{O}_X(D)$ is (the sheaf of sections of) the line bundle associated to D. Setting $D = C$ we obtain

$$C^2 = \deg \mathcal{O}_X(C)|_C = \deg \mathcal{N}_{C/X}$$

where

$$\mathcal{N}_{C/X} = \mathcal{O}_X(C)|_C \tag{5}$$

is the normal bundle of $C \subset X$.

We explain the equality (5). We assume $C \subset X$ is smooth for simplicity. Recall that the *normal bundle* $\mathcal{N}_{C/X}$ of $C \subset X$ is the line bundle defined by the exact sequence of vector bundles on C

$$0 \to T_C \to T_X|_C \to \mathcal{N}_{C/X} \to 0$$

where T_C, T_X denote the tangent bundles of C, X. Dually,

$$0 \to \mathcal{N}^*_{C/X} \to \Omega_X|_C \to \Omega_C \to 0.$$

Locally, let $C = (x = 0) \subset X$, then $\mathcal{N}^*_{C/X}$ is generated by dx. We have an exact sequence of sheaves

$$0 \to \mathcal{I}_{C/X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

where $\mathcal{I}_{C/X} \subset \mathcal{O}_X$ is the ideal sheaf of regular functions vanishing on C. There is a natural isomorphism

$$\mathcal{I}_{C/X}|_C \xrightarrow{\sim} \mathcal{N}^*_{C/X}$$

given locally by $x \mapsto dx$. Finally, observe that $\mathcal{I}_{C/X} = \mathcal{O}_X(-C)$ — a section of $\mathcal{O}_X(-C)$ over $U \subset X$ is a rational function f such that $(f) - C \geq 0$ on U, equivalently, f is regular on U and vanishes on C. So $\mathcal{O}_X(-C)|_C \simeq \mathcal{N}^*_{C/X}$, and dualising gives (5).

For X a smooth variety, the *canonical line bundle* $\omega_X = \wedge^{\dim X}\Omega_X$ is the top exterior power of the cotangent bundle. The *canonical divisor class* K_X is the associated divisor class. Now let X be a smooth projective surface and $C \subset X$ a smooth curve. Then we have the *adjunction formula*

$$K_C = (K_X + C)|_C. \tag{6}$$

Taking degrees,

$$2g - 2 = \deg K_C = (K_X + C) \cdot C$$

where g is the genus of C. The adjunction formula is deduced from the exact sequence

$$0 \to \mathcal{N}^*_{C/X} \to \Omega_X|_C \to \Omega_C \to 0 \tag{7}$$

as follows. If

$$0 \to U \to V \to W \to 0$$

is an exact sequence of vector spaces of dimensions r, s, t, we have a natural isomorphism

$$\wedge^s V \simeq \wedge^r U \otimes \wedge^t W.$$

This induces a corresponding isomorphism for an exact sequence of vector bundles. In particular, from (7) we obtain

$$\wedge^2\Omega_X|_C \simeq \mathcal{N}^*_{C/X} \otimes \Omega_C.$$

Rearranging and using $\mathcal{N}_{C/X} = \mathcal{O}_X(C)|_C$ gives

$$\omega_C = \omega_X \otimes \mathcal{O}_X(C)|_C$$

and passing to the associated divisors we obtain the adjunction formula (6).

Example 4.5. Let X be a smooth surface, $P \in X$ a point, and $\pi\colon \tilde{X} \to X$ the blowup of $P \in X$. So $\pi^{-1}P = E$ is a copy of $\mathbb{P}^1$, the *exceptional curve*, and π restricts to an isomorphism

$$\pi\colon \tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{P\}.$$

We compute that the self intersection $E^2 = -1$. Let $D \subset X$ be a smooth curve through P. Let $D' \subset \tilde{X}$ be the *strict transform* of D, that is, D' is the closure of the preimage of $D \setminus \{P\}$. Then D' is a smooth curve which intersects E transversely in one point. In particular, $D' \cdot E = 1$. Now consider the pullback π^*D of D. (In general, if $f\colon X \to Y$ is a morphism of smooth varieties and D is a divisor on Y, then locally on Y the divisor D is principal, say $D|_{U_i} = (g_i)$ for some open cover $\mathcal{U} = \{U_i\}$, and we define f^*D by $f^*D|_{f^{-1}U_i} = (g_i \circ f)$.) Let x, y be local coordinates at $P \in X$ such that $D = (y = 0)$. Over $P \in X$ we have a chart of the blowup π of the form

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (u, uy').$$

We deduce that $\pi^*D = D' + E$ (because in this chart $\pi^*D = \pi^*(y = 0) = (uy' = 0) = E + D'$). Now $\pi^*D \cdot E = 0$ because we can write $D \sim B$, where B is a divisor not containing P, then $\pi^*D \cdot E = \pi^*B \cdot E = 0$ since π^*B is disjoint from E. Combining we deduce that $E^2 = -1$ as claimed.

We make a few more comments about the self-intersection C^2 of a smooth curve $C \subset X$. If $C^2 < 0$ then C cannot move in a family, that is, there does not exist a non-trivial family $\{C_t\}$ of curves with $C_0 = C$. Indeed, given such a family we have $C^2 = C \cdot C_t \geq 0$, a contradiction. Now suppose $C^2 \geq 0$ and $\{C_t\}$ is a family with $C_0 = C$. The family determines a section $s \in \Gamma(C, \mathcal{N}_{C/X})$ of the normal bundle of C in X as follows. Locally, write $C = (f = 0) \subset X$, and $C_t = (f + tg + \cdots = 0)$ where $\cdots$ denotes higher order terms in t. Then the section s of $\mathcal{N}_{C/X} = \mathcal{I}^*_{C/X}|_C$ is locally given by $f \mapsto \bar{g}$, where $\bar{g} \in \mathcal{O}_C$ is the image of $g \in \mathcal{O}_X$. (So, s corresponds to the *first order deformation of* C given by the $\{C_t\}$.) This gives a geometric explanation for the formula $C^2 = \deg \mathcal{N}_{C/X}$ in this case — the zero locus of s is approximately equal to $C \cap C_t$ for small t.

5 The Riemann–Roch theorem

If X is an algebraic variety over a field k and $\mathcal{F}$ is a coherent sheaf on X then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. If in addition X is projective then $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space for all i. See [Serre55, p. 259, Thm. 1]. We write $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$.

Now assume X is a projective variety over k of dimension n and $\mathcal{F}$ is a coherent sheaf on X. We define the *Euler-Poincaré characteristic* $\chi(X, \mathcal{F})$ of $\mathcal{F}$ by

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

If

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is an exact sequence of sheaves on X then we have

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{E}) + \chi(X, \mathcal{G}). \tag{8}$$

Indeed, we have the long exact sequence of cohomology groups

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to \cdots \to H^n(X, \mathcal{G}) \to 0.$$

By linear algebra the alternating sum of the dimensions of the terms in this exact sequence is zero. This gives (8).

Because of the additivity property (8) it is usually much easier to compute $\chi(X, \mathcal{F})$ than the individual terms $h^i(X, \mathcal{F})$. In fact, $\chi(X, \mathcal{F})$ can be expressed solely in terms of topological invariants of X and $\mathcal{F}$. This is the content of the Hirzebruch–Riemann–Roch formula. We describe the formula in case $\mathcal{F}$ is a line bundle $\mathcal{L} = \mathcal{O}_X(D)$ and $\dim X = 1, 2$.

5.1 Curves

Let X be a smooth projective curve over $k = \mathbb{C}$. The cohomological form of the Riemann–Roch formula for $\mathcal{L} = \mathcal{O}_X(D)$ is

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D.$$

That is,

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + \deg D.$$

Serre duality defines an isomorphism $H^1(\mathcal{O}_X(D)) \simeq H^0(\mathcal{O}_X(K_X - D))^*$, so we can rewrite the above formula as

$$h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K_X - D)) = h^0(\mathcal{O}_X) - h^0(\mathcal{O}_X(K_X)) + \deg D.$$

Finally $h^0(\mathcal{O}_X) = 1$ and $h^0(\mathcal{O}_X(K_X)) = g$, the genus of X, so we obtain

Theorem 5.1. *(Riemann–Roch for curves) Let X be a smooth projective curve of genus g over $k = \mathbb{C}$. Then*

$$h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K_X - D)) = 1 - g + \deg D.$$

The Riemann–Roch formula in this form was proved in 508A (notes available on my website). In that course we used the notation $L(D) = H^0(\mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(D))$ and $l(D) = \dim_k L(D)$.

5.2 Surfaces

Theorem 5.2. *(Riemann Roch for surfaces) Let X be a smooth projective surface over $k = \mathbb{C}$. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D(D - K_X).$$

Proof. First suppose that $D = C$ is a smooth curve on X. We have the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

on X, where $\mathcal{O}_X(-C) = \mathcal{I}_{C/X} \subset \mathcal{O}_X$ is the ideal sheaf of $C \subset X$ (the sheaf of regular functions on X vanishing on C). We tensor this exact sequence with the line bundle $\mathcal{O}_X(C)$ to obtain

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0$$

(note that tensor product of line bundles corresponds to addition of divisors). Now by additivity of χ we have

$$\chi(\mathcal{O}_X(C)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C)|_C).$$

So, to verify the Riemann–Roch formula in this case, we need to show that

$$\chi(\mathcal{O}_X(C)|_C) = \frac{1}{2}C(C - K_X).$$

By Riemann–Roch on C we have

$$\chi(\mathcal{O}_X(C)|_C) = 1 - g + \deg \mathcal{O}_X(C)|_C = 1 - g + C^2$$

where g is the genus of C, and

$$2g - 2 = \deg K_C = (K + C)C$$

by the adjunction formula. Combining we obtain

$$\chi(\mathcal{O}_X(C)|_C) = -\frac{1}{2}(K + C)C + C^2 = \frac{1}{2}C(C - K_X)$$

as required.

Now suppose D is arbitrary. Let H be a hyperplane section of X in some projective embedding $X \subset \mathbb{P}^N$. Then, for $n \gg 0$, $D + nH$ is a hyperplane section in some embedding $X \subset \mathbb{P}^M$. This follows from *Serre vanishing* (if $\mathcal{F}$ is a coherent sheaf on a projective variety X and H is a hyperplane section, then $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(nH)) = 0$ for $i > 0$ and $n \gg 0$), together with the *embedding criterion* (a map $X \to \mathbb{P}^N$ from a projective variety to projective space defined by a line bundle $\mathcal{L}$ is an embedding if the global sections $s \in \Gamma(X, \mathcal{L})$ separate points and tangent vectors). We omit the details here. We write

$$D = (D + nH) - nH \sim A - B,$$

where A and B are general hyperplane sections of X in two (different) embeddings in projective space. In particular, A and B are smooth and irreducible. (This is *Bertini's theorem*: if $X \subset \mathbb{P}^N$ is a smooth closed subvariety of projective space then a general hyperplane section of X is smooth, and is irreducible if $\dim X > 1$. See [GH, p. 137].) Now consider the exact sequence

$$0 \to \mathcal{O}_X(A - B) \to \mathcal{O}_X(A) \to \mathcal{O}_X(A)|_B \to 0.$$

Note that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(A - B)$ because $D \sim A - B$. So

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(A)) - \chi(\mathcal{O}_X(A)|_B),$$

and we can compute $\chi(\mathcal{O}_X(A))$ using the Riemann–Roch formula for an irreducible smooth divisor on a surface proved above and $\chi(\mathcal{O}_X(A)|_B)$ using the Riemann-Roch formula on B. We deduce the Riemann–Roch formula on a surface in the general case. □

As in the case of curves, the Riemann–Roch formula for surfaces is used together with Serre duality.

Theorem 5.3. *(Serre duality) Let X be a smooth projective variety over $k = \mathbb{C}$ of dimension n and D a divisor on X. Then there is a k-bilinear map*

$$H^i(X, \mathcal{O}_X(D)) \times H^{n-i}(X, \mathcal{O}_X(K_X - D)) \to H^n(\mathcal{O}_X(K_X)) \simeq k.$$

which induces an isomorphism $H^i(X, \mathcal{O}_X(D)) \simeq H^{n-i}(X, \mathcal{O}_X(K_X - D))^*$.

This is proved in [Hartshorne, III.7] using projective methods and in [GH, p. 153] and [Voisin, p. 135, Thm. 5.32] using Hodge theory. (Note that the analytic approach is closely related to Poincaré duality.) It was also proved for curves in 508A using residues of differential forms (there we used the notation $I(D)$ for $H^1(\mathcal{O}_X(D))$).

Combining the Riemann–Roch formula for surfaces with Serre duality we obtain

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(K_X - D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D(D - K_X).$$

In applications we often use the resulting inequality

$$h^0(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(K_X - D)) \geq \chi(\mathcal{O}_X) + \frac{1}{2}D(D - K_X)$$

because $h^1(\mathcal{O}_X(D))$ does not have a direct geometric meaning.

If X is a smooth projective curve then $\chi(\mathcal{O}_X) = 1 - g$ where g is the genus of X. The analogous result for surfaces is

Theorem 5.4. *(Noether's formula)[GH, p. 600] Let X be a smooth projective surface over $k = \mathbb{C}$. Then*

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)),$$

where

$$e(X) := \sum(-1)^i \dim_{\mathbb{R}} H^i(X, \mathbb{R})$$

is the (topological) Euler characteristic of X.

6 Hodge index theorem

Let X be a smooth complex projective surface (or more generally a compact complex Kähler manifold). Consider the wedge product

$$\wedge \colon H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}.$$

Recall that we have the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

and $H^{0,2} = \overline{H^{2,0}}$, $H^{1,1} = \overline{H^{1,1}}$. Thus $H^{2,0} \oplus H^{0,2}$ and $H^{1,1}$ are preserved by complex conjugation, so are obtained from $\mathbb{R}$-vector subspaces $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ and $H^{1,1}_{\mathbb{R}}$ of $H^2(X, \mathbb{R})$ by extension of scalars from $\mathbb{R}$ to $\mathbb{C}$, and

$$H^2(X, \mathbb{R}) = (H^{2,0} \oplus H^{0,2})_{\mathbb{R}} \oplus H^{1,1}_{\mathbb{R}}.$$

Theorem 6.1. *(Hodge index theorem for surfaces) [GH, p. 123–125] [Voisin, p. 152] The wedge product is positive definite on $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ and has signature $(1, h^{1,1} - 1)$ on $H^{1,1}_{\mathbb{R}}$.*

The theorem is proved using the Lefschetz decomposition of cohomology. We will prove a weaker assertion in the algebraic context shortly.

Let X be a smooth projective surface over $k = \mathbb{C}$. We prove the first statement in the Hodge index theorem: the wedge product is positive definite on $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$. (The proof of the second statement, that the signature of $H^{1,1}_{\mathbb{R}}$ is $(1, h^{1,1} - 1)$, is more involved.) Let $0 \neq [\omega] \in (H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$. Then $\omega = \alpha + \bar{\alpha}$ where α is a $(2,0)$-form, and

$$\omega^2 = \omega \wedge \omega = 2\alpha \wedge \bar{\alpha}$$

(Note $\alpha^2 = \bar{\alpha}^2 = 0$ since α is a $(2,0)$-form.) Locally on X, write $\alpha = f dz_1 \wedge dz_2$ where f is a smooth $\mathbb{C}$-valued function and z_1, z_2 are complex coordinates on X. Then

$$\omega^2 = 2\alpha \wedge \bar{\alpha} = 2|f|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$
$$= -2|f|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = -2|f|^2(-2i)^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$$
$$= 8|f|^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

Here we wrote $z_j = x_j + iy_j$ and used the identity

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy.$$

We deduce that $[\omega]^2 = \int_X \omega^2 > 0$ as required.

We now describe the algebraic consequences of the Hodge index theorem. Let X be a smooth projective surface over $k = \mathbb{C}$ as above. We say two divisors D, D' on X are *numerically equivalent* and write $D \equiv D'$ if $D \cdot C = D' \cdot C$ for every curve $C \subset X$. We write $\operatorname{Num} X$ for the (free) abelian group of divisors modulo numerical equivalence, and $\rho(X)$ for the rank of $\operatorname{Num} X$.

Corollary 6.2. $\operatorname{Num} X = H^{1,1} \cap (H^2(X, \mathbb{Z})/\operatorname{Tors})$ *and the intersection product on* $\operatorname{Num} X$ *has signature* $(1, \rho - 1)$.

Proof. Recall that the intersection product on $\operatorname{Cl}(X) = \operatorname{Pic}(X)$ is induced from the wedge product via the first Chern class

$$c_1 \colon \operatorname{Pic} X \to H^2(X, \mathbb{Z}),$$

and the image of c_1 is $H^{1,1} \cap H^2(X, \mathbb{Z})$. Let H be a hyperplane section of X in some embedding in projective space. Then $H^2 > 0$, so the wedge product on the orthogonal complement $H^{\perp} \subset H^{1,1}_{\mathbb{R}}$ is negative definite by the Hodge index theorem. In particular, the form is negative definite on $H^{\perp} \subset H^{1,1} \cap (H^2(X, \mathbb{Z})/\operatorname{Tors})$, and so is nondegenerate of hyperbolic signature $(1, r - 1)$ on $H^{1,1} \cap (H^2(X, \mathbb{Z})/\operatorname{Tors})$. We deduce that $\operatorname{Num} X = H^{1,1} \cap (H^2(X, \mathbb{Z})/\operatorname{Tors})$, of signature $(1, \rho - 1)$, as required. □

We give an algebraic proof that $\operatorname{Num} X$ has hyperbolic signature $(1, \rho-1)$ using the Riemann–Roch formula. Let H be a hyperplane section of X. Then $H^2 > 0$. If the signature is not hyperbolic, there exists a divisor D such that $D \cdot H = 0$ and $D^2 > 0$. (Note that the intersection product on $\operatorname{Num} X$ is nondegenerate by construction.) We show that such a divisor cannot exist. Consider the Riemann–Roch formula for nD, $n \in \mathbb{Z}$:

$$h^0(\mathcal{O}_X(nD)) + h^0(\mathcal{O}_X(K_X - nD)) \geq \chi(\mathcal{O}_X(nD)) = \chi(\mathcal{O}_X) + \frac{1}{2}nD(nD - K_X).$$

In particular, as $n \to \pm\infty$, $\chi(\mathcal{O}_X(nD)) \sim \frac{1}{2}n^2D^2 \to \infty$. We claim that $h^0(\mathcal{O}_X(nD)) = 0$ for all $n \in \mathbb{Z}$. Indeed if $0 \neq s \in H^0(\mathcal{O}_X(nD))$, then $D' = (s = 0)$ is an effective divisor linearly equivalent to D. But then $D' \cdot H = D \cdot H = 0$, and H is a hyperplane section, so $D' = 0$. This contradicts $D^2 > 0$. We deduce that $h^0(\mathcal{O}_X(K_X - nD)) \to \infty$ as $n \to \pm\infty$. For $n \gg 0$, pick $0 \neq s \in H^0(K_X - nD)$, then tensor product with s defines an inclusion $H^0(K_X + nD) \subset H^0(2K_X)$. So $h^0(K_X + nD) \leq h^0(2K_X)$ for $n \gg 0$, a contradiction.

7 Birational geometry of surfaces

7.1 The blowup of a point on a smooth surface

Let X be a smooth complex surface and $P \in X$ a point. The *blowup* of $P \in X$ is a birational morphism $\pi \colon \tilde{X} \to X$ such that $E := \pi^{-1}P$ is a copy of $\mathbb{P}^1$, the *exceptional curve*, and π restricts to an isomorphism

$$\tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{P\}.$$

If x, y are local coordinates at $P \in X$, then, working locally analytically at $P \in X$, we can identify $P \in X$ with $0 \in \mathbb{A}^2_{x,y}$. Then the blowup $\pi \colon \tilde{X} \to X$ is given by two charts as follows:

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y} \quad (u, y') \mapsto (x, y) = (u, uy')$$

$$\mathbb{A}^2_{x',v} \to \mathbb{A}^2_{x,y} \quad (x', v) \mapsto (x, y) = (vx', v)$$

Here the exceptional curve E is given by $(u = 0)$ and $(v = 0)$ in the two charts. We observed earlier that $E^2 = -1$.

Proposition 7.1. *We have an isomorphism*

$$\operatorname{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} \tilde{X}, \quad (D, n) \mapsto \pi^* D + nE.$$

Proof. By the definition of the divisor class group, we have an exact sequence

$$\mathbb{Z}E \to \mathrm{Cl}(\tilde{X}) \to \mathrm{Cl}(\tilde{X} \setminus E) \to 0$$

where the second arrow is given by restriction to the open set $\tilde{X} \setminus E \subset \tilde{X}$. (See [Hartshorne, p. 133, II.6.5] for more details). Now $\tilde{X} \setminus E \simeq X \setminus \{P\}$ and $\mathrm{Cl}(X \setminus \{P\}) = \mathrm{Cl}(X)$ because $P \in X$ has codimension 2. So we obtain an exact sequence

$$\mathbb{Z}E \to \mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}(X) \to 0.$$

It remains to show that the first arrow is injective. This follows from $E^2 = -1$. ☐

Next we describe the blowup topologically. Let $P \in B \subset X$ be a ball around $P \in X$ (in the Euclidean topology) and $N = \pi^{-1}B$, a tubular neighbourhood of $E \subset \tilde{X}$. We consider the Mayer–Vietoris sequence in (integral) cohomology for $\tilde{X} = (\tilde{X} \setminus E) \cup N$ and $X = (X \setminus \{P\}) \cup B$. Recall that the Mayer–Vietoris sequence for a union $K \cup L$ is

$$\cdots \to H^i(K \cup L) \to H^i(K) \oplus H^i(L) \to H^i(K \cap L) \to H^{i+1}(K \cup L) \to \cdots .$$

In our case we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & H^i(\tilde{X}) & \to & H^i(\tilde{X} \setminus E) \oplus H^i(N) & \to & H^i(\partial N) & \to & H^{i+1}(\tilde{X}) & \to \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & H^i(X) & \to & H^i(X \setminus \{P\}) \oplus H^i(B) & \to & H^i(\partial B) & \to & H^{i+1}(X) & \to \cdots
\end{array}$$

Now B is contractible, so $H^i(B) = H^i(\mathrm{pt})$, and N is homotopy equivalent to $E \simeq \mathbb{P}^1$, so $H^i(N) \simeq H^i(\mathbb{P}^1)$. Also $\tilde{X} \setminus E \simeq X \setminus \{P\}$ and $\partial N \simeq \partial B$. We deduce that $H^2(\tilde{X}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus \mathbb{Z}$ and $H^i(\tilde{X}, \mathbb{Z}) \simeq H^i(X, \mathbb{Z})$ for $i \neq 2$. Note that the isomorphism $H^2(\tilde{X}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus \mathbb{Z}$ is compatible with the isomorphism $\mathrm{Pic}(\tilde{X}) \simeq \mathrm{Pic}\, X \oplus \mathbb{Z}$ obtained above under c_1.

Proposition 7.2. *Let X be a smooth surface, $\pi\colon \tilde{X} \to X$ the blowup of a point $P \in X$, and $E = \pi^{-1}P$ the exceptional curve. Then $K_{\tilde{X}} = \pi^* K_X + E$.*

Proof. Let x, y be local coordinates at $P \in X$ and $\omega = dx \wedge dy$, a rational 2-form on X. Then $K_X = (\omega)$, the divisor of zeroes and poles of ω. (Here $(\omega) := \sum_{C \subset X} \nu_C(\omega) C$ where the sum is over irreducible divisors $C \subset X$ and $\nu_C(\omega)$ is the order of vanishing of ω along C defined as follows: let $Q \in C$ be a general point, z, w local coordinates at Q, and write $\omega = f dz \wedge dw$ where $f \in k(X)$, then $\nu_C(\omega) := \nu_C(f)$. Equivalently, (ω) is the divisor of zeroes and poles of ω regarded as a rational section of the canonical line bundle ω_X). Also $K_{\tilde{X}} = (\pi^*\omega) = (d(\pi^*x) \wedge d(\pi^*y))$. (If $f\colon X \to Y$ is a morphism of smooth varieties and ω is a k-form on Y then the *pullback* $f^*\omega$ of ω is defined in the obvious way: locally on Y, $\omega = \sum g_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $x_1, \ldots, x_n$ are local coordinates and $f^*\omega := \sum f^*g_{i_1\cdots i_k} d(f^*x_{i_1}) \wedge \cdots \wedge d(f^*x_{i_k})$, where $f^*g = g \circ f$ denotes pullback of functions.)

We need to compare $\pi^*K_X = \pi^*(\omega)$ and $K_{\tilde{X}} = (\pi^*\omega)$. Clearly these divisors coincide over $\tilde{X} \setminus E$ (because π restricts to an isomorphism $\tilde{X} \setminus E \simeq X \setminus \{P\}$.) The divisor $\pi^*(\omega)$ does not contain $E = \pi^{-1}P$ because ω is regular and nonzero at P. We compute the coefficent of E in $(\pi^*\omega)$ by a local calculation: one chart of the blowup is

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (x, y) = (u, uy').$$

So, on this chart,

$$\pi^*\omega = d(\pi^*x) \wedge d(\pi^*y) = du \wedge d(uy') = du \wedge (y'du + udy') = udu \wedge dy'.$$

Thus $(\pi^*\omega) = (u = 0) = E$ in this chart. Combining, we deduce that $K_{\tilde{X}} = \pi^*K_X + E$. □

7.2 Elimination of indeterminacy of rational maps

Let X and Y be varieties and $f\colon X \dashrightarrow Y$ a rational map. (Recall that a *rational map* $f\colon X \to Y$ is a morphism $f\colon U \to Y$ from a Zariski open subset $U \subset X$, and we regard two rational maps $f_1, f_2\colon X \dashrightarrow Y$ as equivalent if they agree on $U_1 \cap U_2$.) Assume that X is smooth. Then f restricts to a morphism $f\colon U \to Y$ where $U = X \setminus Z$ and $Z \subset X$ is closed of codimension ≥ 2. To see this, assume (for simplicity) Y is projective, and write $Y \subset \mathbb{P}^N$, where Y is not contained in a hyperplane. Then $f\colon X \dashrightarrow Y \subset \mathbb{P}^N$ is given by $f = (f_0\colon \cdots f_N)$ where $f_i \in k(X)^\times$ are nonzero rational functions on X. In particular, f is well defined at $P \in X$ if each f_i is regular at P and some f_j is nonzero at P. Moreover, the $(N+1)$-tuples $(f_0\colon \cdots : f_N)$ and $(gf_0\colon \cdots : gf_N)$ define the same rational map for any $g \in k(X)^\times$. So, locally near a given point $P \in X$, we can clear denominators so that each f_i is regular, and cancel common factors so that the locus of common zeroes has codimension ≥ 2. Thus f is well defined outside a locus $Z \subset X$ of codimension ≥ 2 as claimed. In particular, if X is a smooth surface, a rational map $f\colon X \dashrightarrow Y$ is well defined outside a finite set of points $Z \subset X$.

Recall that for X a smooth variety and D a divisor on X the *complete linear system* $|D|$ on X is the set of effective divisors D' linearly equivalent to D. It is identified with the projectivisation $\mathbb{P}\Gamma(\mathcal{O}_X(D))$ of the space of global sections s of the associated line bundle $\mathcal{O}_X(D)$ via $s \mapsto D' = (s = 0)$. A *linear system* is a projective subspace of a complete linear system. We say an irreducible divisor $F \subset X$ is a *fixed component* of a linear system δ if every $D \in \delta$ contains F.

A rational map $f\colon X \dashrightarrow Y \subset \mathbb{P}^N$ as above corresponds to the linear system δ on X without fixed components given by

$$\delta := \{f^*H \mid H \subset \mathbb{P}^N \text{a hyperplane}\}.$$

Note that f^*H makes sense because f is well defined outside a codimension 2 locus $Z \subset X$. Explicitly, write $f = (f_0\colon \cdots : f_N)$, $D_i = (f_i)$, and $D'_i = D_i - \min_j D_j$. (Here by $\min_j D_j$ we mean: write $D_j = \sum_k n_{jk} Y_k$, then $\min_j D_j := \sum_k (\min_j n_{jk}) Y_k$.) Then $D'_i = f^*(X_i = 0)$, and δ is the linear system generated (as a projective space) by the D'_i. Equivalently, in terms of line bundles, let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^N}(1)$ be the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$, and $s_i = f^*X_i \in \Gamma(X, \mathcal{L})$ the global sections of $\mathcal{L}$ given by the pullback of the global sections X_i of $\mathcal{O}_{\mathbb{P}^N}(1)$. Then $f = (s_0\colon \cdots : s_N)$, and $D'_i = (s_i = 0)$.

The *base locus* of δ is

$$\operatorname{Bs}\delta := \{P \in X \mid P \in D \text{ for all } D \in \delta\} \subset X,$$

It is the locus $Z \subset X$ where f is undefined. In the above notation

$$Z = \operatorname{Bs}\delta = \bigcap_{i=0}^{N} D'_i = (s_0 = \cdots = s_N = 0) \subset X.$$

Proposition 7.3. *Let X be a smooth surface and $f\colon X \dashrightarrow Y$ a rational map to a variety Y. Then there exists a sequence of blowups*

$$W = X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that the induced map $g = f \circ \pi_1 \circ \cdots \pi_n$ is a morphism. That is, we have a commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{g} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

where $p = \pi_1 \circ \cdots \circ \pi_n$ is the composite of a sequence of blowups and g is a morphism.

Proof. Let δ be the linear system defining the rational map f and suppose $P \in X$ is a basepoint of δ. Let $D \in \delta$ be a general element. Let x, y be local coordinates at P and write $D = (f(x, y) = 0)$ near P. Let

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots$$

where $f_k(x, y)$ is homogeneous of degree k. So m is the *multiplicity* of D at $P \in X$. We compute that $\pi^*D = D' + mE$ where D' is the strict transform of D. (Recall that the *strict transform* D' of D is defined as follows: D' is the closure in $\tilde{X}$ of the inverse image of the restriction of D to $X \setminus \{P\}$ under the isomorphism $\tilde{X} \setminus E \simeq X \setminus \{P\}$.) We use the chart

$$\mathbb{A}^2_{u,y'} \to \mathbb{A}^2_{x,y}, \quad (u, y') \mapsto (x, y) = (u, uy').$$

In this chart, $\pi^*D = (\pi^*f) = (f(u, uy') = 0)$, and

$$f(u, uy') = (f_m(1, y') + uf_{m+1}(u, y') + \cdots)u^m,$$

so $\pi^*D = D' + mE$ as claimed. The composite rational map $\tilde{f} = f \circ \pi$ is defined by the linear system

$$\tilde{\delta} = \{\tilde{D} := \pi^*D - mE \mid D \in \delta\}.$$

Note that

$$\tilde{D}^2 = (\pi^*D - mE)^2 = D^2 - m < D^2.$$

(Here we used $E^2 = -1$, and $\pi^*A \cdot E = 0$, $\pi^*A \cdot \pi^*B = A \cdot B$ for A, B divisors on X.) If $\tilde{f}$ is not a morphism we repeat this process.

If δ is a linear system without fixed components then $D^2 \geq 0$ for $D \in \delta$ (because there exist effective divisors $D_1, D_2 \in \delta$ with no common components, so $D^2 = D_1 \cdot D_2 \geq 0$). Hence the above procedure stops after a finite number of blowups. □

Example 7.4. Consider the rational map

$$f \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad (X_0 : X_1 : X_2) \mapsto (X_0 : X_1).$$

The map f corresponds to the linear system

$$\delta = \{(a_0X_0 + a_1X_1 = 0) \mid (a_0 : a_1) \in \mathbb{P}^1\}$$

on $\mathbb{P}^2$ which has a single basepoint $P = (0 : 0 : 1) \in \mathbb{P}^2$. Let $\pi \colon \tilde{X} \to \mathbb{P}^2$ be the blowup of $P \in \mathbb{P}^2$. Then the composite $\tilde{f} := f \circ \pi \colon \tilde{X} \to \mathbb{P}^1$ is a morphism.

7.3 Negativity of contracted locus

Before stating the result we quickly review the notion of a normal variety. A variety X is *normal* if for every point $Q \in X$ the local ring $\mathcal{O}_{X,Q}$ is integrally closed. Equivalently, X satisfies the following two conditions:

(R_1) The singular locus of X has codimension ≥ 2.

(S_2) If $U \subset X$ is open, $Z \subset X$ is closed of codimension ≥ 2, and f is a rational function which is regular on $U \setminus Z$, then f is regular on U.

See [Matsumura, p. 183, Thm. 23.8]. In particular, a normal surface has a finite number of singular points, and (for example) a surface obtained from a smooth surface by glueing two points together is not normal (since it does not satisfy the condition S_2). Smooth varieties are normal, and a curve is normal iff it is smooth. If X is a variety then the *normalisation* of X is a normal variety X^ν together with a finite birational morphism $\nu \colon X^\nu \to X$. Any dominant map $f \colon Y \to X$ from a normal variety Y factors uniquely through the normalisation of X.

Theorem 7.5. *Let $f \colon X \to Y$ be a birational morphism from a smooth projective surface X to a normal projective surface Y. Let $E_1, \ldots, E_r$ be the exceptional curves over a point $Q \in Y$ (the curves contracted by f to Q). Then $(\sum a_i E_i)^2 < 0$ for $(a_1, \ldots, a_r) \neq 0$.*

Proof. We work locally over $Q \in Y$. Let $C \subset X$ be a irreducible curve such that $C \cap E_i \neq \emptyset$ (and $C \neq E_i$) for all i. For example, we can take C a general hyperplane section of X. Choose $g \in \mathcal{O}_{Y,Q}$ such that $f(C) \subset F := (g = 0)$. Then $(f^*g) = f^*F = F' + \sum \mu_i E_i$ where F' denotes the strict transform of F and $\mu_i > 0$ for each i. Write $D = \sum \mu_i E_i$. Then $D \cdot E_i = -F' \cdot E_i < 0$ for all i because $D + F' = (f^*g) \sim 0$ and F' contains C (which intersects each E_i) by construction. Now the result follows from the algebraic lemma below. □

Lemma 7.6. *Let M be a free abelian group spanned by elements $e_1, \ldots, e_r$ and*

$$M \times M \to \mathbb{Z}, \quad (a, b) \mapsto a \cdot b$$

a symmetric bilinear form such that

- *$e_i \cdot e_j \geq 0$ for all $i \neq j$, and*
- *there exists a linear combination $d = \sum \mu_j e_j$, with $\mu_j > 0$ for all j, such that $d \cdot e_i < 0$ for all i.*

Then $e_1, \ldots, e_r$ are linearly independent and $m^2 < 0$ for $0 \neq m \in M$.

Proof. Write $a_{ij} = e_i \cdot e_j$ and let

$$\phi\colon \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}, \quad \phi(x, y) = \sum a_{ij} x_i y_j$$

be the symmetric bilinear form with matrix $A = (a_{ij})$. We need to show that ϕ is negative definite. The symmetric matrix A can be diagonalised by an orthogonal matrix P. That is, there exists P such that $P^T AP$ is diagonal and $P^T P = I$. Let $\lambda_1 \leq \ldots \leq \lambda_r$ be the diagonal entries of $P^T AP$ (the eigenvalues of A). Then, writing $x = Px'$, we have $\phi(x, x) = \sum \lambda_i {x'_i}^2$ and $\|x'\| = \|x\|$. We deduce that, for $0 \neq x \in \mathbb{R}^r$, we have

$$\frac{\phi(x, x)}{\|x\|^2} \leq \lambda_r,$$

with equality iff x is an eigenvector of A with eigenvalue λ_r.

Let $0 \neq x \in \mathbb{R}^r$ be a vector such that $\phi(x, x)/\|x\|^2$ is maximal. Note that replacing $x = (x_1, \ldots, x_r)$ by $(|x_1|, \ldots, |x_r|)$ does not decrease $\phi(x, x)/|x|^2$. Indeed

$$\sum a_{ij} x_i x_j \leq \sum a_{ij} |x_i||x_j|$$

because $a_{ij} = e_i \cdot e_j \geq 0$ for $i \neq j$ by assumption. So, we may assume $x_i \geq 0$ for all i. For $m \in \mathbb{R}^r$ we have

$$\phi(m, x) = \sum_{i,j} a_{ij} m_i x_j = \sum_i (\sum_j a_{ij} x_j) m_i = \lambda_r \sum_i x_i m_i \tag{9}$$

because $Ax = \lambda_r x$ as noted above. Now set $m = d$, where $d = (\mu_1, \ldots, \mu_r)$ is as in the statement. Then $\phi(d, x) < 0$ because $d \cdot e_i < 0$ and $x_i \geq 0$ for all i, and $\sum x_i \mu_i > 0$, so $\lambda_r < 0$ by (9). Thus ϕ is negative definite as required. □

Corollary 7.7. *Let $f\colon X \to Y$ be a birational morphism from a smooth projective surface X to a normal projective surface Y. Let $\{E_i\}$ be the exceptional curves. Then we have an exact sequence*

$$0 \to \oplus \mathbb{Z} E_i \to \operatorname{Cl} X \to \operatorname{Cl} Y \to 0.$$

Proof. We have an exact sequence

$$\oplus \mathbb{Z} E_i \to \operatorname{Cl} X \to \operatorname{Cl}(X \setminus \cup E_i) \to 0$$

by the definition of the divisor class group (see [Hartshorne, p. 133, II.6.5]). Let $f(\cup E_i) = \{Q_1, \dots, Q_s\}$, then f restricts to an isomorphism

$$X \setminus \cup E_i \xrightarrow{\sim} Y \setminus \{Q_1, \dots, Q_s\}.$$

So

$$\mathrm{Cl}(X \setminus \cup E_i) \simeq \mathrm{Cl}(Y \setminus \{Q_1, \dots, Q_s\}) \simeq \mathrm{Cl}(Y).$$

Finally, injectivity of the map $\oplus \mathbb{Z} E_i \to \mathrm{Cl}\, X$ follows from the theorem. □

7.4 Factorisation of birational morphisms

Lemma 7.8. *Let $f\colon X \to Y$ be a birational morphism of smooth projective surface. Let E_i be the exceptional curves of f. Then*

$$K_X = f^* K_Y + \sum a_i E_i$$

where $a_i \in \mathbb{Z}$ and $a_i > 0$ for all i.

Proof. By Cor. 7.7 we have an exact sequence

$$0 \to \oplus \mathbb{Z} E_i \to \mathrm{Cl}\, X \to \mathrm{Cl}\, Y \to 0.$$

So $K_X = f^* K_Y + \sum a_i E_i$ for some $a_i \in \mathbb{Z}$. It remains to show that $a_i > 0$ for each i. Let $f(\cup E_i) = \{Q_1, \dots, Q_s\}$, so f restricts to an isomorphism

$$X \setminus \cup E_i \xrightarrow{\sim} Y \setminus \{Q_1, \dots, Q_s\},$$

and let ω be a rational 2-form on Y which is regular and nonzero at each Q_j. Then

$$(\pi^* \omega) = \pi^*(\omega) + \sum a_i E_i$$

and the divisor $\pi^*(\omega)$ is disjoint from the E_i by construction, so a_i is the order of vanishing of $\pi^*\omega$ along E_i. In particular $a_i \geq 0$ because $\pi^*\omega$ is regular near E_i. We show $a_i > 0$. Let $f(E_i) = Q_j$, let $P \in E_i$ be a general point, and choose local coordinates u, v at $P \in X$ and x, y at $Q_j \in Y$. Then $\omega = g dx \wedge dy$ where $g \in \mathcal{O}_{X,P}$ and $g(P) \neq 0$. Now $\pi^*\omega = \pi^* g \cdot J du \wedge dv$ where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of the map f with respect to these coordinates. But $J(P) = 0$ by the inverse function theorem (if $J(P) \neq 0$, then f is a local isomorphism near $P \in X$). Thus $\pi^*\omega$ vanishes along E_i, so $a_i > 0$. □

Corollary 7.9. *$K_X \cdot E_j < 0$ for some j.*

Proof. We have $K_X \cdot (\sum a_i E_i) = (\sum a_i E_i)^2 < 0$ by Thm. 7.5 and $a_i > 0$ for all i. So $K_X \cdot E_j < 0$ for some j. □

7.5 Numerical characterisation of (-1)-curves

Lemma 7.10. *Let X be a smooth projective surface and $C \subset X$ an irreducible curve (not necessarily smooth). Then*

$$(K_X + C) \cdot C = 2p_a(C) - 2$$

where $p_a(C) := h^1(\mathcal{O}_C)$ is the arithmetic genus *of C. Let $\nu \colon C^\nu \to C$ be the normalisation of C. Then $p_a(C) \geq g(C^\nu)$ with equality iff C is smooth.*

Proof. The exact sequence of sheaves

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

gives

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_C)$$

and

$$\chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2}(-C) \cdot (-C - K_X)$$

by Riemann–Roch. Combining we deduce that $\chi(\mathcal{O}_C) = -\frac{1}{2} C \cdot (C + K_X)$, and rearranging gives $2p_a(C) - 2 = (K_X + C) \cdot C$.

To relate the arithmetic genus of C to the genus of its normalisation C^ν, we consider the exact sequence of sheaves on C

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{C^\nu} \to \mathcal{T} \to 0.$$

(If $f \colon X \to Y$ is a morphism and $\mathcal{F}$ is a sheaf on X then $f_* \mathcal{F}$ is the sheaf on Y defined by $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}U)$. If $\mathcal{F}$ is coherent and f is finite then $H^i(Y, f_* \mathcal{F}) = H^i(X, \mathcal{F})$. This follows because the Cech cohomology can be computed using an affine open covering, and $f^{-1}U \subset X$ is affine if $U \subset Y$ is affine and f is finite.) The sheaf $\mathcal{T} := \nu_* \mathcal{O}_{C^\nu} / \mathcal{O}_C$ is a direct sum of skyscraper sheaves supported at the singular points of C. For example, if $P \in C$ is a node then $\mathcal{T}$ has stalk k at P. The long exact sequence of cohomology gives an exact sequence

$$0 \to H^0(\mathcal{T}) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{C^\nu}) \to 0.$$

Here we used $H^0(\mathcal{O}_C) = H^0(\mathcal{O}_{C^\nu}) = k$ and $H^1(\mathcal{T}) = 0$ because $\mathcal{T}$ is a direct sum of skyscraper sheaves. We deduce that

$$p_a(C) = h^1(\mathcal{O}_{C^\nu}) = h^1(\mathcal{O}_C) + h^0(\mathcal{T}) \geq h^1(\mathcal{O}_C) = g(C),$$

with equality iff $\mathcal{T} = 0$ or, equivalently, $C = C^\nu$ is smooth. □

Remark 7.11. If C is smooth then the equality $(K_X + C) \cdot C = 2g(C) - 2$ follows from the adjunction formula $(K_X + C)|_C = K_C$. More generally, if C is an irreducible curve, the *dualising sheaf* is a line bundle ω_C on C such that $\omega_X(C)|_C = \omega_C$ where $\omega_X = \wedge^2 \Omega_X$ is the canonical line bundle on X, and $\deg \omega_C = 2p_a(C) - 2$. (If C is smooth then $\omega_C = \Omega_C$ is the canonical line bundle.) This gives a more direct proof of the equality in the singular case. See [Hartshorne, III.7] for more details.

Corollary 7.12. *Let $f\colon X \to Y$ be a birational morphism from a smooth surface X to a normal surface Y. Suppose $E \subset X$ is a curve contracted by f such that $K_X \cdot E < 0$. Then E is a* (-1)-curve*, that is, $E \simeq \mathbb{P}^1$ and $E^2 = -1$.*

Proof. We have $K_X \cdot E < 0$ by assumption and $E^2 < 0$ by negativity of the contracted locus. So

$$2p_a(E) - 2 = K_X \cdot E + E^2 < 0.$$

We deduce that $p_a(E) = 0$ and $K_X \cdot E = E^2 = -1$. Now $g(E^\nu) \leq p_a(E)$ so $g(E^\nu) = 0$ and $E = E^\nu \simeq \mathbb{P}^1$. □

7.6 Castelnuovo's contractibility criterion

Theorem 7.13. *Let X be a smooth projective surface and $E \subset X$ a (-1)-curve. Then there is a birational morphism $\pi\colon X \to Y$ to a smooth surface Y such that π is the blowup of a point $P \in Y$ and $E = \pi^{-1}P$.*

Proof. The idea of the proof is to write down a line bundle on X whose sections define the desired map $\pi\colon X \to Y \subset \mathbb{P}^N$. Let H be a hyperplane section of X in some embedding. Assume that $H^1(\mathcal{O}_X(H)) = 0$. (This can be achieved by replacing H by a sufficiently large multiple nH, by Serre vanishing.) Let $H \cdot E = k$, a positive integer. We define $D = H + kE$. Then

$$D \cdot E = k + kE^2 = 0$$

and $D \cdot C > 0$ for $C \subset X$ a curve, $C \neq E$. (So, D looks like the pullback of a hyperplane section under a map π which contracts E but does not contract any other curves.) We show that the global sections of $\mathcal{O}_X(D)$ define a morphism π as in the statement.

We first describe the global sections of $\mathcal{O}_X(D)$ explicitly. Let $\{s_0, \ldots, s_n\}$ be a basis of $H^0(\mathcal{O}_X(H))$. Note that $H^0(\mathcal{O}_X(E))$ is 1-dimensional. (Equivalently, if $D \sim E$ and $D \geq 0$ then $D = E$. To see this, note that $D \cdot E = E^2 < 0$, so D contains E. Then $D - E$ is effective and $D - E \sim 0$, so $D = E$.) Let $0 \neq s \in H^0(\mathcal{O}_X(E))$ be a nonzero section, then $E = (s = 0) \subset X$. Tensoring the exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

with $\mathcal{O}_X(H + iE)$, $0 \leq i \leq k$, we obtain the exact sequence

$$0 \to \mathcal{O}_X(H + (i-1)E) \to \mathcal{O}_X(H + iE) \to \mathcal{O}_E(k - i) \to 0.$$

Here since $E \simeq \mathbb{P}^1$ we write $\mathcal{O}_E(d)$ for the unique line bundle on $E \simeq \mathbb{P}^1$ of degree d, and we have $\mathcal{O}_X(H+iE)|_E = \mathcal{O}_E(d)$ where $d = (H+iE) \cdot E = k-i$. Now $H^1(\mathcal{O}_E(k-i)) = 0$ because $k - i \geq 0$. So the long exact sequence of cohomology gives

$$0 \to H^0(\mathcal{O}_X(H + (i-1)E)) \to H^0(\mathcal{O}_X(H + iE)) \to H^0(\mathcal{O}_E(k - i))$$
$$\to H^1(\mathcal{O}_X(H + (i-1)E)) \to H^1(\mathcal{O}_X(H + iE)) \to 0.$$

Recall that $H^1(\mathcal{O}_X(H)) = 0$ by assumption. It follows by induction that $H^0(\mathcal{O}_X(H + iE)) = 0$ for $0 \leq i \leq k$. So the sequence

$$0 \to H^0(\mathcal{O}_X(H + (i-1)E) \to H^0(\mathcal{O}_X(H + iE)) \to H^0(\mathcal{O}_E(k - i)) \to 0$$

is exact for $1 \leq i \leq k$. We have

$$H^0(\mathcal{O}_E(d)) = H^0(\mathcal{O}_{\mathbb{P}^1}(d)) = \langle X_0^d, X_0^{d-1}X_1, \ldots, X_1^d \rangle_k.$$

Let $a_{i,0}, \ldots, a_{i,k-i} \in H^0(\mathcal{O}_X(H + iE))$ be a lift of a basis of $H^0(\mathcal{O}_E(k - i))$ for each $1 \leq i \leq k$. Then $H^0(\mathcal{O}_X(H + kE))$ has basis

$$s^k s_0, \ldots, s^k s_n, s^{k-1}a_{1,0}, \ldots, s^{k-1}a_{1,k-1}, \ldots, sa_{k-1,0}, sa_{k-1,1}, a_{k,0}.$$

Now let $\pi \colon X \dashrightarrow Y \subset \mathbb{P}^N$ be the rational map defined by $H^0(\mathcal{O}_X(D))$. Then the restriction of π to $X \setminus E$ is an isomorphism onto its image. Indeed, s is nonzero on $X \setminus E$ and $s_0, \ldots, s_n$ define an embedding of X by assumption, so $s^k s_0, \ldots, s^k s_n$ define an embedding on $X \setminus E$. Second, π is a morphism. To see this, note that $a_{k,0}$ is nonzero along E, and we already observed that π is defined on $X \setminus E$. Third, π contracts E to the point $P = (0 : \cdots : 0 : 1) \in Y$. (Note: we do not know yet that $P \in Y$ is a smooth point.)

Let $V = (X_N \neq 0) \subset Y$, an open neighbourhood of $P \in Y$, and $U = \pi^{-1}V = (a_{k,0} \neq 0) \subset X$ the corresponding neighbourhood of $E \subset X$. Let $X = a_{k-1,0}/a_{k,0}, Y = a_{k-1,1}/a_{k,0} \in \Gamma(U, \mathcal{O}_X(-E))$ and $x = sX, y = sY \in \Gamma(U, \mathcal{O}_X)$. Then x, y, X, Y define a morphism

$$U \to \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{(X:Y)}$$

which factors through

$$\mathrm{Bl}_0 \mathbb{A}^2_{x,y} = (xY - yX = 0) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{(X:Y)}.$$

This gives a commutative diagram

$$\begin{array}{ccc} (E \subset U) & \longrightarrow & (F \subset \mathrm{Bl}_0 \mathbb{A}^2) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ (P \in V) & \longrightarrow & (0 \in \mathbb{A}^2) \end{array}$$

where $F \subset \mathrm{Bl}_0 \mathbb{A}^2$ is the exceptional curve. Assume for simplicity that $k = \mathbb{C}$. We claim that $f \colon (E \subset U) \to (F \subset \mathrm{Bl}_0 \mathbb{A}^2)$ restricts to an isomorphism over a neighbourhood of E in the Euclidean topology. Indeed, we observe that the map $E \to F$ is an isomorphism, and the derivative df is an isomorphism at each point $Q \in E$, so f is an isomorphism near E by the inverse function theorem. It follows that $g \colon (P \in V) \to (0 \in \mathbb{A}^2)$ restricts to an isomorphism over $B^\times = B \setminus \{P\}$ for some Euclidean neighbourhood B of $P \in V$. But $0 \in \mathbb{A}^2$ is normal, so the restriction to B is also an isomorphism. We conclude that $P \in Y$ is smooth and $\pi \colon X \to Y$ is the blowup of $P \in Y$. This completes the proof. □

7.7 Decomposition of birational maps

Theorem 7.14. *Let $f \colon X \dashrightarrow Y$ be a birational map between smooth surfaces X and Y. Then there exists a commutative diagram*

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & \\ X & & \overset{f}{\dashrightarrow} & & Y \end{array}$$

where p and q are compositions of sequences of blowups.

Proof. We just need to put our previous results together. By elimination of indeterminacy for rational maps of surfaces, there exists a sequence of blowups $p\colon Z \to X$ such that the induced rational map

$$q = f \circ p\colon Z \dashrightarrow Y$$

is a morphism. Now consider $q\colon Z \to Y$. There exists a curve $E \subset Z$ such that E is contracted by q and $K_Z \cdot E < 0$. So E is a (-1)-curve by Cor. 7.12. By Castelnuovo's contractibility criterion, there exists a morphism $\pi\colon Z \to Z'$ to a smooth surface Z' such that π is the blowup of a point $P \in Z'$ and $E = \pi^{-1}P$. Now $q\colon Z \to Y$ factors through Z', that is, $q = q' \circ \pi$ where $q'\colon Z' \to Y$. Indeed, we certainly have a continuous map $q'\colon Z' \to Y$ of topological spaces such that $q = q' \circ \pi$, we just need to show that this a morphism, that is, regular functions pull back to regular functions. Let $f \in \mathcal{O}_Y(U)$. Then $q^*f \in \mathcal{O}_Z(q^{-1}U)$, and π restricts to an isomorphism

$$Z \setminus E \xrightarrow{\sim} Z' \setminus \{P\}.$$

So $q'^*f \in \mathcal{O}_{Z'}(q'^{-1}U - \{P\})$. But Z' is normal, so $q'^*f \in \mathcal{O}_{Z'}(q'^{-1}U)$ as required. It follows by induction that q is the composition of a sequence of blowups. (Note that the process must stop, since $\operatorname{rk} H^2(Z',\mathbb{Z}) = \operatorname{rk} H^2(Z,\mathbb{Z}) - 1$, and $\operatorname{rk} H^2(Z,\mathbb{Z}) < \infty$.) This completes the proof. $\square$

Example 7.15. Let $X \subset \mathbb{P}^3$ be a smooth quadric surface. There is only one such surface up to automorphisms of $\mathbb{P}^3$ — it is the image

$$(XT = YZ) \subset \mathbb{P}^3_{(X:Y:Z:T)}$$

of the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3, \quad ((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0Y_0 : X_0Y_1 : X_1Y_0 : X_1Y_1).$$

The fibres of the two projections $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ give two families of lines in $\mathbb{P}^3$ which cover the surface X, with distinct lines in a given family being disjoint and lines from different families intersecting transversely in one point.

Let $P \in X$ be a point and consider the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by projection from P. In coordinates, if $P = (0 : 0 : 0 : 1) \in \mathbb{P}^3$ then the projection is given by

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2, (X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : X_1 : X_2).$$

The induced rational map $f\colon X \dashrightarrow \mathbb{P}^2$ is birational. To see this, note that the preimage of a point $Q \in \mathbb{P}^2$ is $L \cap X \setminus \{P\}$, where L is the corresponding line in $\mathbb{P}^3$ through the point P. This will be a single point unless L is contained in X, so f is birational. The rational map f is not defined at $P \in X$: if P' approaches P along a smooth curve $C \subset X$ then $f(P')$ approaches the point Q corresponding to the tangent line $T_P C \subset \mathbb{P}^3$ to C at P. However, if $\pi\colon \tilde{X} \to X$ is the blowup of P, then the composite $f' = f \circ \pi\colon \tilde{X} \dashrightarrow \mathbb{P}^2$ is a morphism. (The same is true whenever we project from a point on a smooth surface.) Let L_1, L_2 be the two lines on X passing through P. Then the strict transforms $L_1', L_2' \subset \tilde{X}$ are contracted by f', and $f'\colon \tilde{X} \to \mathbb{P}^2$ is the birational morphism given by blowing up two distinct points on $\mathbb{P}^2$, with exceptional curves L_1', L_2'.

One can picture this decomposition as follows. The curves $(X_0 = 0), (Y_0 = 0), (X_1 = 0), (Y_1 = 0)$ form a cycle of 4 smooth rational curves on $X = \mathbb{P}^1 \times \mathbb{P}^1$ of self-intersection zero. (Note that, if X is a smooth projective surface, $p\colon X \to C$ is a morphism to a curve, and F is a fibre, then $F^2 = 0$. Indeed, the associated 2-cycle F is homologous to any other fibre F', that is $[F] = [F'] \in H_2(X, \mathbb{Z})$. So $F^2 = F \cdot F' = 0$.) Let P be the intersection point $(X_0 = 0) \cap (Y_0 = 0)$. Then, in the above notation, $L_1, L_2 = (X_0 = 0), (Y_0 = 0)$. In general, if X is a smooth surface, $\pi\colon \tilde{X} \to X$ is the blowup of a point $P \in X$, C is a smooth curve through $P \in X$, and $C' \subset \tilde{X}$ is the strict transform of C, then $C' = \pi^* C - E$ and so $C'^2 = C^2 - 1$. Thus ${L_1'}^2 = {L_2'}^2 = -1$. Now we contract the two disjoint curves L_1', L_2', and the remaining curves (the exceptional curve E of π and the inverse images of the curves $(X_1 = 0)$,$(Y_1 = 0)$) map to a set of coordinate hyperplanes on $\mathbb{P}^2$. Note that you can check that the images have self-intersection 1 using the formula $C'^2 = C^2 - 1$ given above.

Example 7.16. The *Cremona transformation* is the birational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (X_0 : X_1 : X_2) \mapsto (X_0^{-1} : X_1^{-1} : X_2^{-1}).$$

It can be decomposed into blowups as follows. Consider the set of coordinate hyperplanes $(X_i = 0)$, $i = 0, 1, 2$, in $\mathbb{P}^2$. Let $\pi\colon \tilde{X} \to \mathbb{P}^2$ be the blowup of the 3 intersection points. The strict transforms of the coordinate hyperplanes are (-1)-curves, so can be blown down. We obtain another copy of $\mathbb{P}^2$, with the images of the exceptional curves of π giving a set of coordinate hyperplanes.

Remark 7.17. A theorem of Castelnuovo–Noether asserts that any birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be decomposed into a sequence of Cremona transformations and automorphisms of $\mathbb{P}^2$ (recall $\operatorname{Aut} \mathbb{P}^2 = \mathrm{PGL}(3)$). Algebraically, this theorem gives a set of generators for the group $\operatorname{Aut} k(x_1, x_2)$ of automorphisms of the field of rational functions in 2 variables.

Example 7.18. A *ruled surface* is a smooth projective surface X together with a morphism $f\colon X \to C$ to a smooth curve C such that every fibre of F is isomorphic to $\mathbb{P}^1$. (Warning: some authors define a ruled surface as a surface which is birational to a surface of this form.) As observed earlier, in this situation $F^2 = 0$. Let $P \in F \subset X$ be a point and $\pi\colon \tilde{X} \to X$ the blowup of $P \in X$. Then the strict transform of F is a (-1)-curve so can be blown down to obtain another ruled surface $f'\colon X' \to C$. The birational map $g\colon X \dashrightarrow X'$ is called an *elementary transformation* of ruled surfaces.

Remark 7.19. If $f\colon X \to C$ and $f'\colon X' \to C'$ are ruled surfaces and $g\colon X \dashrightarrow X'$ is a birational map then $C \simeq C'$. If in addition C is not isomorphic to $\mathbb{P}^1$ then there is a commutative diagram

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ C & \overset{\sim}{\longrightarrow} & C' \end{array}$$

and g can be decomposed into a sequence of elementary transformations.

8 The Kodaira vanishing theorem

If X is a smooth projective variety and $\mathcal{L}$ is a line bundle on X then the Riemann–Roch formula expresses $\chi(X, \mathcal{L})$ in terms of topological data associated to X and $\mathcal{L}$. This means that $\chi(X, \mathcal{L})$ is usually easy to compute. However, we are typically interested in $h^0(X, \mathcal{L})$, the dimension of the space of global sections of $\mathcal{L}$, and

$$\chi(X, \mathcal{L}) = \sum (-1)^i h^i(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + \cdots + (-1)^n h^n(\mathcal{L})$$

where $n = \dim X$. So it is very useful to know conditions on $\mathcal{L}$ which imply $h^i(\mathcal{L}) = 0$ for all $i > 0$, since then $h^0(X, \mathcal{L}) = \chi(X, \mathcal{L})$. The Kodaira vanishing theorem is the most important result of this type.

Theorem 8.1. *(Kodaira vanishing) Let X be a smooth projective variety over $k = \mathbb{C}$ of dimension n and D an ample divisor on X (that is, mD is a hyperplane section of X in some embedding for some $m > 0$.) Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for $i > 0$. Equivalently, by Serre duality, $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i < n$.*

Remark 8.2. If X is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$ then the analogous statement is false in general.

The usual proof of this result uses an identification of cohomology classes in $H^i(X, \mathcal{O}_X(K_X+D))$ with harmonic forms on X of type (n,i) with values in $\mathcal{L} = \mathcal{O}_X(D)$. See [GH, p. 154].

Alternatively, one can prove the result by considering a cyclic covering $Y \to X$ of order $N \gg 0$ defined by $\mathcal{L}$ and a section $s \in \Gamma(X, \mathcal{L}^N)$, and using the following results:

(1) $H^i(X, \mathcal{O}_X(K_X + mD)) = 0$ for $m \gg 0$ (by Serre vanishing).

(2) The inclusion $\underline{\mathbb{C}} \subset \mathcal{O}_{Y^{\text{an}}}$ of the constant sheaf with stalk $\mathbb{C}$ in the sheaf of holomorphic functions on the complex manifold Y^{an} induces a surjection
$$H^i(Y, \mathbb{C}) \to H^i(Y, \mathcal{O}_Y).$$
(This map is given by projection onto the factor $H^{0,i}$ of the Hodge decomposition.)

See [KM, p. 62, Thm. 2.47] for details.

9 Characterisation of the projective plane

Recall that we say a divisor D is *ample* if mD is linearly equivalent to a hyperplane section H in some embedding, for some $m > 0$. Also $\rho(X)$ denotes the rank of the group $\operatorname{Num} X$ of divisors modulo numerical equivalence.

Theorem 9.1. *Let X be a smooth projective surface over $k = \mathbb{C}$ and suppose $\rho(X) = 1$ and $-K_X$ is ample. Then X is isomorphic to $\mathbb{P}^2$.*

Remark 9.2. If $\rho(X) = 1$ then $\operatorname{Num}(X) \simeq \mathbb{Z}$ so either K_X is ample, $-K_X$ is ample, or K_X is numerically equivalent to 0. Indeed, since X is projective, we can pick an ample generator H of $\operatorname{Num}(X)$ and write $K_X \equiv mH$, then the 3 cases correspond to $m > 0$, $m < 0$, and $m = 0$. (Note that if H is ample and D is numerically equivalent to H, then D is ample. This follows for example from the Nakai–Moishezon criterion for ampleness [Hartshorne, V.1.10, p.365], [KM, 1.37, p. 31] or Kodaira's metric characterisation of ampleness [GH, Prop., p.148, Thm., p. 181]).

Proof of Thm. 9.1. By the Kodaira vanishing theorem applied to $D = -K_X$, we have $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. The exponential sequence
$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{X^{\text{an}}} \to \mathcal{O}^{\times}_{X^{\text{an}}} \to 0$$

gives the exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(\mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X,\mathbb{Z}) \to H^2(\mathcal{O}_X) \to \cdots$$

Thus $H^1(X,\mathbb{Z}) = 0$ and $c_1\colon \operatorname{Pic} X \xrightarrow{\sim} H^2(X,\mathbb{Z})$ is an isomorphism. So $H^2(X,\mathbb{Z})/\operatorname{Tors} = \operatorname{Num} X = \mathbb{Z}$ because $\rho(X) = 1$ by assumption. We have $H^3(X,\mathbb{Z})/\operatorname{Tors} = H^1(X,\mathbb{Z})^* = 0$ by Poincaré duality, so the Euler number

$$e(X) := \sum (-1)^i \dim_{\mathbb{R}} H^i(X,\mathbb{R}) = 1 - 0 + 1 - 0 + 1 = 3.$$

So Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X))$$

gives $K_X^2 = 9$. The pairing on $\operatorname{Num} X = H^2(X,\mathbb{Z})/\operatorname{Tors}$ is unimodular by Poincaré duality. So, if H is an ample generator of $\operatorname{Num} X$ then $H^2 = 1$, and $K_X \equiv -3H$ (because $K_X^2 = 9$ and $-K_X$ is ample).

We want to show that the global sections of the line bundle $\mathcal{O}_X(H)$ define an isomorphism $X \xrightarrow{\sim} \mathbb{P}^2$. (Under this isomorphism, H will correspond to a hyperplane in $\mathbb{P}^2$.) We first compute the dimension of the space of global sections of $\mathcal{O}_X(H)$. By Riemann–Roch,

$$\chi(\mathcal{O}_X(H)) = \chi(\mathcal{O}_X) + \frac{1}{2}H\cdot(H - K_X) = 1 + \frac{1}{2}H \cdot 4H = 3.$$

If we write $H = K_X + D$, then $D = H - K_X \equiv 4H$ is ample, so $H^i(\mathcal{O}_X(H)) = 0$ for $i > 0$ by Kodaira vanishing. Thus $h^0(\mathcal{O}_X(H)) = 3$. In particular, we may assume H is an effective divisor.

Let $\phi\colon X \dashrightarrow \mathbb{P}^2$ be the rational map defined by the global sections of $\mathcal{O}_X(H)$. We show that $\mathcal{O}_X(H)$ is generated by global sections (that is, for every point $P \in X$, there exists a section $s \in H^0(\mathcal{O}_X(H))$ such that $s(P) \neq 0$), so ϕ is a morphism. Equivalently, the linear system $|H|$ is basepoint free. We consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_X(H)|_H \to 0$$

Since $H^1(\mathcal{O}_X) = 0$ the restriction map $H^0(\mathcal{O}_X(H)) \to H^0(\mathcal{O}_X(H)|_H)$ is surjective. So, it is enough to show that $\mathcal{O}_X(H)|_H$ is generated by global sections. Now, H is irreducible (because it is a generator of $\operatorname{Num}(X)$) and

$$2p_a(H) - 2 = (K_X + H)\cdot H = -2H \cdot H = -2$$

so $p_a(H) = 0$ and $H \simeq \mathbb{P}^1$. The line bundle $\mathcal{O}_X(H)|_H$ has degree $H^2 = 1$, so corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$. In particular, $\mathcal{O}_X(H)|_H$ is generated by global sections.

The divisor $H \subset X$ is the pullback of a hyperplane $L \subset \mathbb{P}^2$ under the morphism $\phi \colon X \to \mathbb{P}^2$, and $H^2 = 1$. It follows that the inverse image of a general point $P = L_1 \cap L_2 \in \mathbb{P}^2$ is a single point $H_1 \cap H_2 \in X$. Thus ϕ is a birational morphism. But ϕ is a finite morphism because H is ample (if $C \subset X$ is a curve then $H \cdot C > 0$, so C is not contracted by ϕ). So ϕ is an isomorphism. $\square$

10 Ruled surfaces

A *ruled surface* $f \colon X \to C$ is a $\mathbb{P}^1$-bundle over a curve C for the Zariski topology. That is, there exists a Zariski open covering $\mathcal{U} = \{U_i\}$ of C and isomorphisms

$$\begin{array}{ccc} f^{-1}U_i & \xrightarrow{\quad\sim\quad} & U_i \times \mathbb{P}^1 \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U_i & \end{array}$$

Theorem 10.1. *Let X be a smooth projective surface over $k = \mathbb{C}$ and $f \colon X \to C$ a morphism to a smooth curve C with connected fibres. Suppose $K_X \cdot F < 0$ for F a fibre and $\rho(X) = 2$. Then $f \colon X \to C$ is a ruled surface.*

Proof. Let $P \in C$ be a point and let $E_1, \ldots, E_r$ be the irreducible components of $f^{-1}P$. By the fibre F over $P \in C$ we mean the divisor with support $f^{-1}P$ defined as follows. Let t be a local parameter at $P \in C$, then, working locally over $P \in C$,

$$F := (f^*t) = \sum m_i E_i.$$

Here $m_i = \nu_{E_i}(f^*t)$ is the *multiplicity* of the component E_i of the fibre F. Note that a general fibre is irreducible and smooth, and has multiplicity 1.

If F is an irreducible fibre of multiplicity 1, then $F \simeq \mathbb{P}^1$. Indeed, $K_X \cdot F < 0$ by assumption and $F^2 = 0$, so

$$2p_a(F) - 2 = (K_X + F) \cdot F < 0.$$

Thus $p_a(F) = 0$, $K_X \cdot F = -2$, and $F \simeq \mathbb{P}^1$. Note also that there are no irreducible fibres of multiplicity greater than one. Indeed, suppose $F = mE$ with E irreducible, $m \geq 1$. Then as above we find $K_X \cdot E = -2$. But $K_X \cdot (mE) = K_X \cdot F = -2$, so $m = 1$.

We show that there are no reducible fibres. We use Lem. 10.3 below. Let $F = \sum m_i E_i$ be a fibre and H a horizontal curve on X (that is, an irreducible curve which is not contained in a fibre). Then $H, E_1, \ldots, E_r$ are linearly independent in $\mathrm{Num}(X)$. Indeed, suppose $D = aH + \sum a_i E_i \equiv 0$. We have $E_i \cdot F = 0$ for all i (because F is numerically equivalent to any other fibre F', and $F \cap F' = \emptyset$) and $H \cdot F > 0$. So, $D \cdot F = 0$ gives $a = 0$. Now $D = \sum a_i E_i$, and since $D^2 = 0$ we have $D = \lambda F$ for some $\lambda \in \mathbb{Q}$ by Lem. 10.3. Finally $D \cdot H = 0$ gives $\lambda = 0$, so $a_i = 0$ for all i as required. In our situation $\rho(X) := \operatorname{rk} \mathrm{Num}(X) = 2$ by assumption. We deduce that $r = 1$, that is, F is irreducible.

Let $P \in C$ be a point and F the fibre of f over P. (So by our previous results $F \simeq \mathbb{P}^1$ is a smooth rational curve of multiplicity 1.) We show that there exists a Euclidean neighbourhood $P \in U \subset C^{\mathrm{an}}$ of P and an isomorphism

$$\begin{array}{ccc} f^{-1}U & \xrightarrow{\quad\sim\quad} & U \times \mathbb{P}^1 \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

An isomorphism as in the diagram is given by $(f,g)\colon f^{-1}U \to U \times \mathbb{P}^1$ where $g\colon f^{-1}U \to \mathbb{P}^1$ is a morphism which restricts to an isomorphism $F \xrightarrow{\sim} \mathbb{P}^1$. (Given such a morphism g, (f,g) is an isomorphism over some smaller neighbourhood $P \in U' \subset U$ by the inverse function theorem.) The morphism g corresponds to a line bundle $\mathcal{L} = g^*\mathcal{O}_{\mathbb{P}^1}(1)$ on $f^{-1}U$ and sections $s_0, s_1 \in \Gamma(f^{-1}U, \mathcal{L})$, $s_i = g^* X_i$, such that $\mathcal{L}|_F = \mathcal{O}_F(1)$ and $s_0|_F, s_1|_F$ form a basis of $\Gamma(F, \mathcal{O}_F(1))$. So, we need to show that (1) we can lift the line bundle $\mathcal{O}_F(1)$ on F to a line bundle $\mathcal{L}$ on a tubular neighbourhood $Y := f^{-1}U$ of the fibre F in X, and (2) we can lift the global sections of $\mathcal{O}_F(1)$ to sections of $\mathcal{L}$ over Y. We will give two proofs of these facts. The first uses the behaviour of cohomology under restriction to a fibre [Hartshorne, III.12.11], and is more widely applicable. The second is more elementary.

From the exponential sequences on $Y = f^{-1}U$ and F, we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^1(\mathcal{O}_Y) & \longrightarrow & \operatorname{Pic} Y & \longrightarrow & H^2(Y,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_Y) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^1(\mathcal{O}_F) & \longrightarrow & \operatorname{Pic} F & \longrightarrow & H^2(F,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_F) & \longrightarrow & \cdots \end{array}$$

The fibre $F \simeq \mathbb{P}^1$ so $H^1(\mathcal{O}_F) = H^2(\mathcal{O}_F) = 0$, and $\operatorname{Pic} F \xrightarrow{\sim} H^2(F,\mathbb{Z}) = \mathbb{Z}$. The space Y is a tubular neighbourhood of F, so the inclusion $F \subset Y$

is a homotopy equivalence and the restriction map $H^2(Y,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ is an isomorphism. Finally, since $H^i(\mathcal{O}_F) = 0$ for $i > 0$, it follows that $H^i(\mathcal{O}_Y) = 0$ for $i > 0$ (shrinking U if necessary) by [Hartshorne, III.12.11]. So the restriction map $\operatorname{Pic} Y \to \operatorname{Pic} F$ is an isomorphism. In particular, there exists a (unique) line bundle $\mathcal{L}$ on Y such that $\mathcal{L}|_F = \mathcal{O}_F(1)$. Now we lift sections. Consider the exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{O}_Y(-F) \to \mathcal{L} \to \mathcal{L}|_F \to 0.$$

Note that the ideal $\mathcal{O}_Y(-F)$ of F in Y is generated by a local parameter t at $P \in C$ (shrinking U if necessary), in particular $\mathcal{O}_Y(-F) \simeq \mathcal{O}_Y$. We can thus rewrite the exact sequence as

$$0 \to \mathcal{L} \to \mathcal{L} \to \mathcal{L}|_F \to 0$$

where the first arrow is given by multiplication by t. Now consider the associated long exact sequence of cohomology

$$\cdots \to H^0(Y,\mathcal{L}) \to H^0(F,\mathcal{L}|_F) \to H^1(Y,\mathcal{L}) \to \cdots$$

Since $H^1(F,\mathcal{L}|_F) = H^1(F,\mathcal{O}_F(1)) = 0$, it follows that $H^1(Y,\mathcal{L}) = 0$ by [Hartshorne, III.12.11] again. Thus the restriction map

$$H^0(Y,\mathcal{L}) \to H^0(F,\mathcal{L}|_F) = H^0(F,\mathcal{O}_F(1))$$

is surjective. So, we can lift a basis of $H^0(F,\mathcal{O}_F(1))$ to sections s_0, s_1 of the line bundle $\mathcal{L}$ over Y as required.

We give a more elementary proof of the two facts needed to show that the fibration $f\colon X \to C$ is locally trivial in the Euclidean topology, given that each fibre is a smooth rational curve of multiplicity one. Let $P \in C$ be a point and $F \simeq \mathbb{P}^1$ the fibre over P. We first show that there exists a line bundle $\mathcal{L}$ on X such that $\mathcal{L}|_F \simeq \mathcal{O}_F(1)$. We have $K_X \cdot F < 0$ and $F^2 = 0$, so K_X is not linearly equivalent to an effective divisor (because if D is an effective divisor, C is an irreducible curve, and $D \cdot C < 0$, then C is contained in the support of D and $C^2 < 0$). Equivalently, $h^0(K_X) = 0$. Thus $H^2(X,\mathbb{C}) = H^{1,1}$, and so $\operatorname{Num} X = H^2(X,\mathbb{Z})/\operatorname{Tors}$. Now, by Poincaré duality, the intersection product gives an isomorphism $\operatorname{Num} X \simeq (\operatorname{Num} X)^*$. The image of the map

$$\theta\colon \operatorname{Num} X \to \mathbb{Z}, \quad D \mapsto D \cdot F$$

is a subgroup $d\mathbb{Z} \subset \mathbb{Z}$, some $d \in \mathbb{N}$. Then $\frac{1}{d}\theta \in (\operatorname{Num} X)^*$ corresponds to some $G \in \operatorname{Num} X$, that is, $D \cdot F = d(D \cdot G)$ for all D. Thus $F \equiv dG$. Now

$K_X \cdot F = -2$, so $d = 1$ or 2. Also, for X a smooth projective surface and D a divisor on X, $D^2 - D \cdot K_X$ is even. Indeed,

$$D^2 - D \cdot K_X = 2(\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X))$$

by the Riemann–Roch formula. We deduce that $d = 1$, that is, there exists a divisor H such that $H \cdot F = 1$. Let $\mathcal{L} = \mathcal{O}_X(H + rF)$ for some $r \in \mathbb{Z}$, then $\mathcal{L}|_F \simeq \mathcal{O}_F(1)$. We next show that, for $r \gg 0$, the map $\Gamma(X, \mathcal{L}) \to \Gamma(F, \mathcal{O}_F(1))$ on global sections is surjective. Consider the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(H + (r-1)F) \to \mathcal{O}_X(H + rF) \to \mathcal{O}_F(1) \to 0$$

and the associated long exact sequence of cohomology

$$\begin{array}{ccccccc} & & H^0(\mathcal{O}_X(H+rF)) & \to & H^0(\mathcal{O}_F(1)) & \to \\ H^1(\mathcal{O}_X(H+(r-1)F)) & \to & H^1(\mathcal{O}_X(H+rF)) & \to & 0 & \end{array} \tag{10}$$

(here we used $H^1(\mathcal{O}_F(1)) = 0$). Now, the space $H^1(\mathcal{O}_X(H))$ is finite dimensional, and the map

$$\theta_r \colon H^1(\mathcal{O}_X(H + (r-1)F)) \to H^1(\mathcal{O}_X(H + rF))$$

is surjective for each r by the exact sequence (10). Hence θ_r is an isomorphism for $r \gg 0$. So the restriction map $H^0(\mathcal{O}_X(H + rF)) \to H^0(\mathcal{O}_F(1))$ is surjective for $r \gg 0$ by (10).

Finally, we show that the fibration $f \colon X \to C$ is locally trivial in the Zariski topology. Recall that X is a $\mathbb{P}^1$-bundle over C for the Euclidean topology. We show that $X \to C$ is the projectivisation of a rank 2 (analytic) vector bundle E on C^{an}. Then by GAGA, E is actually an algebraic vector bundle on C. It follows that $f \colon X \to C$ is a $\mathbb{P}^1$-bundle for the Zariski topology. We have an exact sequence of sheaves of (non-abelian) groups on C^{an} (we omit the superscript "an" in what follows)

$$0 \to \mathcal{O}_C^\times \to \mathrm{GL}_2(\mathcal{O}_C) \to \mathrm{PGL}_2(\mathcal{O}_C) \to 0.$$

Note that this is a *central extension* of sheaves of groups: that is, the kernel is abelian and contained in the centre of the second term. In this situation, we obtain a long exact sequence of cohomology

$$\cdots \to H^1(\mathcal{O}_C^\times) \to H^1(\mathrm{GL}_2(\mathcal{O}_C)) \to H^1(\mathrm{PGL}_2(\mathcal{O}_C)) \to H^2(\mathcal{O}_C^\times). \tag{11}$$

This sequence does *not* continue further to the right (it is not possible to make sense of Cech cohomology H^i of a sheaf of nonabelian groups for $i \geq 2$).

The set $H^1(\mathrm{GL}_2(\mathcal{O}_C))$ is the set of isomorphism classes of rank 2 vector bundles over C. This is analogous to the identification $\operatorname{Pic} C \simeq H^1(\mathcal{O}_C^\times)$. Explicitly, if E is a rank r vector bundle over a complex manifold X, let $\mathcal{U} = \{U_i\}$ be an open covering of X and

$$\phi_i \colon E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^r$$

local trivialisations. Then

$$\phi_j \circ \phi_i^{-1} \colon U_{ij} \times \mathbb{C}^r \xrightarrow{\sim} U_{ij} \times \mathbb{C}^r, \quad (x, v) \mapsto (x, g_{ij}(x) \cdot v),$$

where $g_{ij} \in \Gamma(U_{ij}, \mathrm{GL}_r(\mathcal{O}_X))$ are the *transition functions*. The g_{ij} satisfy the cocycle condition $g_{jk} g_{ij} = g_{ik}$, equivalently, $g_{jk} g_{ij} g_{ik}^{-1} = 1$ (note that the order of the factors is important here for $r > 1$). If we change the trivialisations ϕ_i by multiplication by $f_i \in \Gamma(U_i, \mathrm{GL}_r(\mathcal{O}_X))$, then the new transition functions are given by $g'_{ij} = f_j g_{ij} f_i^{-1}$. The Cech cohomology set $H^1(\mathcal{U}, \mathrm{GL}_r(\mathcal{O}_X))$ with respect to the open covering $\mathcal{U}$ is by definition the set of tuples $(g_{ij}) \in \bigoplus \Gamma(U_{ij}, \mathrm{GL}_r(\mathcal{O}_X))$ satisfying $g_{jk} g_{ij} g_{ik}^{-1} = 1$, modulo the equivalence relation $(g_{ij}) \sim (f_j g_{ij} f_i^{-1})$ for all $(f_i) \in \bigoplus \Gamma(U_i, \mathrm{GL}_r(\mathcal{O}_X))$. The Cech cohomology set $H^1(X, \mathrm{GL}_r(\mathcal{O}_X))$ is obtained by taking the direct limit $\varinjlim H^1(\mathcal{U}, \mathrm{GL}_r(\mathcal{O}_X))$ over all open coverings $\mathcal{U}$ as usual. By the previous discussion, this set is identified with the set of isomorphism classes of rank r vector bundles over X. Returning to our example, a similar analysis shows that the set $H^1(\mathrm{PGL}_2(\mathcal{O}_C))$ is the set of isomorphism classes of $\mathbb{P}^1$-bundles over C (because $\operatorname{Aut} \mathbb{P}^1_{\mathbb{C}} = \mathrm{PGL}_2(\mathbb{C})$). The map $H^1(\mathrm{GL}_2(\mathcal{O}_C)) \to H^1(\mathrm{PGL}_2(\mathcal{O}_C))$ sends a vector bundle to its projectivisation. We claim that this map is surjective. Equivalently, by the exact sequence (11), $H^2(\mathcal{O}_C^\times) = 0$. The exponential sequence on C

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_C \to \mathcal{O}_C^\times \to 0$$

yields the long exact sequence of cohomology

$$\cdots \to H^2(\mathcal{O}_C) \to H^2(\mathcal{O}_C^\times) \to H^3(C, \mathbb{Z}) \to \cdots$$

Now $H^2(\mathcal{O}_C) = 0$ because $\mathcal{O}_C$ is coherent and $\dim_{\mathbb{C}} C = 1 < 2$, and $H^3(C, \mathbb{Z}) = 0$, so $H^2(\mathcal{O}_C^\times) = 0$ as required. Thus the $\mathbb{P}^1$-bundle $f \colon X \to C$ is the projectivisation of a rank 2 vector bundle over C. This completes the proof. □

Remark 10.2. We can make the last step of the proof more explicit as follows. Let $\mathcal{U} = \{U_i\}$ be an open covering of C by small discs and let

$h_{ij} \in \mathrm{PGL}_2(\mathcal{O}_X(U_{ij}))$ be transition functions for the $\mathbb{P}^1$-bundle $f\colon X \to C$ with respect to this covering. Lift h_{ij} to $g_{ij} \in \mathrm{GL}_2(\mathcal{O}_C(U_{ij}))$. Then $g_{jk}g_{ij}g_{ik}^{-1} = \alpha_{ijk} \in \mathcal{O}_C^\times(U_{ijk})$ (because $h_{jk}h_{ij}h_{ik}^{-1} = 1$ in $\mathrm{PGL}_2(\mathcal{O}_C(U_{ijk}))$). One checks that $\alpha = (\alpha_{ijk})$ is a Cech 2-cocycle for $\mathcal{O}_C^\times$. Since $H^2(\mathcal{O}_C^\times) = 0$ we can write $\alpha = d\beta$, that is, $\alpha_{ijk} = \beta_{jk}\beta_{ij}\beta_{ik}^{-1}$. Now define $g'_{ij} = g_{ij}\beta_{ij}^{-1}$, then g'_{ij} is a lift of h_{ij} and $g'_{jk}g'_{ij}{g'_{ik}}^{-1} = 1$. Thus the g'_{ij} define a vector bundle E over C such that the projectivisation of E is isomorphic to the $\mathbb{P}^1$-bundle $f\colon X \to C$.

The proof of the following lemma is similar to the proof of negativity of the contracted locus for a birational morphism.

Lemma 10.3. *[Shafarevich, p. 270, Thm. 4] Let X be a smooth projective surface and $f\colon X \to C$ a morphism to a smooth curve C with connected fibres. Let $F = \sum_{i=1}^r m_iE_i$ be a fibre of f, and $D = \sum a_iE_i$, $a_i \in \mathbb{Z}$. Then $D^2 \le 0$, with equality iff $D = \lambda F$ for some $\lambda \in \mathbb{Q}$.*

10.1 Invariants of ruled surfaces

Let $f\colon X \to C$ be a ruled surface (a $\mathbb{P}^1$-bundle over a curve C).

We first observe that the map $f_*\colon \pi_1(X) \to \pi_1(C)$ is an isomorphism. In general, if $p\colon E \to B$ is a (topological) fibre bundle with fibre F, then there is a homotopy long exact sequence

$$\cdots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \cdots$$

(recall that $\pi_0(X)$ is the set of connected components of X). See [Hatcher, p. 376, Thm. 4.41]. For example, given a loop γ in B based at a point $b \in B$, we can lift it to a path in E, whose end points lie in the fibre $F = p^{-1}b$. Then if F is connected we can join the end points by a path in F to obtain a loop $\tilde{\gamma}$ in E such that $p_*\tilde{\gamma} = \gamma$. This shows that $\pi_1(E) \to \pi_1(B)$ is surjective if $\pi_0(F)$ is trivial. In our situation, the map $f\colon X \to C$ is a fibre bundle with fibre $F \simeq \mathbb{P}^1$. In particular, F is connected and $\pi_1(F) = 0$. So $f_*\colon \pi_1(X) \to \pi_1(C)$ is an isomorphism as claimed. Passing to abelianisations, we deduce that $f_*\colon H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is an isomorphism.

We show that the pullback map on 1-forms

$$f^*\colon \Gamma(\Omega_C) \to \Gamma(\Omega_X)$$

is an isomorphism. We give two proofs. The first proof uses analytic methods. Let $\eta \in \Gamma(\Omega_X)$ be a global 1-form on X. Then the restriction $\eta|_F$ of η to a fibre F equals zero, because $F \simeq \mathbb{P}^1$ and $\Gamma(\Omega_{\mathbb{P}^1}) = 0$. Now let $P \in C$

be a point and $V \subset C^{\mathrm{an}}$ a small disc about P. Let $F = f^{-1}P$ be the fibre over P and $U = f^{-1}V$ the tubular neighbourhood of F over V. Then U is homotopy equivalent to F, so $H_1(U,\mathbb{Z}) = 0$. Also the 1-form η is closed by Prop. 10.4 below. So we can define a holomorphic function $g\colon U \to \mathbb{C}$ such that $dg = \eta|_U$ by

$$g(P) = \int_{P_0}^{P} \eta$$

where $P_0 \in F$ is a fixed basepoint and the notation $\int_{P_0}^{P}$ means the integral over some path from P_0 to P in U. (The choice of path γ is irrelevant. Indeed if γ, γ' are two such paths then $\gamma - \gamma' = \partial\beta$ for some 2-cycle β because $H_1(U,\mathbb{Z}) = 0$. So

$$\int_{\gamma} \eta - \int_{\gamma'} \eta = \int_{\partial\beta} \eta = \int_{\beta} d\eta = 0$$

using $d\eta = 0$.) Now g is constant along fibres because (as noted above) $\eta|_G = 0$ for any fibre G. So $g = f^*h$ is the pullback of a holomorphic function h on $V \subset C$, and $\eta|_U = dg = f^*dh$ is the pullback of a holomorphic 1-form $\omega = dh$ on V. Now, since ω is uniquely determined by η, these local 1-forms patch to give a global 1-form ω on C such that $\eta = f^*\omega$, as required.
The second proof is algebraic. In general, suppose given smooth varieties X, Y, and a submersion (or smooth morphism) $f\colon X \to Y$, that is, a morphism such that the derivative

$$df_P\colon T_{X,P} \to T_{Y,f(P)}$$

of f at P is surjective for all $P \in X$. (Here $T_{X,P}$ denotes the tangent space to X at P, or, equivalently, the fibre of the tangent bundle T_X over $P \in X$.) Then we have an exact sequence of vector bundles on X

$$0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0$$

where the map $T_X \to f^*T_Y$ is the derivative of f and the kernel $T_{X/Y}$ is the bundle of tangent vectors on X which are parallel to the fibres of f. Dualising we obtain an exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0,$$

and so an exact sequence of k-vector spaces

$$0 \to \Gamma(f^*\Omega_Y) \to \Gamma(\Omega_X) \to \Gamma(\Omega_{X/Y})$$

Note that $\Omega_{X/Y}|_F = \Omega_F$ for F a fibre. So, in our example $f\colon X \to C$, we find $\Gamma(\Omega_{X/C}) = 0$ because $\Gamma(\Omega_F) = 0$ (cf. the first proof). Thus $\Gamma(f^*\Omega_C) \to \Gamma(\Omega_X)$ is an isomorphism. Now $\Gamma(f^*\Omega_C) = \Gamma(f_*f^*\Omega_C)$ by the definition of f_* for sheaves, $f_*f^*\Omega_C = \Omega_C \otimes f_*\mathcal{O}_X$ by the projection formula [Hartshorne, p. 124, Ex. II.5.1(d)], and $f_*\mathcal{O}_X = \mathcal{O}_C$ by Stein factorisation [Hartshorne, p. 280, III.11.5]. So $\Gamma(f^*\Omega_C) = \Gamma(\Omega_C)$ and $\Gamma(\Omega_C) \to \Gamma(\Omega_X)$ is an isomorphism as required.

Proposition 10.4. *Let X be a smooth complex projective variety (or, more generally, a compact Kähler manifold). Then a holomorphic k-form η on X is closed. Moreover, the map*

$$H^0(\Omega_X^k) \to H^k_{\mathrm{dR}}(X, \mathbb{C}), \quad \eta \mapsto [\eta]$$

is the natural inclusion

$$H^0(\Omega_X^k) = H^{k,0} \subset H^k_{\mathrm{dR}}(X, \mathbb{C}).$$

Proof. This follows from the description of the Hodge decomposition in terms of harmonic forms. See [CMSP, p. 95, Prop. 3.1.1]. □

Remark 10.5. Note that it is essential that X is compact. For example, the holomorphic 1-form $z_1 dz_2$ on $\mathbb{C}^2_{z_1,z_2}$ is not closed.

In general, if $f\colon X \to B$ is a (topological) fibre bundle with fibre F, then

$$e(X) = e(B)e(F)$$

where as usual $e(X)$ denotes the Euler number. (Proof: By decomposing B into pieces and using the Mayer–Vietoris sequence we can assume that f is a trivial bundle, so $X \simeq B \times F$. Recall that the Euler number can be computed from a cellular subdivision of X as $e(X) = \sum(-1)^i N_i$ where N_i is the number of cells of dimension i. Now taking triangulations of B and F we obtain a cellular subdivision of $X = B \times F$ with cells $\sigma \times \tau$ where σ and τ are simplices in the triangulations of B and F respectively. We deduce that $e(X) = e(B)e(F)$.) In our example $f\colon X \to C$ we obtain

$$e(X) = e(C)e(F) = (2 - 2g)(2) = 4 - 4g \tag{12}$$

where $g = g(C)$ is the genus of C. Now we can compute all the Betti numbers $b_i(X) = \dim_{\mathbb{R}} H^i(X, \mathbb{R})$. Recall that $H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is an isomorphism, so $b_1(X) = b_1(C) = 2g$. Also, we always have $b_0 = 1$, and $b_i = b_{4-i}$ by Poincaré duality. So

$$e(X) := \sum(-1)^i b_i = 2 - 2b_1 + b_2 = 2 - 4g + b_2$$

and combining with (12) we obtain $b_2 = 2$. We also note that the integral homology and cohomology of X has no torsion. Indeed, $H_1(X, \mathbb{Z}) \simeq H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ is torsion free, and it follows that $H_i(X, \mathbb{Z})$ and $H^i(X, \mathbb{Z})$ are torsion free for each i by the universal coefficient theorem and Poincaré duality.

As noted earlier, since $K_X \cdot F < 0$ and $F^2 = 0$, we have $h^0(K_X) = 0$. (More generally, $h^0(nK_X) = 0$ for all $n > 0$.) In particular, $H^{2,0} = h^0(K_X) = 0$, so $H^2(X, \mathbb{C}) = H^{1,1}$ and

$$\operatorname{Num} X = H^{1,1} \cap H^2(X, \mathbb{Z}) / \operatorname{Tors} = H^2(X, \mathbb{Z}).$$

Thus $\operatorname{Num} X \simeq \mathbb{Z}^2$, that is, $\rho(X) = 2$. We describe a basis of $\operatorname{Num} X$. We observe that, since $f : X \to C$ is a $\mathbb{P}^1$-bundle for the Zariski topology on C, there exists a section S of f. (Strictly speaking a *section* of $f : X \to C$ is a morphism $s : C \to X$ such that $f \circ s = \mathrm{id}_C$. Here we identify a section with its image $S = s(C) \subset X$.) Indeed, let $U \subset C$ be a Zariski open subset of C such that the restriction $X|_U$ is a trivial $\mathbb{P}^1$-bundle, that is, there is an isomorphism

$$\phi : X|_U \xrightarrow{\sim} U \times \mathbb{P}^1$$

over U. Now let S_U be a section of $X|_U$ (for example, $S_U = \phi^{-1}(U \times \{P\})$ for some fixed $P \in \mathbb{P}^1$), and define $S = \overline{S_U} \subset X$, the closure of S_U in X. Then S is a section of f. Let F be a fibre of f. We claim that S, F is a basis of $\operatorname{Num} X$. Indeed, it suffices to observe that the determinant of the matrix

$$\begin{pmatrix} S^2 & S \cdot F \\ S \cdot F & F^2 \end{pmatrix} = \begin{pmatrix} ? & 1 \\ 1 & 0 \end{pmatrix}$$

equals -1.

Remark 10.6. Note that $S^2 \mod 2$ is a topological invariant of F. Indeed, S^2 is even iff the intersection form on $\operatorname{Num} X = H^2(X, \mathbb{Z})$ is even, that is, x^2 is even for all $x \in H^2(X, \mathbb{Z})$.

For example, the (holomorphic or algebraic) classification of ruled surfaces $f : X \to C$ over $C = \mathbb{P}^1$ can be described as follows. For each n there exists a ruled surface $\mathbb{F}_n \to \mathbb{P}^1$ with a section $S \subset \mathbb{F}_n$ such that $S^2 = -n$. If $n > 0$ then the section S is the unique section with negative self-intersection. (If $n = 0$ then $f : \mathbb{F}_0 \to \mathbb{P}^1$ is given by $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, any fibre S of the other projection $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a section of f with $S^2 = 0$, and any other section has strictly positive self-intersection.) Every ruled surface over $\mathbb{P}^1$ is isomorphic to $\mathbb{F}_n$ for some n. Now we describe the topological classification: The ruled surfaces $\mathbb{F}_n$, $\mathbb{F}_m$ are homeomorphic iff $n \equiv m \mod 2$.

In fact, if $n \equiv m \mod 2$, and $n > m$, there is a family $\mathcal{X} \to \Delta$ of smooth surfaces over the disc $\Delta = (|t| < 1) \subset \mathbb{C}$ such that $\mathcal{X}_t \simeq \mathbb{F}_m$ for $t \neq 0$ and $\mathcal{X}_0 \simeq \mathbb{F}_n$.

11 Complex tori

Let V be a complex vector space of dimension n. Let $L \subset V$ be a lattice, that is, $L \simeq \mathbb{Z}^{2n}$ is a free abelian group of rank $2n$ and the map

$$L \otimes_{\mathbb{Z}} \mathbb{R} \to V_{\mathbb{R}}$$

is an isomorphism of $\mathbb{R}$-vector spaces. (Here we write $V_{\mathbb{R}}$ for V regarded as an $\mathbb{R}$-vector space.) Then we say $X := V/L$ is a *complex torus.* The space X is a compact complex manifold, and the underlying smooth manifold $X_{\mathbb{R}}$ is diffeomorphic to the real torus $(S^1)^{2n}$ of dimension $2n$. (Proof: picking a basis of L gives an identification $X_{\mathbb{R}} = V_{\mathbb{R}}/L = (\mathbb{R}/\mathbb{Z})^{2n} = (S^1)^{2n}$.) A complex torus $X = V/L$ is a complex Lie group, with group law induced by addition in V.

We say a complex torus X is an *abelian variety* if it is projective, that is, admits an embedding in projective space. This is *not* always the case for $n > 1$.

The following lemma reduces the study of morphisms of complex tori to linear algebra.

Lemma 11.1. *Let $X = V/L$, $X' = V'/L'$ be complex tori, and $f \colon X \to X'$ a morphism. Then f is induced by an affine linear function $\tilde{f} \colon V \to V'$ such that $\tilde{f}(L) \subset L'$.*

Proof. The map $f \colon X \to X'$ lifts to a map $\tilde{f} \colon V \to V'$ because V is simply connected and $V' \to X'$ is a covering space. Pick coordinates w_i, z_j on V', V, and consider the derivative $df = (\frac{\partial f_i}{\partial z_j})$ of f in these coordinates. Each matrix entry $\frac{\partial f_i}{\partial z_j}$ is a holomorphic function on V which is periodic with respect to L. In particular it is bounded, and therefore constant (Liouville's theorem). So df is a constant matrix. It follows that f is affine linear (that is, of the form $f(x) = Ax + b$), as required. $\square$

Let $X = V/L$ be a complex torus. Then the quotient map $V \to X$ is the universal cover of X, with group $\pi_1(X) = L \simeq \mathbb{Z}^{2n}$. In particular $H_1(X, \mathbb{Z}) = L \simeq \mathbb{Z}^{2n}$. More generally, $H_k(X, \mathbb{Z}) = \wedge^k L \simeq \mathbb{Z}^{2n}$. We sketch two proofs of this fact. Write $m = 2n$, so $X \simeq (S^1)^m$. If X and Y are two topological spaces, then the Künneth formula asserts that

$$H_k(X \times Y, \mathbb{Q}) = \bigoplus_{i+j=k} H_i(X, \mathbb{Q}) \otimes H_j(Y, \mathbb{Q})$$

for each k. Moreover, if $H_i(X, \mathbb{Z})$ and $H_j(Y, \mathbb{Z})$ are torsion free for all i, j, then the equality holds with $\mathbb{Z}$-coefficients. Using this inductively we can compute $H_k((S^1)^m, \mathbb{Z})$. Alternatively, we can use an explicit cell decomposition of $(S^1)^m$ as follows. Consider the standard cell decomposition of S^1 given by one 1-cell and one 0-cell, where the two ends of the 1-cell (an interval) are identified with the 0-cell to form S^1. We take the product decomposition of $(S^1)^m$ given by this cell decomposition of each factor. The cells are given by $e_{i_1} \times \cdots \times e_{i_k}$, where $1 \le i_1 < \cdots < i_k \le m$, and e_i denotes the 1-cell in the ith factor. Observe that each cell defines a cycle in $(S^1)^m$, that is, it has no boundary. (It may help to think of $(S^1)^m$ as obtained from the unit cube of dimension m in $\mathbb{R}^m$ by identifying opposite faces. Then the cell $e_{i_1} \times \cdots \times e_{i_k}$ is a face of the cube of dimension k, which maps to a real k-torus $(S^1)^k \subset (S^1)^m$.) Now it follows that the homology $H_k(X, \mathbb{Z})$ is a free abelian group with basis $e_{i_1} \times \cdots \times e_{i_k}$, $1 \le i_1 < \cdots < i_k \le m$. (Note: We can compute homology using a cell decomposition in the same way as for a triangulation.)

Next we consider de Rham cohomology. Let $e_1, \ldots, e_m$ be a basis of L, and $x_1, \ldots, x_m$ the associated real coordinates on $V_{\mathbb{R}}$. Then $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a closed smooth k-form on $V_{\mathbb{R}}$ for $1 \le i_1 < \cdots < i_k \le m$, and these forms give the basis of $H^k_{\text{dR}}(X, \mathbb{R}) = H_k(X, \mathbb{R})^*$ dual to the basis $\{e_{i_1} \times \cdots \times e_{i_k}\}$ of $H_k(X, \mathbb{R})$ described above. Without choosing a basis, $H^k_{\text{dR}}(X, \mathbb{R}) = \wedge^k V^*_{\mathbb{R}}$.

We describe the Hodge decomposition for a complex torus. (Note: although a complex torus is not necessarily projective, it is always Kähler, so the Hodge decomposition theorem holds.) Recall that $H^k_{\text{dR}}(X, \mathbb{R}) = \wedge^k V^*_{\mathbb{R}}$. So

$$H^k_{\text{dR}}(X, \mathbb{C}) = \wedge^k V^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \wedge^k (V^* \oplus \bar{V}^*) = \bigoplus_{p+q=k} \wedge^p V^* \otimes \wedge^q \bar{V}^*.$$

In coordinates, if $z_1, \ldots, z_n$ are complex coordinates on V, then V^* has basis $\{dz_i\}$ and $\bar{V}^*$ has basis $\{d\bar{z}_j\}$. So $\wedge^p V^* \otimes \wedge^q \bar{V}^*$ has basis $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, where $1 \le i_1 < \cdots < i_p \le n$ and $1 \le j_1 < \cdots < j_q \le n$. It follows that

$$H^{p,q} = \wedge^p V^* \otimes \wedge^q \bar{V}^*.$$

Since $X = V/L$ is a complex Lie group, the tangent bundle T_X is trivial, with fibre $T_{X,0} = V$, the tangent space at the identity $0 \in X$. That is,

$T_X = \mathcal{O}_X \otimes V \simeq \mathcal{O}_X^{\oplus n}$. If $z_1, \ldots, z_n$ are complex coordinates on V, then $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a basis of T_X. Similarly, $\Omega_X = T_X^* = \mathcal{O}_X \otimes V^* \simeq \mathcal{O}_X^{\oplus n}$, generated by $dz_1, \ldots, dz_n$, and $\omega_X = \wedge^n \Omega_X = \mathcal{O}_X \otimes \wedge^n V^* \simeq \mathcal{O}_X$, generated by $dz_1 \wedge \cdots \wedge dz_n$. In particular, the canonical divisor $K_X = 0$.

11.1 Projective embeddings of complex tori

We say a compact complex manifold X is *projective* if it admits a closed embedding in complex projective space $\mathbb{P}^N_{\mathbb{C}}$. (Then, according to Chow's theorem [GH, p. 167], $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomial equations, so is the complex manifold associated to a smooth complex projective variety.)

A necessary condition for a compact complex manifold to be projective is that the group $\operatorname{Num} X$ of numerical equivalence classes of line bundles on X is nontrivial. Indeed, if H is a hyperplane section of X in some projective embedding then the line bundle $\mathcal{O}_X(H)$ is not numerically trivial.

Now let $X = V/L$ be a complex torus. Then

$$\operatorname{Num} X = H^{1,1} \cap H^2(X, \mathbb{Z}) / \operatorname{Tors} = V^* \otimes \bar{V}^* \cap \wedge^2 L^*.$$

Equivalently, a class in $\operatorname{Num} X$ is given by a skew $\mathbb{Z}$-bilinear map

$$\phi \colon L \times L \to \mathbb{Z}$$

such that the skew $\mathbb{R}$-bilinear map

$$\phi_{\mathbb{R}} \colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$$

obtained by extension of scalars from $\mathbb{Z}$ to $\mathbb{R}$ satisfies

$$\phi_{\mathbb{R}}(iv, iw) = \phi_{\mathbb{R}}(v, w) \tag{13}$$

for all $v, w \in V$. To see this, recall the decomposition

$$\wedge^2 L^* \subset \wedge^2 L^* \otimes_{\mathbb{Z}} \mathbb{C} = \wedge^2 V_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = \wedge^2 (V^* \oplus \bar{V}^*) = \wedge^2 V^* \oplus V^* \otimes \bar{V}^* \oplus \wedge^2 \bar{V}^*,$$

where

$$V_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^* \oplus \bar{V}^*$$

is the decomposition of $\mathbb{R}$-linear forms $V \to \mathbb{C}$ into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear forms. Now $\psi(iu, iv) = -\psi(u, v)$ if $\psi \in \wedge^2 V^*$ or $\psi \in \wedge^2 \bar{V}^*$ and $\psi(iu, iv) = \psi(u, v)$ if $\psi \in V \otimes V^*$. So the condition (13) is equivalent to $\phi_{\mathbb{R}} \in V \otimes V^*$.

A skew $\mathbb{R}$-bilinear form $\phi\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that $\phi(iu, iv) = \phi(u, v)$ corresponds to a hermitian form $h\colon V \times V \to \mathbb{C}$ with $\phi = -\operatorname{Im} h$. Indeed, recall that $h\colon V \times V \to \mathbb{C}$ is *hermitian* if $h(\lambda u, v) = \lambda h(u, v)$, $h(u, \lambda v) = \bar{\lambda} h(u, v)$, and $h(v, u) = \overline{h(u, v)}$ for $u, v \in V$, $\lambda \in \mathbb{C}$. So, writing $h = g - i\phi$ where g and ϕ are $\mathbb{R}$-valued, we find $g\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ is a symmetric $\mathbb{R}$-bilinear form and $\phi\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ is a skew $\mathbb{R}$-bilinear form. Moreover,

$$g(u, v) = \operatorname{Re} h(u, v) = \operatorname{Im} h(iu, v) = -\phi(iu, v)$$

and

$$\phi(iu, iv) = -\operatorname{Im} h(iu, iv) = -\operatorname{Im} h(u, v) = \phi(u, v).$$

Conversely, given $\phi\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ a skew $\mathbb{R}$-bilinear form such that $\phi(iu, iv) = \phi(u, v)$, we define $h = g - i\phi$ where $g(u, v) := -\phi(iu, v)$, then $h\colon V \times V \to \mathbb{C}$ is hermitian.

Theorem 11.2. *Let $X = V/L$ be a complex torus. Then X is projective iff there exists $\phi \in \operatorname{Num} X$ such that the corresponding hermitian form $h\colon V \times V \to \mathbb{C}$ is positive definite.*

We give a brief sketch of the proof. Given $\phi \in \operatorname{Num} X$, let $\mathcal{L} \in \operatorname{Pic} X$ be a line bundle with numerical equivalence class ϕ. One proves that $\mathcal{L}$ is ample iff the hermitian form h associated to ϕ is positive definite. The first proof uses the Kodaira embedding theorem, which provides a differential geometric characterisation of ample line bundles. See [GH, p. 181]. The second proof is more direct: if h is positive definite, we explicitly construct global sections of powers of $\mathcal{L}$ given by holomorphic functions $\theta\colon V \to \mathbb{C}$ (called *theta functions*) which satisfy transformation laws of the form

$$\theta(z + \lambda) = e_\lambda(z)\theta(z)$$

for $\lambda \in L$, where the multipliers $e_\lambda(z)$ are nowhere zero holomorphic functions on V which give transition functions for the line bundle. (The theta functions are obtained as power series in $q_j = e^{2\pi i z_j}$, $j = 1, \ldots, n$, where $z_1, \ldots, z_n$ are appropriate complex coordinates on V. In this setup the positive definiteness of h implies that the power series converge.) Finally one proves that $\mathcal{L}^k$ defines an embedding for $k \geq 3$ using the theta functions (Lefschetz' theorem). See [GH, p. 317–324].

Example 11.3. Recall first that if E is a smooth projective complex curve of genus 1 then E is a complex torus of dimension 1, $E = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, and if $\mathcal{L}$ is a line bundle on E of degree 3 then $\mathcal{L}$ defines an embedding $E \subset \mathbb{P}^2$ of E as a smooth plane cubic curve.

If now $X = V/L$ is a complex torus of dimension 2 and $\mathcal{L}$ is an ample line bundle on X of degree $\mathcal{L}^2 = 2$ (this is the minimum possible degree), then $\mathcal{L}^{\otimes 3}$ defines an embedding $X \subset \mathbb{P}^8$ of codimension 6. In the special case that $X = E_1 \times E_2$ is the product of two elliptic curves, we can take $\mathcal{L} = \mathrm{pr}_1^* \mathcal{L}_1 \otimes \mathrm{pr}_2^* \mathcal{L}_2$ where $\mathcal{L}_i$ is a line bundle of degree 1 on E_i. Then the embedding $X \subset \mathbb{P}^8$ defined by $\mathcal{L}^{\otimes 3}$ is given by the embeddings $E_i \subset \mathbb{P}^2$ defined by $\mathcal{L}_i^{\otimes 3}$ and the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, that is,

$$X = E_1 \times E_2 \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$$

Note that an embedded complex torus $X \subset \mathbb{P}^N$ of dimension $n > 1$ is never a complete intersection. This follows from the Lefschetz hyperplane theorem [GH, p. 156], which shows (in particular) that a complete intersection $X \subset \mathbb{P}^N$ of dimension $n > 1$ has $\pi_1(X) \simeq \pi_1(\mathbb{P}^N) = 1$. (Alternatively, one can show algebraically that a complete intersection $X \subset \mathbb{P}^N$ of dimension $n > 1$ has $h^1(\mathcal{O}_X) = 0$.)

11.2 The Albanese variety

Let X be a smooth complex projective variety (or, more generally, a compact Kähler manifold). The *Albanese variety* $\mathrm{Alb}\, X$ is the complex torus of dimension $q = h^0(\Omega_X)$ given by

$$\mathrm{Alb}\, X := \Gamma(\Omega_X)^* / H_1(X, \mathbb{Z}).$$

Here the map $H_1(X, \mathbb{Z}) \to \Gamma(\Omega_X)^*$ is given by integration:

$$H_1(X, \mathbb{Z}) \to \Gamma(\Omega_X)^*, \quad \gamma \mapsto \left(\omega \mapsto \int_\gamma \omega \right).$$

Note that the map

$$H_1(X, \mathbb{Z}) \otimes \mathbb{R} \to \Gamma(\Omega_X)^*$$

is equal to the composition

$$H_1(X, \mathbb{R}) = H^1_{\mathrm{dR}}(X, \mathbb{R})^* \subset H^1_{\mathrm{dR}}(X, \mathbb{C})^* = H^{1,0^*} \oplus H^{0,1^*} \to H^{1,0^*}$$

where the last map is the projection onto the first factor. This is an isomorphism of $\mathbb{R}$-vector spaces: the dimensions are equal, and it is injective because $\alpha = \alpha_{1,0} + \alpha_{0,1} \mapsto \alpha_{1,0}$ and $\alpha_{0,1} = \overline{\alpha_{1,0}}$ since α is real. So the Albanese variety is a complex torus as claimed.

The *Albanese morphism* $\alpha\colon X \to \operatorname{Alb} X$ is defined as follows. Choose a basepoint $P_0 \in X$. We define

$$\alpha\colon X \to \operatorname{Alb} X, \quad P \mapsto \left(\omega \mapsto \int_{P_0}^{P} \omega\right).$$

Changing the choice of basepoint corresponds to composing α with a translation of $\operatorname{Alb} X$

$$t_v\colon \operatorname{Alb} X \to \operatorname{Alb} X, \quad x \mapsto x + v.$$

We describe the Albanese morphism in coordinates. Let $\omega_1, \ldots, \omega_q$ be a basis of the complex vector space $\Gamma(\Omega_X)$ and $\gamma_1, \ldots, \gamma_{2q}$ a basis of the free abelian group $H_1(X, \mathbb{Z})/\operatorname{Tors}$. Then $\operatorname{Alb} X = \mathbb{C}^q/L$ where $L \subset \mathbb{C}^q$ is the lattice generated by the vectors

$$\left(\int_{\gamma_i} \omega_1, \ldots, \int_{\gamma_i} \omega_q\right), \quad 1 \leq i \leq 2q,$$

and $\alpha\colon X \to \operatorname{Alb} X$ is the map

$$\alpha\colon X \to \mathbb{C}^q/L, \quad P \mapsto \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_q\right)$$

If $f\colon X \to Y$ is a morphism between smooth complex projective varieties, then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & \operatorname{Alb} X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \operatorname{Alb} f} \\ Y & \xrightarrow{\alpha_Y} & \operatorname{Alb} Y \end{array} \tag{14}$$

where the morphism $\operatorname{Alb} f\colon \operatorname{Alb} X \to \operatorname{Alb} Y$ is defined as follows. We have the pullback map $f^*\colon \Gamma(\Omega_Y) \to \Gamma(\Omega_X)$ on 1-forms. The dual $(f^*)^*$ of this map is compatible with the pushforward map $f_*\colon H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ on 1-cycles (that is, $\int_\gamma f^*\omega = \int_{f_*\gamma} \omega$). So $(f^*)^*$ defines a morphism $\operatorname{Alb} f\colon \operatorname{Alb} X \to \operatorname{Alb} Y$.

The Albanese morphism $\alpha\colon X \to \operatorname{Alb} X$ satisfies the following universal property: if $\beta\colon X \to T$ is a morphism from X to a complex torus, then there exists a unique morphism $f\colon \operatorname{Alb} X \to T$ such that $\beta = f \circ \alpha$. Indeed, if T is a complex torus, then $\alpha_T\colon T \to \operatorname{Alb} T$ is an isomorphism. Now the existence of f follows from the functoriality (14) of Alb. To get uniqueness, observe that the pullback map on 1-forms

$$\alpha^*\colon \Gamma(\Omega_{\operatorname{Alb} X}) \to \Gamma(\Omega_X)$$

is an isomorphism. Thus $f^*\colon \Gamma(\Omega_T) \to \Gamma(\Omega_{\operatorname{Alb} X})$ is uniquely determined, equivalently, $f\colon \operatorname{Alb} X \to T$ is uniquely determined up to a translation. Finally the translation is determined by the equality $\beta = f \circ \alpha$.

The image $\alpha(X)$ affinely generates $\operatorname{Alb} X$ as a group. That is, the differences $\alpha(x) - \alpha(y)$ for $x, y \in X$ generate $\operatorname{Alb} X$. Indeed, as observed above, the pullback map α^* on 1-forms is an isomorphism. This shows that $\alpha(X)$ is not contained in a translate of a proper subtorus of $\operatorname{Alb} X$. In particular, if $q(X) \neq 0$, then $\alpha(X)$ is not a point.

Example 11.4. Let $f\colon X \to C$ be a ruled surface. We already observed that $f^*\colon \Gamma(\Omega_C) \to \Gamma(\Omega_X)$ and $f_*\colon H_1(X,\mathbb{Z}) \to H_1(C,\mathbb{Z})$ are isomorphisms. Thus $\operatorname{Alb} f\colon \operatorname{Alb} X \to \operatorname{Alb} C$ is an isomorphism. Using this, we can identify the Albanese morphism $\alpha_X\colon X \to \operatorname{Alb} X$ with the composition of f and the Albanese morphism $\alpha_C\colon C \to \operatorname{Alb} C$.

Now, for a curve C the Albanese morphism α_C can be identified with *Abel–Jacobi morphism*

$$C \to \operatorname{Pic}^0 C, \quad P \mapsto \mathcal{O}_C(P - P_0)$$

via a canonical isomorphism $\operatorname{Pic}^0 C \simeq \operatorname{Alb} C$, where $P_0 \in C$ is a fixed basepoint. See [GH, p. 224–237]. Here we only describe the isomorphism $\operatorname{Pic}^0 C \simeq \operatorname{Alb} C$. Recall that

$$\operatorname{Pic}^0 C = H^1(\mathcal{O}_C)/H^1(C,\mathbb{Z}) = H^{0,1}/H^1(C,\mathbb{Z})$$

by the exponential sequence, and

$$\operatorname{Alb} C = \Gamma(\Omega_C)^*/H_1(C,\mathbb{Z}) = (H^{1,0})^*/H^1(C,\mathbb{Z})^*.$$

The cup product

$$\cup\colon H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z}$$

defines an isomorphism $H^1(C,\mathbb{Z}) \xrightarrow{\sim} H^1(C,\mathbb{Z})^*$. The bilinear form

$$b\colon H^{0,1} \times H^{1,0} \to \mathbb{C}, \quad (\omega, \eta) \mapsto \int_X \omega \wedge \eta$$

is non-degenerate (by Poincaré duality) and so defines an isomorphism $H^{0,1} \xrightarrow{\sim} (H^{1,0})^*$. These two isomorphisms are compatible and give the desired isomorphism $\operatorname{Pic}^0 C \simeq \operatorname{Alb} C$. To see the compatibility, note that the map $H^1(C,\mathbb{R}) \to H^{0,1}$ is given by $\omega = \overline{\omega^{0,1}} + \omega^{0,1} \mapsto \omega^{0,1}$, and similarly $H^1(C,\mathbb{R}) \to H^{1,0}$ is given by $\eta = \eta^{1,0} + \overline{\eta^{1,0}} \mapsto \eta^{1,0}$. The map of dual spaces $H^1(C,\mathbb{R})^* \to$

$(H^{1,0})^*$ is given by $\theta = \theta_{1,0} + \overline{\theta_{1,0}} \mapsto \theta_{1,0}$, and so $\theta(\eta) = \theta_{1,0}(\eta^{1,0}) + \overline{\theta_{1,0}}(\overline{\eta^{1,0}}) = 2\,\mathrm{Re}\,\theta_{1,0}(\eta^{1,0})$. Now

$$
\begin{aligned}
[\omega] \cup [\eta] &= \int_X \omega \wedge \eta = \int_X \omega^{0,1} \wedge \eta^{1,0} + \overline{\omega^{0,1}} \wedge \overline{\eta^{1,0}} \\
&= 2\,\mathrm{Re} \int_X \omega^{0,1} \wedge \eta^{1,0} = 2\,\mathrm{Re}\, b(\omega^{0,1}, \eta^{1,0}),
\end{aligned}
$$

so the functional $\theta = ([\omega] \cup (\cdot)) \in H^1(C, \mathbb{R})^*$ corresponds to the functional $b(\omega^{0,1}, \cdot) \in (H^{1,0})^*$, as required.

The Albanese morphism for a curve C is an embedding if $g(C) > 0$. (If $g(C) = 0$ then $C \simeq \mathbb{P}^1$ and $\mathrm{Alb}\, C$ is a point.) Indeed, α_C is injective as a map of sets by the above identification of the Albanese morphism with the Abel–Jacobi map: if $\alpha_C(P) = \alpha_C(Q)$ then $P \sim Q$, so $P = Q$ because C is not isomorphic to $\mathbb{P}^1$. Also, by the fundamental theorem of calculus, the derivative of α_C at a point $P \in C$ is given in coordinates by

$$(d\alpha_C)_P = (\omega_1, \ldots, \omega_q)$$

where $\omega_1, \ldots, \omega_q$ is a basis of $\Gamma(\Omega_C)$. Now K_C is basepoint free by Riemann–Roch: for $P \in C$ a point we have

$$h^0(K_C - P) - h^0(P) = 1 - g + (2g - 2 - 1) = g - 2$$

and $h^0(P) = 1$ since C is not isomorphic to $\mathbb{P}^1$, so $h^0(K_C - P) = g - 1 = h^0(K_C) - 1$. So $(d\alpha_C)_P \neq 0$ for all $P \in C$, and α_C is an embedding.

Remark 11.5. If X is a smooth complex projective variety then $\mathrm{Alb}\, X$ is projective. In fact one can explicitly write down an ample class $\phi \in \mathrm{Num}(\mathrm{Alb}\, X)$ in terms of an ample class on X. Alternatively, consider the map

$$\alpha^k \colon X^k = X \times \cdots \times X \to \mathrm{Alb}\, X, \quad (x_1, \ldots, x_k) \mapsto \alpha(x_1) + \cdots + \alpha(x_k).$$

One shows that α^k is surjective for large k. Then, since $\mathrm{Alb}\, X$ is Kähler and is covered by a projective variety, it follows from a result of Moishezon that $\mathrm{Alb}\, X$ is projective. See [Voisin, p. 298, Lem. 12.11, Cor. 12.12].

12 K3 surfaces

A *K3 surface* is a compact complex surface such that $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$. (The name K3 refers to Kummer, Kähler, and Kodaira.) We do *not* assume that X is algebraic.

The main results are the following:

(1) Any 2 K3's are diffeomorphic as smooth manifolds (Kodaira).

(2) Every K3 is a Kähler manifold, so the Hodge decomposition theorem holds (Siu).

(3) The Torelli theorem for K3 surfaces: an explicit description of the moduli space of K3 surfaces in terms of the Hodge decomposition (Piatetski-Shapiro–Shafarevich).

Remark 12.1. The Torelli theorem for K3s is so called because it is analogous to the Torelli theorem for curves. See [GH, p. 359].

We give some examples of K3 surfaces. We first consider complete intersections in projective space. Recall the adjunction formula: if X is a smooth variety and $Y \subset X$ is a smooth closed subvariety of codimension 1, then

$$K_Y = K_X + Y|_Y$$

Now suppose $X \subset \mathbb{P}^N$ is a complete intersection. Let c be the codimension of $X \subset \mathbb{P}^N$ and $d_1, \ldots, d_c$ the degrees of the defining equations. Recall that $K_{\mathbb{P}^N} = -(N+1)H$ where H is the class of a hyperplane. Now by the adjunction formula and induction we deduce

$$K_X = (d_1 + \cdots + d_c - (N+1))H|_X.$$

In particular $K_X = 0$ iff $d_1 + \cdots + d_c = N + 1$. We also need the following

Lemma 12.2. *Let $X \subset \mathbb{P}^N$ be a complete intersection. Then $H^i(\mathcal{O}_X(n)) = 0$ for $0 < i < \dim X$ and $n \in \mathbb{Z}$. (Here $\mathcal{O}_X(n) := \mathcal{O}_{\mathbb{P}^N}(n)|_X$.)*

Proof. We use induction on the codimension c of $X \subset \mathbb{P}^N$. If $c = 0$, that is, $X = \mathbb{P}^N$, this follows from an explicit computation of the Cech cohomology groups with respect to the standard affine open covering $U_i = (X_i \neq 0)$ of $\mathbb{P}^N$, see [Hartshorne, III.5.1]. Suppose the result is true for codimension $c-1$, and let $X \subset \mathbb{P}^N$ be codimension c, cut out by equations of degrees $d_1, \ldots, d_c$. Consider $X \subset Y \subset \mathbb{P}^N$, where $Y \subset \mathbb{P}^N$ is cut out by the first $c-1$ of the c equations defining X. The ideal sheaf $\mathcal{I}_{X/Y}$ of $X \subset Y$ is isomorphic to $\mathcal{O}_Y(-d_c)$, so we have an exact sequence

$$0 \to \mathcal{O}_Y(-d_c) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0,$$

and tensoring by $\mathcal{O}_Y(n)$ we obtain an exact sequence

$$0 \to \mathcal{O}_Y(n - d_c) \to \mathcal{O}_Y(n) \to \mathcal{O}_X(n) \to 0$$

Now the long exact sequence of cohomology

$$\cdots \to H^i(\mathcal{O}_Y(n)) \to H^i(\mathcal{O}_X(n)) \to H^{i+1}(\mathcal{O}_Y(n-d_c)) \to \cdots$$

together with the induction hypothesis shows that $H^i(\mathcal{O}_X(n)) = 0$ for $0 < i < \dim X = \dim Y - 1$ and $n \in \mathbb{Z}$, as required. □

In particular, if $X \subset \mathbb{P}^N$ is a complete intersection, then $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. (This can also be deduced from the Lefschetz hyperplane theorem and the Hodge decomposition.) We deduce that the following complete intersections are K3 surfaces.

(1) $X_4 \subset \mathbb{P}^3$

(2) $X_{2,3} \subset \mathbb{P}^4$

(3) $X_{2,2,2} \subset \mathbb{P}^5$.

Here $X_{d_1,\ldots,d_c} \subset \mathbb{P}^N$ denotes a complete intersection of codimension c defined by equations of degrees $d_1, \ldots, d_c$. Let $L = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. The degree of $X \subset \mathbb{P}^N$ is $L^{\dim X} = d_1 d_2 \cdots d_c$. In the cases above, we have $L^2 = 4, 6, 8$.

A similar example is the double cover $f\colon X \to \mathbb{P}^2$ branched over a smooth plane curve $B \subset \mathbb{P}^2$ of degree 6. Here

$$X = (Y^2 = F(X_0, X_1, X_2)) \subset L,$$

where $p\colon L \to \mathbb{P}^2$ is the line bundle with sheaf of sections $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$, $Y \in \Gamma(L, p^*\mathcal{L})$ is the tautological homogeneous vertical coordinate on L (that is, for $t \in L$, $p(t) = x$, we define $Y(t) = t \in L_x$), and $F \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(6))$ is the equation of $B \subset \mathbb{P}^2$. Explicitly, over $U_0 = (X_0 \neq 0) \subset \mathbb{P}^2$,

$$X = (y^2 = f(x_1, x_2)) \subset \mathbb{A}^3_{x_1,x_2,y} \xrightarrow{f} \mathbb{A}^2_{x_1,x_2}$$

where $x_1 = X_1/X_0$, $x_2 = X_2/X_0$, $y = Y/X_0^3$, and $f(x_1, x_2) = F/X_0^6$ is the equation of B in the affine piece $U_0 = \mathbb{A}^2_{x_1,x_2}$. In particular, X is smooth because B is so. We observe that

$$f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \cdot Y$$

Now $H^1(\mathcal{O}_X) = H^1(f_*\mathcal{O}_X)$ because f is finite, so $H^1(\mathcal{O}_X) = 0$. Next we compute the canonical divisor K_X. Let $P \in B$ be a point and choose coordinates x, z at $P \in \mathbb{P}^2$ such that $B = (z = 0)$ near P. Then, locally in the Euclidean topology at P, $f\colon X \to \mathbb{P}^2$ is given by

$$\mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,z}, \quad (x, y) \mapsto (x, y^2).$$

So the 2-form $f^*dx \wedge dz = 2ydx \wedge dy$ on X vanishes along the ramification locus $R := f^{-1}B = (y = 0)$ with multiplicity 1. It follows that

$$\omega_X = f^*\omega_{\mathbb{P}^2} \otimes \mathcal{O}_X(R).$$

(This is an instance of the Riemann–Hurwitz formula, cf. [Hartshorne, IV.2.3].) But $R = (Y = 0) \subset X$ and $Y \in \Gamma(f^*\mathcal{O}_{\mathbb{P}^2}(3))$, so $\mathcal{O}_X(R) \simeq f^*\mathcal{O}_{\mathbb{P}^2}(3)$. We deduce that $\omega_X \simeq \mathcal{O}_X$, that is, $K_X = 0$. Finally, we note that the line bundle $M = f^*\mathcal{O}_{\mathbb{P}^2}(1)$ on X satisfies $M^2 = \deg f = 2$.

12.1 Kummer surfaces

A Kummer surface is a special type of K3 surface constructed from a complex torus as follows. Let $Y = V/L$ be a complex torus of dimension 2. Let $i \colon Y \xrightarrow{\sim} Y$ be the involution given by scalar multiplication by (-1). Let Z be the quotient $Y/\langle i \rangle$. Then Z is a normal surface with $16 = 2^4$ singularities corresponding to the fixed points $\frac{1}{2}L/L \subset Y = V/L$ of i. Let $\tilde{Z} \to Z$ be the blowup of the singular points. Then $X := \tilde{Z}$ is a smooth K3 surface called the *Kummer surface* associated to Y. We describe the construction in more detail and check that X is indeed a K3 surface below.

First we analyse the singularities of $Z = Y/(\mathbb{Z}/2\mathbb{Z})$. Working locally in the Euclidean topology at a fixed point of the involution i, the involution is of the form

$$\mathbb{A}^2_{x,y} \xrightarrow{\sim} \mathbb{A}^2_{x,y}, \quad (x, y) \mapsto (-x, -y).$$

Now, the coordinate ring $k[Z]$ of the quotient $Z = Y/G$ of an affine variety Y by a finite group G is the ring of invariants $k[Y]^G$ in the coordinate ring $k[Y]$ of Y. (This was proved in 508A, see the notes on my webpage.) In our case

$$k[x, y]^{\mathbb{Z}/2\mathbb{Z}} = k[x^2, xy, y^2] = k[u, v, w]/(uw = v^2),$$

so $Z = Y/(\mathbb{Z}/2\mathbb{Z})$ is locally of the form

$$(uw = v^2) \subset \mathbb{A}^3_{u,v,w}$$

This singularity is called an *ordinary double point* or *A_1-singularity.* (Note that any nondegenerate quadratic form in 3 complex variables defines an isomorphic singularity.) Next we study the blowup $\pi \colon \tilde{Z} \to Z$ of the singular point $P \in Z$. Recall that $\tilde{Z}$ can be computed as the strict transform of $Z \subset \mathbb{A}^3$ under the blowup $b \colon \operatorname{Bl}_0 \mathbb{A}^3 \to \mathbb{A}^3$ of $0 \in \mathbb{A}^3$. The exceptional divisor F of b is a copy of $\mathbb{P}^2$ with homogeneous coordinates U, V, W corresponding to the coordinates u, v, w at $0 \in \mathbb{A}^3$. The exceptional divisor E of π is given by $E = (UW = V^2) \subset F = \mathbb{P}^2$. (In general, the exceptional divisor for the blowup of a hypersurface $(f = 0) \subset \mathbb{A}^n$ is given by $(F = 0) \subset \mathbb{P}^{n-1}$ where F is the homogeneous component of f of minimal degree.) In particular, E is a smooth conic, so $E \simeq \mathbb{P}^1$. Now E is smooth and $E \subset \tilde{Z}$ is locally defined by one equation (a general property of the blowup). It follows that $\tilde{Z}$ is smooth. We can also compute directly in charts. Consider the affine piece $(U \neq 0)$ of the blowup $b\colon \mathrm{Bl}_0 \mathbb{A}^3 \to \mathbb{A}^3$, given by

$$\mathbb{A}^3_{p,v',w'} \to \mathbb{A}^3_{u,v,w}, \quad (p, v', w') \mapsto (p, pv', pw').$$

Now $Z = (uw = v^2)$, so in this affine piece $b^*Z = (p^2w' = p^2v'^2)$ and $\tilde{Z} = Z' = (w' = v'^2) \simeq \mathbb{A}^2_{p,v'}$. So in this affine piece $\pi\colon \tilde{Z} \to Z$ is given by

$$\mathbb{A}^2_{p,v'} \to Z \subset \mathbb{A}^2_{u,v,w}, \quad (p, v') \mapsto (p, pv', pv'^2).$$

Note that $\tilde{Z}$ is covered by the two affine charts $(U \neq 0)$, $(W \neq 0)$, and they are symmetric.

We show that $K_{\tilde{Z}} = 0$. We first do a local calculation on the resolution of the singularity. With notation as above, consider the 2-form $dx \wedge dy$ on $\mathbb{A}^2_{x,y}$. Observe that this form is invariant under the involution $(x, y) \mapsto (-x, -y)$. So it defines a 2-form on the quotient Z away from the singular point $P \in Z$, and we obtain a 2-form ω on $\tilde{Z} \setminus E$ by pullback. We show that the rational 2-form ω on $\tilde{Z}$ is actually a nowhere zero regular 2-form on $\tilde{Z}$, in particular, $K_{\tilde{Z}} = (\omega) = 0$. We just compute in the charts described above. We have

$$\tilde{Z} \supset \mathbb{A}^2_{p,v'} \to Z \subset \mathbb{A}^3_{u,v,w}, \quad (p, v') \mapsto (p, pv', pv'^2)$$

$$\mathbb{A}^2_{x,y} \to Z \subset \mathbb{A}^3_{u,v,w}, \quad (x, y) \mapsto (x^2, xy, y^2)$$

So $p = x^2$, $pv' = xy$, $pv'^2 = y^2$. Hence $y = xv'$ and

$$\omega = dx \wedge dy = x dx \wedge dv' = \frac{1}{2} dp \wedge dv',$$

a nowhere vanishing 2-form on the affine piece $\mathbb{A}^2_{p,v'}$ of $\tilde{Z}$. Now, the same construction applied globally to the invariant 2-form $dz_1 \wedge dz_2$ on $Y = V/L$, where z_1, z_2 are complex coordinates on V, produces a nowhere zero regular 2-form ω on $X := \tilde{Z}$, so we obtain $K_X = 0$.

It remains to show that $H^1(\mathcal{O}_X) = 0$, or, equivalently (by the Hodge decomposition), $H^1(X, \mathbb{Q}) = 0$. We first observe that there is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{f} & Z \end{array}$$

where $f\colon Y \to Z = Y/\langle i\rangle$ is the quotient map, $\pi\colon X = \tilde{Z} \to Z$ is the blowup of the singular points, $\phi\colon W \to Y$ is the blowup of the fixed points of the involution i, and $g\colon W \to X$ is a double cover with branch locus the union of the exceptional curves of π. In charts, the diagram is

$$\begin{array}{ccc} \mathbb{A}^2_{q,y'} & \longrightarrow & \mathbb{A}^2_{p,v'} \\ \downarrow & & \downarrow \\ \mathbb{A}^2_{x,y} & \longrightarrow & Z \end{array}$$

where $(q, y') \mapsto (p, v') = (q^2, y')$, $(q, y') \mapsto (x, y) = (q, qy')$, and $(p, v') \mapsto (p, pv', pv'^2)$ and $(x, y) \mapsto (x^2, xy, y^2)$ as before. We check that this does define a commutative diagram. In general, given a finite morphism $f\colon Y \to Z$ and a birational morphism $\pi\colon X \to Z$ there exists a unique normal variety W together with a finite morphism $g\colon W \to X$ and a birational morphism $\phi\colon W \to Y$ such that $\pi g = \phi f$. (W is called the *normalisation of X in the function field of Y*.) In particular, in our example the patching of the two charts over the given singular point $P \in Z$ is automatic.

We can now show $H^1(X, \mathbb{Q}) = 0$ as follows. The morphism $g\colon W \to X$ is the quotient map for the action of $\mathbb{Z}/2\mathbb{Z}$ on W, so the image of $g^*\colon H^1(X, \mathbb{Q}) \to H^1(W, \mathbb{Q})$ lies in $H^1(W, \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$. Moreover, g^* is injective, because $g_! g^* = (\deg g) \cdot \mathrm{id} = 2\,\mathrm{id}$. (Here $g_!\colon H^1(W, \mathbb{Q}) \to H^1(X, \mathbb{Q})$ denotes the *Gysin map* which corresponds to the map $g_*\colon H_3(W, \mathbb{Q}) \to H_3(X, \mathbb{Q})$ under the Poincaré duality isomorphisms $H^1(W, \mathbb{Q}) \simeq H_3(W, \mathbb{Q})$, $H^1(X, \mathbb{Q}) \simeq H_3(X, \mathbb{Q})$.) Now, the map $f^*\colon H^1(Y, \mathbb{Q}) \to H^1(W, \mathbb{Q})$ is an isomorphism because $W \to Y$ is a composition of blowups. So $H^1(W, \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}} \simeq H^1(Y, \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$. Finally $H^1(Y, \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}} = 0$ because the involution i acts by multiplication by (-1) on $H^1(Y, \mathbb{Q})$. We deduce that $H^1(X, \mathbb{Q}) = 0$, as required.

12.2 Topological and analytic invariants of K3 surfaces

Let X be a K3 surface. We first compute the Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q} = h^q(\Omega^p_X)$ of X.

If X is a complex smooth projective variety (or Kähler manifold) of dimension n, the Hodge numbers $h^{p,q}$ of X are usually displayed in the so called *Hodge diamond* — that is, we arrange the numbers $h^{p,q}$ in the xy-plane with $h^{p,q}$ at the point with coordinates $(p-q, p+q)$. (Note that the possible types (p, q) are given by $0 \leq p, q \leq n$, so we do get a diamond.) Then the kth row corresponds to the decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$ of the kth cohomology group. Now $H^{q,p} = \overline{H^{p,q}}$ and also $H^{n-p,n-q} \simeq (H^{p,q})^*$ by Poincaré duality (or Serre duality). So the diamond is symmetric under reflection in the vertical line $x = p - q = 0$ and reflection in the horizontal line $y = p+q = n$. (The first symmetry is given by complex conjugation, the second by the composition of the Poincaré duality isomorphism and complex conjugation.)

For a K3 surface X, we have $h^1(\mathcal{O}_X) = 0$ and $h^2(\mathcal{O}_X) = h^0(\omega_X) = 1$ because $\omega_X \simeq \mathcal{O}_X$. Now Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X))$$

gives $e(X) = 12\chi(\mathcal{O}_X) = 24$. So the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proposition 12.3. *Let X be a K3 surface. Then $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$ and $(H^2(X, \mathbb{Z}), \cup) \simeq L := H^3 \oplus (-E_8)^2$*

Proof. We first show that there is no torsion in the integral cohomology of X. For X a compact smooth 4-manifold, H^1 is torsion free and $\operatorname{Tors} H^3 \simeq \operatorname{Tors} H^2 \simeq \operatorname{Tors} H_1$ by Poincaré duality and the universal coefficient theorem. So, it's enough to show that H_1 is torsion free. Suppose not, and let $H_1(X, \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ be a surjection for some $n > 1$. The composition $\pi_1(X) \to H_1(X, \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ defines a covering space $p\colon Y \to X$ of degree n. (Explicitly, Y is the quotient of the universal covering $\tilde{X}$ of X by the kernel of the surjection $\pi_1(X) \to \mathbb{Z}/n\mathbb{Z}$). Then Y inherits the structure of a complex manifold from X such that $p\colon Y \to X$ is étale, that is, everywhere locally on Y the map p induces an isomorphism of complex manifolds. Now $e(Y) = n \cdot e(X) = 24n$ because $p\colon Y \to X$ is a covering space of degree n. Also $K_Y = p^*K_X = 0$ because p is étale, so $h^2(\mathcal{O}_Y) = h^0(\omega_Y) = 1$. Now

Noether's formula gives

$$2 - h^1(\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = \frac{1}{12}(K_Y^2 + e(Y)) = 2n.$$

But $n > 1$, so this is a contradiction. So $H_1(X, \mathbb{Z})$ is torsion-free, as required. (In fact, a K3 surface X is simply connected, but we do not need this here.) It follows that the integral cohomology of X is torsion free, and so by our previous calculation $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is free of rank 22.

It remains to determine the isomorphism type of the lattice $(H^2(X, \mathbb{Z}), \cup)$. (Here, a *lattice* is a free abelian group together with a symmetric $\mathbb{Z}$-valued bilinear form.) First, the cup product $\cup$ on $H^2(X, \mathbb{Z})$ is unimodular by Poincaré duality. Second, it has signature $(1 + 2h^{2,0}, h^{1,1} - 1) = (3, 19)$ by the Hodge index theorem. Third, it is even, that is $\alpha^2 := \alpha \cup \alpha$ is even for all $\alpha \in H^2(X, \mathbb{Z})$. Indeed, we already observed that, for X a compact complex surface and D a divisor on X, $D^2 - D \cdot K_X$ is even (by the Riemann–Roch formula for $\chi(\mathcal{O}_X(D))$). More generally, for $\alpha \in H^2(X, \mathbb{Z})$, $\alpha^2 - \alpha \cdot K_X$ is even. This follows from the Wu formula [MS, p. 132]. If now X is a K3 surface then $K_X = 0$, so α^2 is even for all $\alpha \in H^2(X, \mathbb{Z})$ as claimed. Finally, by the classification of indefinite unimodular lattices described in Sec. 1.6, it follows that $(H^2(X, \mathbb{Z}), \cup)$ is isomorphic to the lattice $L := H^3 \oplus (-E_8)^2$, the direct sum of 3 copies of the hyperbolic plane and 2 copies of the negative definite E_8 lattice. □

12.3 The Torelli theorem for K3 surfaces

In this section we give the statement of the Torelli theorem and surjectivity of the period mapping for K3 surfaces.

Let X be a K3 surface. Recall that the lattice $(H^2(X, \mathbb{Z}), \cup)$ is isomorphic to $L := H^3 \oplus (-E_8)^2$ and the summands in the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

have dimensions $1, 20, 1$.

The Hodge decomposition of $H^2(X, \mathbb{C})$ is determined by $H^{2,0} \subset H^2(X, \mathbb{C})$. Indeed, we have $H^{0,2} = \overline{H^{2,0}}$ and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$. (For a k-vector space V with a bilinear pairing $V \times V \to k$ and a subspace $W \subset V$, we define $W^{\perp} = \{v \in V \mid (v, w) = 0 \ \forall\ w \in W\}$.) To see the second equality, recall that the cup product on $H^2(X, \mathbb{C}) = H^2_{\mathrm{dR}}(X, \mathbb{C})$ is given by

$$([\omega], [\eta]) \mapsto \int_X \omega \wedge \eta$$

so $H^{p,q}$ is orthogonal to $H^{p',q'}$ unless $p + p' = q + q' = 2$. Thus $H^{1,1} \subseteq (H^{2,0} \oplus H^{0,2})^{\perp}$, and the dimensions agree, so we have equality.

Theorem 12.4. *(Torelli theorem for K3s)[BHPV, Cor. 11.2, p. 333] Suppose X, X' are K3 surfaces and $\phi\colon (H^2(X,\mathbb{Z}),\cup) \to (H^2(X',\mathbb{Z}),\cup)$ is an isomorphism of lattices such that $\phi_{\mathbb{C}}(H^{2,0}(X)) = H^{2,0}(X')$. Then X is isomorphic to X'.*

Let X be a K3 surface and Ω a nonzero holomorphic 2-form on X. So

$$H^{2,0}(X) = H^0(\omega_X) = \mathbb{C}\cdot[\Omega] \subset H^2_{\mathrm{dR}}(X,\mathbb{C}).$$

Then $[\Omega]^2 = 0$ and $[\Omega][\overline{\Omega}] = \int_X \Omega \wedge \bar{\Omega} > 0$.

Theorem 12.5. *(Surjectivity of the period mapping)[BHPV, Cor 14.2, p. 339] Let $v \in L_{\mathbb{C}}$ be a vector such that $v^2 = 0$ and $v \cdot \bar{v} > 0$. Then there exists a K3 surface X and an isomorphism $\phi\colon (H^2(X,\mathbb{Z}),\cup) \xrightarrow{\sim} L$ such that $\phi_{\mathbb{C}}(H^{2,0}(X)) = \mathbb{C}\cdot v$. (Note that X is uniquely determined by the Torelli theorem).*

12.4 Elliptic fibrations of K3 surfaces

Recall the Riemann-Roch formula for a divisor D on a compact complex surface X

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D(D - K_X)$$

If now X is a K3 surface the formula simplifies considerably: we have $\chi(\mathcal{O}_X) = 2$ and $K_X = 0$, so

$$\chi(\mathcal{O}_X(D)) = 2 + \frac{1}{2}D^2$$

In particular, using Serre duality $h^2(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(K_X - D))$ and $K_X = 0$, we obtain the inequality

$$h^0(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(-D)) \geq 2 + \frac{1}{2}D^2.$$

Theorem 12.6. *Let X be a K3 surface and D a divisor on X*

(1) If $D^2 \geq -2$ then either $h^0(\mathcal{O}_X(D)) \neq 0$ or $h^0(\mathcal{O}_X(-D)) \neq 0$. (Note: if both are nonzero then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$.)

(2) If $C \subset X$ is an irreducible curve then $C^2 \geq -2$, with equality iff $C \simeq \mathbb{P}^1$. In this case C is called a (-2)-curve.

(3) Suppose $D^2 = 0$, $D \neq 0$, and $h^0(\mathcal{O}_X(D)) \neq 0$. Then D is linearly equivalent to a positive linear combination

$$aE + \sum m_i C_i$$

where E is a smooth curve of genus 1 such that $E^2 = 0$ and the C_i are (-2)-curves. The linear system $|E|$ is basepoint free and defines a morphism $\phi\colon X \to \mathbb{P}^1$ with general fibre a smooth curve of genus 1.

Proof. (1) follows immediately from the Riemann–Roch inequality derived above. To explain the note: $h^0(\mathcal{O}_X(D)) \neq 0$ iff D is linearly equivalent to an effective divisor D'. If X is projective, let H be a hyperplane section, then $H \cdot D = H \cdot D' \geq 0$, with equality iff $D' = 0$. So if $h^0(\mathcal{O}_X(D)) \neq 0$ and $h^0(\mathcal{O}_X(-D)) \neq 0$ then $D \sim 0$. More generally, if X is a Kähler manifold, the same argument works with H replaced by a Kähler class $\kappa \in H^2(X, \mathbb{R})$.

(2) Recall that

$$2p_a(C) - 2 = (K_X + C) \cdot C$$

and $p_a(C) \geq 0$ with equality iff $C \simeq \mathbb{P}^1$. Now $K_X = 0$ gives $2p_a(C) - 2 = C^2$ and the result follows.

(3) We first show that there exists a positive linear combination $\sum m_i C_i$ of (-2)-curves such that $D' := D - \sum m_i C_i$ is nef, $D'^2 = 0$, and $h^0(\mathcal{O}_X(D')) \neq 0$. (Recall that a divisor D on a variety X is *nef* if $D \cdot C \geq 0$ for every curve $C \subset X$.) If D is nef, we are done. So suppose $D \cdot C < 0$ for some curve C. Since $h^0(\mathcal{O}_X(D)) \neq 0$, we may assume D is effective (replacing D by a linearly equivalent divisor). Then, since $D \cdot C < 0$, the curve C is contained in the support of D and $C^2 < 0$. Hence by (2) $C^2 = -2$ and C is a (-2)-curve. We now consider the *Picard–Lefschetz reflection*

$$s_C\colon \operatorname{Pic} X \to \operatorname{Pic} X, \quad B \mapsto B + (B \cdot C)C$$

defined by the (-2)-curve C. This is an isomorphism of lattices for the intersection product on $\operatorname{Pic} X$. Indeed,

$$s_C(B)^2 = B^2 + 2(B \cdot C)(B \cdot C) + (B \cdot C)^2 C^2 = B^2$$

using $C^2 = -2$. We replace D by $D_1 := s_C(D)$. Then $D_1 = D - mC$ where $m = -(D \cdot C) > 0$, $D_1^2 = 0$, and $h^0(\mathcal{O}_X(D_1)) \neq 0$ by Lemma 12.7 below. This process cannot continue indefinitely because the degree of D with respect to a hyperplane section (or a Kähler class) decreases at each stage. So, after a finite number of steps we obtain $D' = D - \sum m_i C_i$ such that D' is nef, $D'^2 = 0$, and $h^0(\mathcal{O}_X(D')) \neq 0$, as required.

Let D be an effective divisor on a smooth K3 X such that $D^2 = 0$, $D \neq 0$ and D is nef. By the Riemann–Roch inequality, $h^0(\mathcal{O}_X(D)) \geq 2$. Consider the complete linear system $|D|$. Write $|D| = |M| + F$ where F is the fixed part and M is mobile (that is, $|M|$ does not have fixed components). Then $D^2 = D(M + F) = 0$ and $DM, DF \geq 0$, so $DM = DF = 0$. Also M is nef because it is mobile. So from $0 = DM = (M+F)M$ and $M^2, MF \geq 0$ we get $M^2 = MF = 0$, and finally $0 = DF = (M+F)F$ gives $F^2 = 0$. If $F \neq 0$ then $h^0(\mathcal{O}_X(F)) \geq 2$ by the Riemann–Roch inequality. This is a contradiction because F is fixed. So $F = 0$. If M_1, M_2 are two general elements of $|M|$ then M_1, M_2 do not contain common components, and $M_1 \cdot M_2 = M^2 = 0$, so M_1 and M_2 are disjoint. Thus $|D| = |M|$ is basepoint free.

Let $\phi\colon X \to \mathbb{P}^N$ be the morphism defined by $|D|$. The image of ϕ is a curve C because $D^2 = 0$ and $D \neq 0$. Indeed, $D = \phi^*H$ where H is a hyperplane in $\mathbb{P}^N$, so $\dim \phi(X) \leq 1$ because $D^2 = 0$ (and $\phi(X)$ is not a point because $D \neq 0$). The Stein factorisation of ϕ ([Hartshorne, III.11.5, p. 280]) gives $\phi = p \circ f$ where $f\colon X \to \tilde{C}$ is a morphism to a smooth curve $\tilde{C}$ with connected fibres and $p\colon \tilde{C} \to C$ is a finite morphism. We claim that $\tilde{C} \simeq \mathbb{P}^1$. Indeed, the map $f^*\colon \Gamma(\Omega_{\tilde{C}}) \to \Gamma(\Omega_X)$ is injective, and $\Gamma(\Omega_X) = H^{1,0}(X) = 0$ because X is a K3, so $\Gamma(\Omega_{\tilde{C}}) = 0$ and thus $\tilde{C} \simeq \mathbb{P}^1$. The morphism $\phi\colon X \to \mathbb{P}^N$ was defined by the complete linear system $|D|$. So also $p\colon \tilde{C} \to C \subset \mathbb{P}^N$ is defined by a complete linear system. But $\tilde{C} \simeq \mathbb{P}^1$, so $p\colon \tilde{C} \to C$ is an isomorphism. Finally, $D = \phi^*H \sim f^*(NP)$, where $P \in C \simeq \mathbb{P}^1$ is any point, so $D \sim NE$ where E is any fibre of f. A general fibre E of f is smooth by Bertini's theorem, and $E^2 = 0$ because E is a fibre of a morphism from a surface to a curve. Now the genus formula $2g(E) - 2 = (K_X + E)E$ and $K_X = 0$ gives $g(E) = 1$. □

Lemma 12.7. *Let X be a K3 surface, $C \subset X$ a (-2)-curve and*

$$s_C\colon \operatorname{Pic} X \to \operatorname{Pic} X, \quad D \mapsto D + (D \cdot C)C$$

the associated Picard–Lefschetz reflection. If D is an effective divisor on X such that $D^2 \geq 0$, then $s_C(D)$ is linearly equivalent to an effective divisor.

Proof. If $D \cdot C \geq 0$ the result is clear. So suppose $D \cdot C < 0$. By Thm. 12.6(1) either $s_C(D)$ or $-s_C(D)$ is linearly equivalent to an effective divisor. Suppose (for a contradiction) that $-s_C(D)$ is linearly equivalent to an effective divisor B. Then

$$B + D \sim -s_C(D) + D = -(D \cdot C)C = mC$$

where $m > 0$. Now since $C^2 < 0$ it follows that $B + D = mC$. Thus $D = kC$, some $0 < k < m$. This contradicts $D^2 \geq 0$. □

Let $f\colon X \to B$ be a morphism from a compact complex surface X to a smooth curve B. Then the general fibre F of f is a smooth curve. Let $F_i = f^{-1}P_i$, $i = 1, \dots, r$, be the singular fibres of f, and write $B^0 = B \setminus \{P_i\}$ and $X^0 = f^{-1}B^0 = X \setminus \bigcup F_i$. Then

$$e(X) = e(X^0) + \sum e(F_i) = e(B^0)e(F) + \sum e(F_i)$$

$$= e(B)e(F) + \sum (e(F_i) - e(F)).$$

Indeed $e(X^0) = e(B^0)e(F)$ because $X^0 \to B^0$ is a topological fibration, and $e(B^0) = e(B) - r$. To see the first equality, we use the formula

$$e(K \cup L) = e(K) + e(L) - e(K \cap L)$$

given by the Mayer–Vietoris sequence as follows. Let $P_i \in D_i \subset B$ be a small open disc around P_i, $N_i = f^{-1}D_i$, and $Y = X \setminus \bigcup N_i$. Then $X = Y \cup \bigcup \overline{N_i}$, and $Y \subset X^0$ and $F_i \subset \overline{N_i}$ are homotopy equivalences. Also $Y \cap \overline{N_i} = \partial N_i$, the boundary of N_i, and $\partial N_i \to \partial D_i = S^1$ is a fibration. So $e(\partial N_i) = e(F)e(S^1) = 0$ and

$$e(X) = e(Y) + \sum e(\overline{N_i}) - \sum e(\partial N_i) = e(X^0) + \sum e(F_i).$$

Now suppose that $f\colon X \to B$ is an *elliptic fibration*, that is, the general fibre F is a smooth curve of genus 1. Then $e(F) = 0$, so the above formula gives

$$e(X) = \sum e(F_i).$$

Finally, suppose in addition that X is a K3 surface and each singular fibre F_i is a rational nodal curve (this is the generic situation). Here by a *rational nodal curve* we mean a curve obtained from $\mathbb{P}^1$ by glueing two points to form a node. Then $e(X) = 24$ and $e(F_i) = 1$ for each i, so the morphism f has 24 singular fibres.

13 Overview of the classification of surfaces

Let X be a smooth projective surface over $k = \mathbb{C}$.

Theorem 13.1. *If K_X is not nef then there exists an "extremal" curve C with $K_X \cdot C < 0$ and a morphism $\phi\colon X \to Y$ to a normal projective variety Y such that ϕ has connected fibres and a curve $\Gamma \subset X$ is contracted by ϕ iff Γ is numerically equivalent to a rational multiple of C. The morphism ϕ can be described explicitly as follows.*

(1) $C^2 < 0$. *Y is a smooth surface and $\phi\colon X \to Y$ is the blowup of a point $P \in Y$ with exceptional curve C.*

(2) $C^2 = 0$. *Y is a curve and $\phi\colon X \to Y$ is a ruled surface.*

(3) $C^2 > 0$. *Y is a point and $X \simeq \mathbb{P}^2$.*

Proof. (Sketch) The idea of the proof of the first part is the following. If $\phi\colon X \to Y$ is a morphism and H is an ample divisor on Y, then for each curve $\Gamma \subset X$ we have $\phi^*H \cdot \Gamma \geq 0$ with equality iff Γ is contracted by ϕ. Consider the cone of curves

$$\mathcal{C} := \{\sum a_i[C_i] \mid C_i \subset X \text{ curves},\ a_i \in \mathbb{R}_{\geq 0}\} \subset (\operatorname{Num} X)_{\mathbb{R}}.$$

This is a convex cone in the finite dimensional $\mathbb{R}$-vector space $(\operatorname{Num} X)_{\mathbb{R}}$. If $C \subset X$ and $\phi\colon X \to Y$ are as in the statement, write $D := \phi^*H$, then $D \cdot \alpha \geq 0$ for all $\alpha \in \mathcal{C}$, with equality iff $\alpha = \lambda[C]$, some $\lambda \in \mathbb{R}_{\geq 0}$. Geometrically, the hyperplane $(D \cdot \alpha = 0) \subset (\operatorname{Num} X)_{\mathbb{R}}$ intersects the cone $\mathcal{C}$ in the ray $\mathbb{R}_{\geq 0}[C]$, and $\mathcal{C}$ is contained in the halfspace $(D \cdot \alpha \geq 0)$. Conversely, one shows that, given such a D, for some $n > 0$ the linear system $|nD|$ is basepoint free and defines a morphism $\phi\colon X \to Y$ with the desired properties. (We use the assumption $K_X \cdot C < 0$ here.) To prove the existence of C and D, we analyse the structure of the cone $\mathcal{C}$. One shows that, in the half space $(K_X \cdot \alpha < 0)$, the cone is locally polyhedral, that is, the convex hull of finitely many rays. Then we can take C to be a generator of one of the extremal rays of the cone, and D a divisor defining a hyperplane intersecting $\mathcal{C}$ in this ray. For more details see [Reid, Ch. D].

Now suppose given C and $\phi\colon X \to Y$ as in the statement. We derive the explicit description of the possibilities for ϕ. If $\dim Y = 2$ then ϕ is birational so $C^2 < 0$ by negativity of the contracted locus. Now $K_X \cdot C < 0$ implies C is a (-1)-curve by Cor. 7.12 and ϕ is the blowup of a point on a smooth surface by Castelnuovo's contractibility criterion (Thm. 7.13). If $\dim Y = 1$ then $\phi\colon X \to Y$ has irreducible fibres (since every contracted curve is numerically equivalent to a multiple of C). Thus C is a fibre of ϕ and so $C^2 = 0$. Now $K_X \cdot C < 0$ implies that $\phi\colon X \to Y$ is a ruled surface by Thm. 10.1. If Y is a point then every curve is contracted by ϕ, so $\rho(X) = 1$, $C^2 > 0$, and $-K_X$ is ample (since $K_X \cdot C < 0$). Hence $X \simeq \mathbb{P}^2$ by Thm. 9.1. □

Corollary 13.2. *Let X be a smooth projective surface over $k = \mathbb{C}$. There exists a finite sequence of contractions of (-1)-curves*

$$X = X_0 \to X_1 \to \cdots \to X_n = Y$$

such that one of the following holds.

(1) K_Y *is nef.*

(2) Y *is a ruled surface.*

(3) $Y \simeq \mathbb{P}^2$.

If K_Y is nef we say Y is a *minimal model.*

Proof. This follows from the theorem by induction. Note that if $\phi\colon X \to X'$ is the contraction of a (-1)-curve (a blowup) then $\rho(X') = \rho(X) - 1$. So the process must stop. □

Proposition 13.3. *(Uniqueness of minimal models) Suppose* Y_1, Y_2 *are smooth projective surfaces with* K_{Y_1}, K_{Y_2} *nef, and* $f\colon Y_1 \dashrightarrow Y_2$ *is a birational map. Then* f *is an isomorphism.*

Proof. See Example Sheet 2, Q4. □

We now describe the classification of surfaces with K_X nef. Recall that we use the symbol '$\equiv$' to denote numerical equivalence of divisors.

Theorem 13.4. *Let* X *be a smooth projective surface over* $k = \mathbb{C}$ *such that* K_X *is nef. Then we have the following cases.*

(1) $K_X \equiv 0$. *If* $K_X = 0$ *then* X *is a K3 surface or an abelian surface (a projective complex torus of dimension* 2*). In general,* $mK_X = 0$ *for some* $m \in \{1, 2, 3, 4, 6\}$*, and* X *is a quotient of a K3 or abelian surface by a free action of* $\mathbb{Z}/m\mathbb{Z}$.

(2) $K_X^2 = 0, K_X \not\equiv 0$. *For some* $n > 0$ *the linear system* $|nK_X|$ *is basepoint free and defines an elliptic fibration* $\phi\colon X \to C$.

(3) $K_X^2 > 0$. *For some* $n > 0$ *the linear system* $|nK_X|$ *is basepoint free and defines a birational morphism* $\phi\colon X \to \overline{X}$.

In case (3) we say X is a *surface of general type.* For example, if $X \subset \mathbb{P}^N$ is a complete intersection, with defining equations of degrees $d_1, \ldots, d_{N-2}$, then $K_X = (\sum d_i - (N+1))H$ where H is a hyperplane section. So X is of general type iff $\sum d_i > N + 1$.

14 Godeaux surfaces

We describe an example of a surface X of general type such that $h^0(K_X) = 0$.

Let $Y = Y_5 \subset \mathbb{P}^3$ be a smooth quintic surface. Consider the action of $G = \mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^3$ given by

$$(X_0 : X_1 : X_2 : X_3) \mapsto (\zeta^0 X_0 : \zeta^1 X_1 : \zeta^2 X_2 : \zeta^3 X_3)$$

where ζ is a 5th root of unity. Assume that Y is preserved by the group action. Assume also that Y does not contain any of the fixed points $(1 : 0 : 0 : 0), \ldots, (0 : 0 : 0 : 1)$. For example, we can take

$$Y = (X_0^5 + X_1^5 + X_2^5 + X_3^5 = 0) \subset \mathbb{P}^3.$$

Then G acts freely on Y, so $X := Y/G$ is smooth and the quotient map $p\colon Y \to X$ is étale. The hypersurface Y is simply connected by the Lefschetz hyperplane theorem, and $p\colon Y \to X$ is a covering map. So Y is the universal cover of X, and $\pi_1(X) = G = \mathbb{Z}/5\mathbb{Z}$. By the adjunction formula

$$K_Y = K_{\mathbb{P}^3} + Y|_Y = -4H + 5H = H$$

where H is a hyperplane section of Y. So K_Y is ample, and $K_Y = p^*K_X$ because p is étale. It follows that K_X is ample. (More generally, if $f\colon X \to Y$ is finite and Y is proper, then a line bundle L on Y is ample iff f^*L is ample. This follows from the Nakai–Moishezon criterion for ampleness [Hartshorne, V.1.10, A.5.1].) Finally, we compute $h^0(K_X) = 0$. We have $\Gamma(\omega_X) = \Gamma(\omega_Y)^G$ because $X = Y/G$ with G acting freely on Y. Also, as noted above, $K_Y = H$, so

$$\Gamma(\omega_Y) \simeq \Gamma(\mathcal{O}_Y(1)) = \Gamma(\mathcal{O}_{\mathbb{P}^3}(1)) = \langle X_0, \ldots, X_3 \rangle_k$$

(To get the equality $\Gamma(\mathcal{O}_Y(1)) = \Gamma(\mathcal{O}_{\mathbb{P}^3}(1))$, we tensor the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-5) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0$$

by $\mathcal{O}_{\mathbb{P}^3}(1)$ to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_Y(1) \to 0.$$

Now the long exact sequence of cohomology together with $H^i(\mathcal{O}_{\mathbb{P}^3}(-4)) = 0$ for $i = 0, 1$ shows that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_Y(1))$ is an isomorphism.) We need to describe the G action on $\Gamma(\omega_Y)$ explicitly to compute the invariants. Consider the affine piece $U_0 = (X_0 \neq 0)$ of

Y and write $x_i = X_i/X_0$ for $i = 1, 2, 3$ and $f = F/X_0^5$, the equation of $U_0 \subset \mathbb{A}^3_{x_1,x_2,x_3}$. Then

$$\omega = \frac{dx_1 \wedge dx_2}{\frac{\partial f}{\partial x_3}} = -\frac{dx_1 \wedge dx_3}{\frac{\partial f}{\partial x_2}} = \frac{dx_2 \wedge dx_3}{\frac{\partial f}{\partial x_1}}$$

is a regular nowhere zero 2-form on U_0. Indeed, on U_0 we have

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$$

and using this gives the equalities above in the definition of ω. Now, since U_0 is smooth, at each point $P \in U_0$ we have $\frac{\partial f}{\partial x_i} \neq 0$ for some i, and then the x_j, $j \neq i$, define local coordinates at P. It follows that ω is regular and nonzero at P. Now consider ω as a rational 2-form on Y. We find $(\omega) = (X_0 = 0)$, and so

$$\Gamma(\omega_Y) = \langle \omega, x_1\omega, x_2\omega, x_3\omega \rangle_k.$$

Finally, we compute the G-action. We have $x_i \mapsto \zeta^i x_i$, $f \mapsto f$, so $\frac{\partial f}{\partial x_i} \mapsto \zeta^{-i}\frac{\partial f}{\partial x_i}$ and $\omega \mapsto \zeta\omega$. We deduce that $\Gamma(\omega_X) = \Gamma(\omega_Y)^G = 0$ as required.

Remark 14.1. A *Godeaux surface* is a surface of general type such that $h^0(K_X) = h^1(\mathcal{O}_X) = 0$ and $K_X^2 = 1$. The first example of such a surface is the one described above. See [BHPV, VII.10] for more details.

15 The Hopf surface

We describe a compact complex surface X such that the Hodge decomposition does not hold.

Consider the action of $G = \mathbb{Z}$ on $Y = \mathbb{C}^2 \setminus \{0\}$ given by

$$z = (z_1, z_2) \mapsto (\frac{1}{2}z_1, \frac{1}{2}z_2).$$

This is a free and properly discontinuous action. That is, for all $P \in Y$ there exists a (Euclidean) neighbourhood $P \in U \subset Y$ such that $gU \cap U = \emptyset$ for $g \in G \setminus \{e\}$. Hence the quotient $X := Y/G$ is a complex manifold. Topologically, we have diffeomorphisms

$$\mathbb{C}^2 - \{0\} = \mathbb{R}^4 - \{0\} \simeq S^3 \times \mathbb{R}_{>0} \simeq S^3 \times \mathbb{R}$$

$$z \mapsto (z/\|z\|, \|z\|) \mapsto (z/\|z\|, \log \|z\|)$$

Taking the quotient we obtain a diffeomorphism

$$X = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z} \simeq S^3 \times (\mathbb{R}/\mathbb{Z} \cdot \log 2) \simeq S^3 \times S^1$$

In particular X is compact, and has Betti numbers $1, 1, 0, 1, 1$ by the Künneth formula. Now we see that the Hodge decomposition $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, $H^{0,1} = \overline{H^{1,0}}$ does not hold because $b_1(X) = 1$ is odd. So X is not a Kähler manifold.

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References

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact complex surfaces, 2nd ed.

[Beauville] A. Beauville, Complex algebraic surfaces.

[CMSP] J. Carlson, S. Müller-Stach, C. Peters, Period mappings and period domains.

[Fulton] W. Fulton, Algebraic topology.

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry.

[Hartshorne] R. Hartshorne, Algebraic geometry.

[Hatcher] A. Hatcher, Algebraic topology, available at www.math.cornell.edu/∼hatcher/AT/ATpage.html

[KM] J. Kollár, S. Mori, Birational geometry of algebraic varieties.

[Matsumura] H. Matsumura, Commutative ring theory.

[MS] J. Milnor, J. Stasheff, Characteristic classes.

[Reid] M. Reid, Chapters on algebraic surfaces, available at arxiv.org

[Serre55] J-P. Serre, Faisceaux algébriques cohérents, available at www.jstor.org

[Serre56] J-P. Serre, Géométrie algébrique et géométrie analytique, available at www.numdam.org

[Serre70] J-P. Serre, A course in arithmetic.

[Shafarevich] I. Shafarevich, Basic algebraic geometry 1.

[Voisin] C. Voisin, Hodge theory and complex algebraic geometry I.