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# Curves, K3 Surfaces and Fano 3-folds of Genus $\leq 10$

Shigeru MUKAI\*

A pair  $(S, L)$  of a K3 surface  $S$  and a pseudo-ample line bundle  $L$  on  $S$  with  $(L^2) = 2g - 2$  is called a (polarized) K3 surface of genus  $g$ . Over the complex number field, the moduli space  $\mathcal{F}_g$  of those  $(S, L)$ 's is irreducible by the Torelli type theorem for K3 surfaces [12]. If  $L$  is very ample, the image  $S_{2g-2}$  of  $\Phi_{|L|}$  is a surface of degree  $2g - 2$  in  $\mathbf{P}^g$  and called the projective model of  $(S, L)$ , [13]. If  $g = 3, 4, 5$  and  $(S, L)$  is general, then the projective model is a complete intersection of  $g - 2$  hypersurfaces in  $\mathbf{P}^g$ . This fact enables us to give an explicit description of the birational type of  $\mathcal{F}_g$  for  $g \leq 5$ . But the projective model is no more complete intersection in  $\mathbf{P}^g$  when  $g \geq 6$ . In this article, we shall show that a general K3 surface of genus  $6 \leq g \leq 10$  is still a complete intersection in a certain homogeneous space and apply this to the description of birational type of  $\mathcal{F}_g$  for  $g \leq 10$  and the study of curves and Fano 3-folds. The homogeneous space  $X$  is the quotient of a simply connected semi-simple complex Lie group  $G$  by a maximal parabolic subgroup  $P$ . For the positive generator  $\mathcal{O}_X(1)$  of  $\text{Pic} X \cong \mathbf{Z}$ , the natural map  $X \rightarrow \mathbf{P}(H^0(X, \mathcal{O}_X(1)))$  is a  $G$ -equivariant embedding and the image coincides with the  $G$ -orbit  $G \cdot \bar{v}$ , where  $v$  is a highest weight vector of the irreducible representation  $H^0(X, \mathcal{O}_X(1))^\vee$  of  $G$ . For each  $6 \leq g \leq 10$ ,  $G$  and the representation  $U = H^0(X, \mathcal{O}_X(1))$  are given as follows:

$g$	6	7	8	9	10
$G$	SL(5)	Spin(10)	SL(6)	Sp(3)	exceptional of type $G_2$
dim $G$	24	45	35	21	14
(0.1) $U$	$\wedge^2 V^5$	half spinor representation	$\wedge^2 V^6$	$\wedge^3 V^6 / \sigma \wedge V^6$	adjoint representation
dim $U$	10	16	15	14	14
dim $X$	6	10	8	6	5

where  $V^i$  denotes an  $i$ -dimensional vector space and  $\sigma \in \wedge^2 V^6$  is a non-degenerate 2-vector of  $V^6$ .

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In the case  $7 \leq g \leq 10$ ,  $\dim U$  is equal to  $g + n - 1$ ,  $n = \dim X$ .  $X$  is of degree  $2g - 2$  in  $\mathbf{P}(U) \cong \mathbf{P}^{g+n-2}$  and the anticanonical (or 1st Chern) class of  $X$  is  $n - 2$  times hyperplane section (cf. (1.5)). Hence a smooth complete intersection of  $X = X_{2g-2}$  and  $n - 2$  hyperplanes is a K3 surface of genus  $g$ . (This has been known classically in the case  $g = 8$  and is first observed by C. Borcea [1] in the case  $g = 10$ .)

**Theorem 0.2.** *If two K3 surfaces  $S$  and  $S'$  are intersections of  $X_{2g-2}$  ( $7 \leq g \leq 10$ ) and  $g$ -dimensional linear subspaces  $P$  and  $P'$ , respectively, and if  $S \subset P$  and  $S' \subset P'$  are projectively equivalent, then  $P$  and  $P'$  are equivalent under the action of  $\bar{G}$  on  $\mathbf{P}(U)$ , where  $\bar{G}$  is the quotient of  $G$  by its center.*

By the theorem there exists a nonempty open subset  $\Xi$  of the Grassmann variety  $G(n - 2, U)$  of  $n - 2$  dimensional subspaces of  $U$  such that the natural morphism  $\Xi/\bar{G} \rightarrow \mathcal{F}_g$  is injective. For each  $7 \leq g \leq 10$ , it is easily checked that  $\dim \Xi/\bar{G} = 19 = \dim \mathcal{F}_g$ . Hence the morphism is birational.

**Corollary 0.3.** *The generic K3 surface of genus  $7 \leq g \leq 10$  is a complete intersection of  $X_{2g-2} \subset \mathbf{P}(U)$  and a  $g$ -dimensional linear subspace in a unique way up to the action  $\bar{G}$  on  $\mathbf{P}(U)$ . In particular, the moduli space  $\mathcal{F}_g$  is birationally equivalent to the orbit space  $G(n - 2, U)/\bar{G}$ .*

In the case  $g = 6$ , the generic K3 surface is a complete intersection of  $X$ , a linear subspace of dimension 6 and a quadratic hypersurface in  $\mathbf{P}(U) \cong \mathbf{P}^9$ . We have a similar result on the uniqueness of this expression of the K3 surface (see (4.1)). In the proof of these results, special vector bundles, instead of line bundles in the case  $g \leq 5$ , play an essential role. For instance, the generic K3 surface  $(S, L)$  of genus 10 has a unique (up to isomorphism) stable rank two vector bundle with  $c_1(E) = c_1(L)$  and  $c_2(E) = 6$  on it and the embedding of  $S$  into  $X = G/P$  is uniquely determined by this vector bundle  $E$ .

The following is the table of the birational type of  $\mathcal{F}_g$  for  $g \leq 10$ :

(0.4)

genus	2	3	4
birational type	$\mathbf{P}(S^6U^3)/\mathrm{PGL}(3)$	$\mathbf{P}(S^4U^4)/\mathrm{PGL}(4)$	$\mathbf{P}(U^{30})/\mathrm{SO}(5)$
5	6	7	
$G(3, S^2U^6)/\mathrm{PGL}(6)$	$(U^{13} \oplus U^9)/\mathrm{PGL}(2)$	$G(8, U^{16})/\mathrm{PSO}(10)$	
8	9	10	
$G(6, \wedge^2V^6)/\mathrm{PGL}(6)$	$G(4, U^{14})/\mathrm{PSp}(3)$	$G(3, \mathfrak{g})/\bar{G}_2$	

to  $g + n - 1$ ,  $n = \dim X$ .  $X$  is of anticanonical (or 1st Chern) class (1.5)). Hence a smooth complete plane is a K3 surface of genus  $g$ .  $g = 8$  and is first observed by C.

' are intersections of  $X_{2g-2}$  ( $7 \leq$  and  $P'$ , respectively, and if  $S \subset P$   $P$  and  $P'$  are equivalent under the  $t$  of  $G$  by its center.

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l type of  $\mathcal{F}_g$  for  $g \leq 10$ :

$$\begin{array}{c} \frac{3}{(U^4)/\mathrm{PGL}(4)} \quad \frac{4}{\mathrm{P}(U^{30})/\mathrm{SO}(5)} \\ \frac{7}{\mathrm{L}(2) \quad G(8, U^{16})/\mathrm{PSO}(10)} \\ \frac{10}{(3) \quad G(3, g)/\bar{G}_2} \end{array}$$

where  $U^d$  is a  $d$ -dimensional irreducible representation of the universal covering group.

**Corollary 0.5.**  $\mathcal{F}_g$  is unirational for every  $g \leq 10$ .

By [5], there exists a Fano 3-fold  $V$  with the property  $\mathrm{Pic} V \cong \mathbf{Z}(-K_V)$  and  $(-K_V)^3 = 22$ . The moduli space of these Fano 3-folds are unirational by their description in [5]. The generic K3 surface of genus 12 is an anticanonical divisor of  $V$  and hence  $\mathcal{F}_{12}$  is also unirational.

**Problem 0.6.** Describe the birational types, e.g., the Kodaira dimensions, of the 19-dimensional varieties  $\mathcal{F}_g$  for  $g \gg 0$ . Are they of general type?

If  $(S, L)$  is a K3 surface of genus  $g$ , then every smooth member of  $|L|$  is a curve of genus  $g$ . Conversely if  $C$  is a smooth curve of genus  $g \geq 2$  on a K3 surface, then  $\mathcal{O}_S(C)$  is pseudo-ample and  $(S, \mathcal{O}_S(C))$  is a K3 surface of the same genus as  $C$ . In the case  $g \leq 9$ , the generic curve lies on a K3 surface, that is, the natural rational map

$$\phi_g : \mathcal{P}_g = \bigcup_{(S, L) \in \mathcal{F}_g} |L| \dashrightarrow \mathcal{M}_g = (\text{the moduli space of curves of genus } g)$$

is generically surjective (§6). The inequality  $\dim \mathcal{M}_g \leq \dim \mathcal{P}_g = 19 + g$  holds if and only if  $g \leq 11$  and  $\psi_{11}$  is generically surjective ([10]). But in spite of  $\dim \mathcal{M}_{10} = 27 < \dim \mathcal{P}_{10} = 29$ , we have

**Theorem 0.7.** The generic curve of genus 10 cannot lie on a K3 surface.

*Proof.* Let  $\mathcal{F}'_{10}$  (resp.  $\mathcal{M}'_{10}$ ) be the subset of  $\mathcal{F}_{10}$  (resp.  $\mathcal{M}_{10}$ ) consisting of K3 surfaces (resp. curves) of genus 10 obtained as a complete intersection in the homogeneous space  $X_{18}^5 \subset \mathbf{P}(g)$ .  $\mathcal{M}'_{10}$  has a dominant morphism from a Zariski open subset  $U$  of  $G(4, g)/\bar{G}$ . Since the automorphism of a curve of genus  $\geq 2$  is finite, the stabilizer group is finite for every 4-dimensional subspace of  $g$  which gives a smooth curve of genus 10. Hence we have  $\dim \mathcal{M}'_{10} \leq \dim U = \dim G(4, g) - \dim G = 26 < \dim \mathcal{M}_{10}$ . On the other hand  $\mathcal{F}'_{10}$  contains a dense open subset of  $\mathcal{F}_{10}$  by Theorem 0.2. Hence the image of  $\psi_{10}$  is contained in the closure of  $\mathcal{M}'_{10} = \psi_{10}(\mathcal{F}'_{10})$  and  $\psi_{10}$  is not generically surjective. q.e.d.

**Remark 0.8.** Every curve of genus 10 has  $g_{12}^4$ , a 4-dimensional linear system of degree 12. If  $C$  is a general linear section of the homogeneous space  $X_{18}^5 \subset \mathbf{P}^{13}$ , then every  $g_{12}^4$  of  $C$  embeds  $C$  into a quadric hypersurface in  $\mathbf{P}^4$ . But if  $C$  is the generic curve of genus 10, then the image  $C_{12} \subset \mathbf{P}^4$  embedded by any  $g_{12}^4$  is not contained in any quadratic hypersurface. This fact gives an alternate proof of the theorem.

In the case  $7 \leq g \leq 10$ , a Fano 3-fold  $V_{2g-2} \subset \mathbf{P}^{g+1}$  is obtained as a complete intersection of the homogeneous space  $X_{2g-2}^n$  and a linear subspace of codimension  $n-3$  in  $\mathbf{P}(U) = \mathbf{P}^{n+g-2}$ . By the Lefschetz theorem, the Fano 3-fold  $V = V_{2g-2}$  has the property  $\text{Pic} V \cong \mathbf{Z}(-K_V)$ . The existence of such  $V$  has been known classically but was shown by totally different construction ([6]). Theorem 0.2 holds for Fano 3-folds, too.

**Theorem 0.9.** *Let  $V_{2g-2}$  and  $V'_{2g-2}$  ( $7 \leq g \leq 10$ ) be two Fano 3-folds which are complete intersections of the homogeneous space  $X_{2g-2}^n \subset \mathbf{P}^{n+g-2}$  and linear subspaces of codimension  $n-3$ . If  $V_{2g-2}$  and  $V'_{2g-2}$  are isomorphic to each other, then they are equivalent under the action of  $\bar{G}$ .*

We note that, by [1], the families of Fano 3-folds in the theorem is locally complete in the sense of [7].

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**Conventions.** Varieties and vector spaces are considered over the complex number field  $\mathbf{C}$ . For a vector space or a vector bundle  $E$ , its dual is denoted by  $E^\vee$ . For a vector space  $V$ ,  $G(r, V)$  (resp.  $G(V, r)$ ) is the Grassmann variety of  $r$ -dimensional subspaces (resp. quotient spaces) of  $V$ .  $G(1, V)$  and  $G(V, 1)$  are denoted by  $\mathbf{P}_*(V)$  and  $\mathbf{P}(V)$ , respectively.

## §1. Preliminary

We study some properties of the Cayley algebra  $\mathcal{C}$  over  $\mathbf{C}$ .  $\mathcal{C}$  is an algebra over  $\mathbf{C}$  with a unit 1 and generated by 7 elements  $e_i$ ,  $i \in \mathbf{Z}/7\mathbf{Z}$ . The multiplication is given by

$$(1.1) \quad \begin{aligned} e_i^2 &= -1 \text{ and } e_i e_{i+a} = -e_{i+a} e_i = e_{i+3a} \\ &\text{for every } i \in \mathbf{Z}/7\mathbf{Z} \text{ and } a = 1, 2, 4. \end{aligned}$$

The algebra  $\mathcal{C}$  is not associative but alternative, i.e.,  $x(xy) = x^2y$  and  $(xy)y = xy^2$  hold for every  $x, y \in \mathcal{C}$ . Let  $\mathcal{C}_0$  be the 7-dimensional subspace of  $\mathcal{C}$  generated by  $e_i$ ,  $i \in \mathbf{Z}/7\mathbf{Z}$  and  $U$  the subspace of  $\mathcal{C}_0$  spanned by  $\alpha = e_3 + \sqrt{-1}e_5$  and  $\beta = e_6 - \sqrt{-1}e_7$ . It is easily checked that  $\alpha^2 = \beta^2 = \alpha\beta = \beta\alpha = 0$ , i.e.,  $U$  is totally isotropic with respect to the multiplication of  $\mathcal{C}$ . Moreover,  $U$  is maximally totally isotropic with respect to the multiplication of  $\mathcal{C}$ , i.e., if  $xU = 0$  or  $Ux = 0$ , then  $x$  belongs to  $U$ . Let  $q$  be the quadratic form  $q(x) = x^2$  on  $\mathcal{C}_0$  and  $b$  the associated symmetric bilinear form.  $b(x, y)$  is equal to  $xy + yx$  for every  $x$  and  $y \in \mathcal{C}_0$ . Let  $V$  be the subspace of  $\mathcal{C}_0$  of vectors orthogonal to  $U$  with respect to  $q$  (or  $b$ ). Since  $U$  is totally isotropic with respect to  $q$ ,  $V$  contains  $U$  and the quotient  $V/U$  carries the quadratic form  $\bar{q}$ .

$V_{2g-2} \subset \mathbf{P}^{g+1}$  is obtained as a space  $X_{2g-2}^n$  and a linear subspace of the Lefschetz theorem, the Fano variety  $(-K_V)$ . The existence of such  $V$  is a totally different construction ([6]).

$\leq 10$ ) be two Fano 3-folds which are linearly isomorphic to each other,  $X_{2g-2}^n \subset \mathbf{P}^{n+g-2}$  and linearly isomorphic to each other,  $X_{2g-2}^n$ .

3-folds in the theorem is locally

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are considered over the complex vector bundle  $E$ , its dual is denoted by  $E^*$ .  $G(r)$  is the Grassmann variety of  $r$ -planes of  $V$ .  $G(1, V)$  and  $G(V, 1)$  are

algebra  $\mathcal{C}$  over  $\mathbf{C}$ .  $\mathcal{C}$  is an algebra with elements  $e_i$ ,  $i \in \mathbf{Z}/7\mathbf{Z}$ . The multipli-

$e_i e_j = e_{i+j}$  if  $i+j \in \mathbf{Z}/7\mathbf{Z}$ ,  $i, j \in \mathbf{Z}/7\mathbf{Z}$ .

native, i.e.,  $x(xy) = x^2y$  and the 7-dimensional subspace of  $\mathcal{C}$  spanned by  $\alpha = e_3 + \sqrt{-1}e_5$ ,  $\alpha^2 = \beta^2 = \alpha\beta = \beta\alpha = 0$ , i.e.,  $U$  is a maximal ideal of  $\mathcal{C}$ . Moreover,  $U$  is maximal with respect to the quadratic form  $q(x) = x^2$  on  $\mathcal{C}_0$  and  $U$  is equal to  $xy + yx$  for every  $x, y \in U$ .  $U$  is orthogonal to  $U$  with respect to  $q$ ,  $V$  contains  $U$  and the

**Lemma 1.2.**  $x'(xy) = b(x, y)x' - b(x', y)x + y(x'x)$  for every  $x, x'$  and  $y \in \mathcal{C}_0$ .

*Proof.* By the alternativity of  $\mathcal{C}$ , we have  $u(vw) + v(uw) = (uv + vu)w$ . Hence, if  $u$  and  $v$  belongs to  $\mathcal{C}_0$ , then we have  $u(vw) + v(uw) = b(u, v)w$ . So we have

$$\begin{aligned} x'(xy) &= x'(b(x, y) - yx) = b(x, y)x' - x'(yx) \\ &= b(x, y)x' - (b(x', y)x - y(x'x)) \\ &= b(x, y)x' - b(x', y)x + y(x'x). \end{aligned}$$

q.e.d.

If  $x \in U$  and  $y \in V$ , then  $U(xy) = 0$  by the above lemma and hence  $xy$  belongs to  $U$ . Hence the right multiplication homomorphism  $R(y)$ ,  $x \mapsto xy$ , by  $y \in V$  maps  $U$  into itself. Since  $R(x)$  is zero on  $U$  if and only if  $x \in U$ ,  $R$  gives an injective homomorphism  $\bar{R}: V/U \rightarrow \text{End}(U)$ .

**Proposition 1.3.** (1)  $\bar{R}(\bar{x})^2 = \bar{q}(\bar{x}) \cdot \text{id}$  for every  $\bar{x} \in V/U$ , and

(2)  $\bar{R}$  is an isomorphism onto  $sl(U)$ , the vector space consisting of trace zero endomorphisms of  $U$ .

*Proof.* (1) follows immediately from the alternativity of  $\mathcal{C}$ . It is easy to check the following fact: if  $r$  is an endomorphism of a 2-dimensional vector space and if  $r^2$  is a constant multiplication, then either  $r$  itself is a constant multiplication or the trace of  $r$  is equal to zero. Hence by (1),  $\bar{R}(\bar{x})$  is a constant multiplication or belongs to  $sl(U)$ , for every  $\bar{x} \in V/U$ . Therefore,  $\bar{R}(V/U)$  is contained in the 1-dimensional vector space consisting of constant multiplications of  $U$  or contained in the 3-dimensional vector space  $sl(U)$ . Since the quadratic form  $q$  is nondegenerate on  $V/U$ , the former is impossible and  $\bar{R}(V/U)$  coincides with  $sl(U)$ . q.e.d.

Let  $G$  be the automorphism group of the Cayley algebra  $\mathcal{C}$ . It is known that  $G$  is a simple algebraic group of type  $G_2$ . The automorphisms which map  $U$  onto itself form a maximal parabolic subgroup  $P$  of  $G$ . The subspace spanned by  $e_1, e_2$  and  $e_4$  (resp. by  $e_3 - \sqrt{-1}e_5$  and  $e_6 + \sqrt{-1}e_7$ ) can be identified with  $sl(U)$  (resp.  $U^\vee$ ) by  $\bar{R}$  (resp.  $b$ ).  $\mathcal{C}$  is isomorphic to  $\mathbf{C} \oplus U \oplus sl(U) \oplus U^\vee$  and if  $f \in GL(U)$ , then  $1 \oplus f \oplus \text{ad}(f) \oplus {}^t f$  is an automorphism of the Cayley algebra  $\mathcal{C}$ . Hence the maximal parabolic subgroup  $P$  contains  $GL(U)$  and  $X = G/P$  can be identified with the set of 2-dimensional subspaces of  $\mathcal{C}_0$  which are equivalent to  $U$  under the action of  $G = \text{Aut } \mathcal{C}$ .

Let  $\mathcal{U}$  be the maximally totally isotropic universal subbundle of  $\mathcal{C}_0 \otimes \mathcal{O}_X$ : the fibre  $\mathcal{U}_x \subset \mathcal{C}_0$  at  $x$  is the 2-dimensional subspace corresponding to  $x \in X$ . Let  $\mathcal{V}$  be the subsheaf of  $\mathcal{C}_0 \otimes \mathcal{O}_X$  consisting of the germs of sections which are orthogonal to  $\mathcal{U}$  with respect to the bilinear form  $b \otimes 1$  on  $\mathcal{C}_0 \otimes \mathcal{O}_X$ .  $\mathcal{V}$  is a rank 5 subbundle of  $\mathcal{C}_0 \otimes \mathcal{O}_X$  and contains  $\mathcal{U}$  as a subbundle. The quotient bundle

$(\mathcal{C}_0 \otimes \mathcal{O}_X)/\mathcal{V}$  is isomorphic to  $\mathcal{U}^\vee$  by  $b \otimes 1$  and  $\mathcal{V}/\mathcal{U}$  has a quadratic form  $\overline{q \otimes 1}$  induced by  $q \otimes 1$  on  $\mathcal{C}_0 \otimes \mathcal{O}_X$ . By Proposition 1.3, we have

**Proposition 1.4.** *The right multiplication induces an isomorphism  $\bar{R}$  from  $\mathcal{V}/\mathcal{U}$  onto the vector bundle  $sl(\mathcal{U})$  of trace zero endomorphisms of  $\mathcal{U}$  and  $\bar{R}(\bar{x})^2$  is equal to  $(\overline{q \otimes 1})(\bar{x}) \cdot \text{id}$  for every  $\bar{x} \in \mathcal{V}/\mathcal{U}$ .*

Next we shall compute the anticanonical class of  $X$  and the degree of  $\mathcal{O}_X(1)$ , the ample generator of  $\text{Pic} X$ , and show some vanishings of the cohomology groups of homogeneous vector bundles  $\mathcal{U}(i)$  and  $(S^2\mathcal{U})(i)$  etc.

Let  $G$  be a simply connected semi-simple algebraic group and  $P$  a maximal parabolic subgroup of  $G$ . Fixing a Borel subgroup  $B$  in  $P$ , the Lie algebra  $\mathfrak{g}$  of  $G$  is the direct sum of  $\mathfrak{b}$  and 1-dimensional eigenspaces  $\mathfrak{g}^\beta$ , where  $\beta$  runs over all negative roots. If we choose a suitable root basis  $\Delta$ , then there exists a positive root  $\alpha \in \Delta$  such that  $\mathfrak{p}$  is equal to the direct sum of  $\bigoplus \mathfrak{g}^\gamma$  and  $\mathfrak{b}$ , where  $\gamma$  runs over all positive roots which are linear combinations of the roots in  $\Delta \setminus \{\alpha\}$  with nonnegative coefficients. A positive root  $\beta$  is said to be *complementary* if  $\mathfrak{g}^\beta \cap \mathfrak{p} = 0$  or equivalently if  $\beta$  cannot be expressed as a linear combination of the roots in  $\Delta \setminus \{\alpha\}$  with nonnegative coefficients.

**Proposition 1.5. (Borel-Hirzebruch [2])** *Let  $G, P, \Delta$  and  $\alpha$  be as above and  $L$  the positive generator of  $\text{Pic}(G/P)$ . Then we have*

- (1) *the quotient  $\mathfrak{g}/\mathfrak{p}$  is isomorphic to  $\bigoplus_{\beta \in R_P} \mathfrak{g}^\beta$ , where  $R_P$  is the set of positive complementary roots. In particular,  $\dim(G/P)$  is equal to the cardinality  $n$  of  $R_P$ ,*
- (2)  *$(L^n) = n! \prod_{\beta \in R_P} \frac{(\beta, w)}{(\beta, \rho)}$ , where  $w$  is the fundamental weight corresponding to  $\alpha$  (or  $L$ ) and  $\rho$  is a half of the sum of all positive roots, and*
- (3) *the sum of all  $\beta \in R_P$  is  $r$  times  $\rho$  for some positive integer  $r$  and  $c_1(G/P)$  (or the anticanonical class of  $G/P$ ) is equal to  $r$  times  $c_1(L)$ .*

A homogeneous vector bundle on  $G/P$  is obtained from a representation of  $P$  and hence from that of reductive part  $G_0$  of  $P$ . Note that the weight spaces of  $G$  and  $G_0$  are naturally identified.

**Theorem 1.6. (Bott [3])** *Let  $E$  be a homogeneous vector bundle over  $G/P$  induced by an irreducible representation of the reductive part of  $P$ . Let  $\gamma$  be the highest weight of the representation and  $\rho$  a half of the sum of all positive roots of  $G$ . Then we have*

- (1) *if  $(\gamma + \rho, \beta) = 0$  for a positive root  $\beta$ , then  $H^i(G/P, E)$  vanishes for every  $i$ , and*
- (2) *let  $i_0$  be the number of positive roots  $\beta$  with  $(\gamma + \rho, \beta)$  negative ( $i_0$  is called the index of  $E$ ). Then  $H^i(G/P, E) = 0$  for all  $i$  except for  $i_0$  and  $H^{i_0}(G/P, E)$  is an irreducible  $G$ -module.*

$\mathcal{V}/\mathcal{U}$  has a quadratic form  $\overline{q \otimes 1}$   
1.3, we have

duces an isomorphism  $\bar{R}$  from  
endomorphisms of  $\mathcal{U}$  and  $\bar{R}(\bar{x})^2$

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e vanishings of the cohomology  
d  $(S^2\mathcal{U})(i)$  etc.

gebraic group and  $P$  a maximal  
oup  $B$  in  $P$ , the Lie algebra  $\mathfrak{g}$  of  
spaces  $\mathfrak{g}^\beta$ , where  $\beta$  runs over all  
is  $\Delta$ , then there exists a positive  
im of  $\bigoplus \mathfrak{g}^\gamma$  and  $\mathfrak{b}$ , where  $\gamma$  runs  
ations of the roots in  $\Delta \setminus \{\alpha\}$   
 $\beta$  is said to be *complementary* if  
essed as a linear combination of  
nts.

et  $G, P, \Delta$  and  $\alpha$  be as above and  
e have

,  $\mathfrak{g}^\beta$ , where  $R_P$  is the set of pos-  
dim( $G/P$ ) is equal to the cardi-

ndamental weight corresponding  
all positive roots, and

ve positive integer  $r$  and  $c_1(G/P)$   
ual to  $r$  times  $c_1(L)$ .

obtained from a representation of  
 $P$ . Note that the weight spaces

eneous vector bundle over  $G/P$   
reductive part of  $P$ . Let  $\gamma$  be the  
lf of the sum of all positive roots

and  $H^i(G/P, E)$  vanishes for every

with  $(\gamma + \rho, \beta)$  negative ( $i_0$  is  
) = 0 for all  $i$  except for  $i_0$  and

Returning to our first situation, our variety  $X$  is the quotient of the excep-  
tional Lie group  $G$  of type  $G_2$  by a maximal parabolic subgroup  $P$ . The root  
system  $G_2$  has two root basis  $\alpha_1$  and  $\alpha_2$  with different lengths and the root  $\alpha$   
corresponding to  $P$  in the above manner is the longer one, say  $\alpha_2$ . The line bun-  
dle  $L = \mathcal{O}_X(1)$  and the vector bundle  $\mathcal{U}^\vee(1)$  on  $X$  come from the representation  
with the highest weights  $w_1 = 3\alpha_1 + 2\alpha_2$  and  $w_2 = 2\alpha_1 + \alpha_2$ , respectively, which  
are the fundamental weights of  $G$ . Since  $\mathcal{U}$  is of rank 2 and  $\wedge^2 \mathcal{U} \cong \mathcal{O}_X(1)$ ,  $\mathcal{U}^\vee$   
is isomorphic to  $\mathcal{U}(1)$ .  $\rho$  is equal to  $w_1 + w_2$  and the inner products of  $\rho, w_1, w_2$   
and the 6 positive roots are as follows:

	$\alpha_1$	$3\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$	$3\alpha_1 + 2\alpha_2$	$\alpha_1 + \alpha_2$	$\alpha_2$
$\rho$	1	6	5	9	4	3
$w_1$	0	3	3	6	3	3
$w_2$	1	3	2	3	1	0

By (1.5),  $X$  has dimension 5,  $c_1(X) = 3c_1(L)$  and has degree

$$(L^5) = 5! \frac{3 \cdot 3 \cdot 6 \cdot 3 \cdot 3}{6 \cdot 5 \cdot 9 \cdot 4 \cdot 3} = 18$$

in  $\mathbf{P}^{13}$ . The homogeneous vector bundles  $(S^m \mathcal{U})(n)$  comes from the irreducible  
representation with the highest weight  $mw_1 + (n - m)w_2$ . Applying (1.6), we  
have

**Proposition 1.7.** *The cohomology groups of  $\mathcal{U}(n)$ ,  $(S^2 \mathcal{U})(n)$  and  $(S^3 \mathcal{U})(n)$  are  
zero except for the following cases:*

- (1)  $H^0(X, \mathcal{U}(n))$  for  $n \geq 1$ ,  $H^0(X, (S^2 \mathcal{U})(n))$  for  $n \geq 2$  and  $H^0(X, (S^3 \mathcal{U})(n))$   
for  $n \geq 3$ ,
- (2)  $H^1(X, (S^3 \mathcal{U})(1))$  and  $H^4(X, (S^3 \mathcal{U})(-1))$ , and
- (3)  $H^5(X, \mathcal{U}(n))$ ,  $H^5(X, (S^2 \mathcal{U})(n))$  and  $H^5(X, (S^3 \mathcal{U})(n))$  for  $n \leq -3$ .

Let  $S$  be a smooth K3 surface which is a complete intersection of 3 members  
of  $|\mathcal{O}_X(1)|$ . By using the Koszul complex

$$(1.8) \quad 0 \longrightarrow \mathcal{O}_X(-3) \longrightarrow \mathcal{O}_X(-2)^{\oplus 3} \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

we have

**Lemma 1.9.** *If  $E$  is a vector bundle on  $X$  and if  $H^{i+j}(X, E(-j)) = 0$  for  
every  $0 \leq j \leq 3$ , then  $H^i(S, E|_S) = 0$ .*

Since  $\mathcal{U}$  is of rank 2,  $sl(\mathcal{U})$  is isomorphic to  $S^2 \mathcal{U} \otimes (\det \mathcal{U})^{-1} \cong (S^2 \mathcal{U})(1)$ . By  
Proposition 1.7 and Lemma 1.9, we have



**Proposition 1.10.** *Let  $S$  be as above. Then  $H^i(S, sl(\mathcal{U})|_S)$  vanishes for every  $i$ ,  $H^1(S, (sl(\mathcal{U}))(n)|_S)$  vanishes for every  $n$  and  $\mathcal{U}|_S$  or  $(S^3\mathcal{U})(1)|_S$  has no nonzero global sections.*

## §2. Proof of Theorem 0.2 in the case $g = 10$

Let  $A$  be a 3-dimensional subspace of  $H^0(X, L)$  and  $S_A$  the intersection of  $X = X_{18}$  and the linear subspace  $\mathbf{P}(H^0(X, L)/A)$  of  $\mathbf{P}(H^0(X, L))$ . Let  $L_A$  and  $U_A$  be the restrictions of  $L$  and  $\mathcal{U}$  to  $S_A$ , respectively. Let  $\Xi$  be the subset of the Grassmann variety  $G(3, H^0(X, L))$  consisting of  $A$ 's such that  $S_A$  are smooth K3 surfaces and that the vector bundles  $U_A$  are stable with respect to the ample line bundles  $L_A$ .

**Proposition 2.1.**  $\Xi$  is a nonempty open subset of  $G(3, H^0(X, L))$ .

*Proof.*  $U_A$  is a rank 2 bundle and  $\det U_A \cong L_A^{-1}$ . By Moishezon's theorem [9],  $\text{Pic } S_A$  is generated by  $L_A$  if  $A$  is general. Since  $H^0(S_A, U_A) = 0$  by Proposition 1.10,  $U_A$  is stable if  $A$  is general. Since the stableness is an open condition [8], we have our proposition. q.e.d.

In this section we shall prove the following:

(2.2) *If two 3-dimensional subspaces  $A$  and  $B$  belong to  $\Xi$  and if the polarized K3 surfaces  $(S_A, L_A)$  and  $(S_B, L_B)$  are isomorphic to each other, then  $S_A$  and  $S_B$ , and hence  $A$  and  $B$ , are equivalent under the action of  $G$ .*

Let  $\varphi : S_A \xrightarrow{\sim} S_B$  be an isomorphism such that  $\varphi^*L_B \cong L_A$ .

*Step I.* There is an isomorphism  $\beta : U_A \xrightarrow{\sim} \varphi^*U_B$ .

*Proof.* Since  $c_1(U_A) = -c_1(L_A)$  and  $c_1(U_B) = -c_1(L_B)$ , the first Chern classes of  $U_A$  and  $\varphi^*U_B$  are same. Since  $(S_B, U_B)$  is a deformation of  $(S_A, U_A)$ ,  $U_B$  and  $U_A$  have the same second Chern number. Hence the two vector bundles  $\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^*U_B)$  and  $\mathcal{E}nd_{\mathcal{O}_S}(U_A)$  have the same first Chern class and the same second Chern number. Therefore, by the Riemann-Roch theorem and Proposition 1.10, we have

$$\begin{aligned} \chi(\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^*U_B)) &= \chi(\mathcal{E}nd_{\mathcal{O}_S}(U_A)) \\ &= \chi(\mathcal{O}_{S_A}) + \chi(sl(U_A)) = 2. \end{aligned}$$

By the Serre duality, we have

$$\begin{aligned} \dim \mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^*U_B) + \dim \mathcal{H}om_{\mathcal{O}_S}(\varphi^*U_B, U_A) \\ \geq \chi(\mathcal{H}om_{\mathcal{O}_S}(U_A, \varphi^*U_B)) = 2. \end{aligned}$$

Hence there is a nonzero homomorphism from  $U_A$  to  $\varphi^*U_B$  or vice versa. Since  $U_A$  and  $\varphi^*U_B$  are stable vector bundles and have the same slope, the nonzero homomorphism is an isomorphism. q.e.d.

$T^i(S, sl(\mathcal{U})|_S)$  vanishes for every  $i$  or  $(S^3\mathcal{U})(1)|_S$  has no nonzero

$g = 10$

$X, L$ ) and  $S_A$  the intersection of  $A$ ) of  $P(H^0(X, L))$ . Let  $L_A$  and  $\Xi$  be the subset of the  $A$ 's such that  $S_A$  are smooth stable with respect to the ample

of  $G(3, H^0(X, L))$ .

$L_A^{-1}$ . By Moishezon's theorem since  $H^0(S_A, U_A) = 0$  by Proposition 1.9, stability is an open condition *q.e.d.*

belong to  $\Xi$  and if the polarized  $\mathcal{U}$  is generic to each other, then  $S_A$  and  $S_B$  are in the action of  $G$ .

that  $\varphi^*L_B \cong L_A$ .

$\varphi^*U_B$ .

$c_1(L_B)$ , the first Chern class of  $(S_A, U_A)$ , is a deformation of  $(S_B, U_B)$ . Hence the two vector bundles have the same first Chern class and the same Riemann-Roch theorem and

$c_2(U_A)$

$) + \chi(sl(U_A)) = 2$ .

$\text{Hom}_{\mathcal{O}_S}(\varphi^*U_B, U_A)$

$\text{Hom}_{\mathcal{O}_S}(U_A, \varphi^*U_B) = 2$ .

$A$  to  $\varphi^*U_B$  or vice versa. Since  $A$  and  $B$  have the same slope, the nonzero *q.e.d.*

*Step II.* There is an isomorphism  $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$  (as  $\mathbb{C}$ -vector spaces) such that the following diagram is commutative:

$$\begin{array}{ccc} U_A & \xrightarrow{\beta} & \varphi^*U_B \\ \cap & & \cap \\ \mathcal{C}_0 \otimes \mathcal{O}_{S_A} & \xrightarrow{\gamma \otimes 1} & \mathcal{C}_0 \otimes \mathcal{O}_{S_A} = \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{S_B}) \end{array}$$

*Proof.* Let  $\gamma_0$  be the dual map of

$$\text{Hom}(\beta, \mathcal{O}_{S_A}) : \text{Hom}_{\mathcal{O}_S}(\varphi^*U_B, \mathcal{O}_{S_A}) \longrightarrow \text{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A}).$$

*Claim:* The inclusion  $U_A \subset \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$  induces an isomorphism  $\text{Hom}(\mathcal{C}_0, \mathbb{C}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A})$ .

Let  $\mathcal{K}$  be the dual of the quotient bundle  $(\mathcal{C}_0 \otimes \mathcal{O}_X)/\mathcal{U}$  on  $X$ . The natural map from  $\text{Hom}(\mathcal{C}_0, \mathbb{C})$  to  $\text{Hom}_{\mathcal{O}_X}(\mathcal{U}, \mathcal{O}_X)$  is an isomorphism because both are irreducible  $G$ -modules. Hence both  $H^0(X, \mathcal{K})$  and  $H^1(X, \mathcal{K})$  are zero. By the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_0^\vee \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{U}^\vee \longrightarrow 0$$

and Proposition 1.7, we have  $H^i(X, \mathcal{K}(-i)) = H^{i+1}(X, \mathcal{K}(-i)) = 0$  for  $i = 1, 2$  and 3. Hence by Lemma 1.9, both  $H^0(S, \mathcal{K}|_S)$  and  $H^1(S, \mathcal{K}|_S)$  are zero and we have our claim.

By the claim and by applying the claim to  $\varphi^*U_B \subset \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{S_B})$ , we have a homomorphism  $\gamma : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$  such that the following diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\gamma} & \mathcal{C}_0 \\ \downarrow \wr & & \downarrow \wr \\ \text{Hom}_{\mathcal{O}_S}(U_A, \mathcal{O}_{S_A})^\vee & \xrightarrow{\gamma_0} & \text{Hom}_{\mathcal{O}_S}(\varphi^*U_B, \mathcal{O}_{S_A})^\vee \end{array}$$

is commutative. Since  $\beta$  is an isomorphism,  $\gamma_0$  and  $\gamma$  are isomorphisms and  $\gamma$  enjoys our requirement. *q.e.d.*

*Step III.* There is an isomorphism  $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$  (as  $\mathbb{C}$ -vector spaces) such that  $(\gamma \otimes 1)(U_A) = \varphi^*U_B \subset \mathcal{C}_0 \otimes \mathcal{O}_X$  and  $x^2 = \gamma(x)^2$  for every  $x \in \mathcal{C}_0$ .

*Proof.* Take an isomorphism  $\gamma$  which satisfies the requirement of Step II. Put  $q(x) = x^2$  and  $q'(x) = \gamma(x)^2$ . Then  $q$  and  $q'$  are quadratic forms on  $\mathcal{C}_0$  and both  $q \otimes 1$  and  $q' \otimes 1$  are identically zero on  $U_A$ . Hence replacing  $\gamma$  by some multiple by a nonzero constant if necessary, we have our assertion by the following:

*Claim:* The quadratic forms  $Q$  on  $\mathcal{C}_0$  such that  $(Q \otimes 1)|_{U_A} = 0$  form at most one dimensional vector space.

Let  $\mathcal{N}$  be the kernel of the homomorphism  $S^2\alpha : S^2\mathcal{C}_0 \otimes \mathcal{O}_X \rightarrow S^2\mathcal{U}^\vee$ . Since  $S^2\mathcal{C}_0$  is a sum of two irreducible  $G$ -modules of dimension 1 and 27 and since  $H^0(S^2\alpha)$  is a homomorphism of  $G$ -modules, we have  $\dim H^0(X, \mathcal{N}) = \dim \text{Ker} H^0(S^2\alpha) = 1$ . By the exact sequence

$$H^{i-1}(X, S^2\mathcal{U}^\vee(-n)) \rightarrow H^i(X, \mathcal{N}(-n)) \rightarrow H^i(X, S^2\mathcal{C}_0 \otimes \mathcal{O}_X(-n))$$

and Proposition 1.7,  $H^i(X, \mathcal{N}(-i))$  is zero for every  $i = 1, 2$  and 3. Hence by the Koszul complex (1.8), the restriction map  $H^0(X, \mathcal{N}) \rightarrow H^0(S, \mathcal{N}|_S)$  is surjective and we have  $\dim H^0(S, \mathcal{N}|_S) \leq \dim H^0(X, \mathcal{N}) = 1$ , which shows our claim. q.e.d.

*Step IV.* There is an isomorphism  $\gamma : \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}_0$  such that  $(\gamma \otimes 1)(U_A) = \varphi^*U_B$ ,  $x^2 = \gamma(x)^2$  for every  $x \in \mathcal{C}_0$  and  $(\gamma \otimes 1)(xy) = ((\gamma \otimes 1)(x))((\gamma \otimes 1)(y))$  for every  $x \in U_A$  and  $y \in V_A$ .

*Proof.* Take an isomorphism  $\gamma$  which satisfies the requirements of Step III. Then  $\gamma \otimes 1$  maps  $V_A$  onto  $\varphi^*V_B \subset \mathcal{C}_0 \otimes \mathcal{O}_X$  and induces an isomorphism  $\Gamma : V_A/U_A \rightarrow \varphi^*(V_B/U_B)$  which is compatible with the quadratic forms on  $V_A/U_A$  and  $V_B/U_B$ . Let  $r_A : V_A/U_A \rightarrow \mathfrak{sl}(U_A)$  be the restriction of  $\bar{R} : \mathcal{V}/\mathcal{U} \rightarrow \mathfrak{sl}(\mathcal{U})$  to  $S_A$ . Consider the following diagram:

$$\begin{array}{ccc} V_A/U_A & \xrightarrow{r_A} & \mathfrak{sl}(U_A) \\ \Gamma \downarrow & & \downarrow \text{ad}(\gamma \otimes 1) \\ \varphi^*(V_B/U_B) & \xrightarrow{\varphi^*r_B} & \varphi^*\mathfrak{sl}(U_B) \end{array}$$

The vector bundles  $\mathfrak{sl}(U_A)$  and  $\mathfrak{sl}(U_B)$  have the quadratic forms  $f \mapsto (\text{tr } f^2)/2$  and all the homomorphisms in the above diagram are isomorphisms compatible with the quadratic forms by Proposition 1.4. If  $g$  is an automorphism of  $\mathfrak{sl}(U_A)$  and preserves the quadratic form, then  $g$  or  $-g$  comes from an automorphism of  $U_A$  because  $H^1(S, \mathbb{Z}/2\mathbb{Z}) = 0$ . Since every endomorphism of  $U_A$  is a constant multiplication,  $g$  is equal to  $\pm \text{id}$ . Therefore, the above diagram is commutative up to sign. Hence, for  $\gamma$  or  $-\gamma$ , the above diagram is commutative. Since  $xy = r_A(\bar{y})(x)$  for every  $x \in U_A$  and  $y \in V_A$ ,  $\gamma$  or  $-\gamma$  satisfies our requirements, where  $\bar{y} \in V_A/U_A$  is the image of  $y \in V_A$ . q.e.d.

We shall show that, for the isomorphism  $\gamma$  in Step IV,  $\tilde{\gamma} = 1 \oplus \gamma : \mathcal{C}_0 \rightarrow \mathcal{C}_0$  satisfies  $\tilde{\gamma}(xy) = \tilde{\gamma}(x)\tilde{\gamma}(y)$  for every  $x, y \in \mathcal{C}_0$ . If  $x, y \in \mathcal{C}_0$ , then  $xy + yx$  is equal to  $b(x, y)$ , where  $b(x, y)$  is the inner product associated to the quadratic form  $q$ . Hence the real part of  $xy$  is equal to  $b(x, y)/2$ , that is,  $xy - b(x, y)/2$  belongs to  $\mathcal{C}_0$ . Since  $\gamma$  preserves the quadratic form  $q$ ,  $\tilde{\gamma}(x, y)$  and  $\tilde{\gamma}(x)\tilde{\gamma}(y)$  have the same real part, that is, their difference belongs to  $\mathcal{C}_0$ . Put  $\delta(x, y) = \tilde{\gamma}(x, y) - \tilde{\gamma}(x)\tilde{\gamma}(y)$  for every  $x, y \in \mathcal{C}_0$ .  $\delta : \mathcal{C}_0 \otimes \mathcal{C}_0 \rightarrow \mathcal{C}_0$  is skew-symmetric and  $\delta \otimes 1$  is identically zero on  $U_A \otimes V_A \subset \mathcal{C}_0 \otimes \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$ .

*Step V.*  $\delta \otimes 1$  is identically zero on  $V_A \otimes V_A \subset \mathcal{C}_0 \otimes \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$ .

$S^2\alpha : S^2\mathcal{C}_0 \otimes \mathcal{O}_X \rightarrow S^2\mathcal{U}^\vee$ .  
 les of dimension 1 and 27 and  
 les, we have  $\dim H^0(X, \mathcal{N}) =$

$$\rightarrow H^i(X, S^2\mathcal{C}_0 \otimes \mathcal{O}_X(-n))$$

every  $i = 1, 2$  and 3. Hence by  
 $H^0(X, \mathcal{N}) \rightarrow H^0(S, \mathcal{N}|_S)$  is  
 $H^0(X, \mathcal{N}) = 1$ , which shows our  
 q.e.d.

$$\rightarrow \mathcal{C}_0 \text{ such that } (\gamma \otimes 1)(U_A) = \\ xy = ((\gamma \otimes 1)(x))((\gamma \otimes 1)(y))$$

as the requirements of Step III.  
 and induces an isomorphism  $\Gamma :$   
 the quadratic forms on  $V_A/U_A$   
 restriction of  $\bar{R} : \mathcal{V}/\mathcal{U} \rightarrow sl(\mathcal{U})$

$$\begin{pmatrix} U_A \\ \downarrow \text{ad}(\gamma \otimes 1) \\ U_B \end{pmatrix}$$

quadratic forms  $f \mapsto (\text{tr } f^2)/2$   
 are isomorphisms compatible  
 is an automorphism of  $sl(U_A)$   
 comes from an automorphism  
 omorphism of  $U_A$  is a constant  
 above diagram is commutative  
 agram is commutative. Since  
 $-\gamma$  satisfies our requirements,  
 q.e.d.

Step IV,  $\tilde{\gamma} = 1 \oplus \gamma : \mathcal{C}_0 \rightarrow \mathcal{C}^0$   
 $x, y \in \mathcal{C}_0$ , then  $xy + yx$  is equal  
 ciated to the quadratic form  $q$ .  
 is,  $xy - b(x, y)/2$  belongs to  $\mathcal{C}_0$ .  
 and  $\tilde{\gamma}(x)\tilde{\gamma}(y)$  have the same real  
 $\delta(x, y) = \tilde{\gamma}(x, y) - \tilde{\gamma}(x)\tilde{\gamma}(y)$  for  
 metric and  $\delta \otimes 1$  is identically

$$\in \mathcal{C}_0 \otimes \mathcal{C}_0 \otimes \mathcal{O}_{S_A}.$$

*Proof.* Since  $\delta \otimes 1$  is skew-symmetric and identically zero on  $U_A \otimes V_A$ ,  $\delta \otimes 1$  induces a skew-symmetric form  $\bar{\delta}$  on  $V_A/U_A$ . Since  $V_A/U_A$  is isomorphic to  $sl(U_A)$ ,  $\wedge^2(V_A/U_A)^\vee$  is also isomorphic to  $sl(U_A)$  and has no nonzero global sections. Hence  $\bar{\delta}$  is zero and  $\delta \otimes 1$  is identically zero on  $V_A \otimes V_A$ . q.e.d.

Step VI. Every homomorphism  $f$  from  $V_A$  to  $U_A$  is zero.

*Proof.* Since  $V_A/U_A$  is isomorphic to  $sl(U_A)$ , there are no nonzero homomorphisms from  $V_A/U_A$  to  $\mathcal{O}_{S_A}$ . Hence  $V_A/U_A$  cannot be a subsheaf of  $\mathcal{C}_0 \otimes \mathcal{O}_{S_A}$ . Therefore, the exact sequence  $0 \rightarrow U_A \rightarrow V_A \rightarrow V_A/U_A \rightarrow 0$  does not split. Hence the restriction  $f|_{U_A} : U_A \rightarrow U_A$  of  $f$  to  $U_A$  is not an isomorphism. Since every endomorphism of  $U_A$  is a constant multiplication,  $f|_{U_A}$  is zero and  $f$  induces a homomorphism  $\bar{f} : V_A/U_A \rightarrow U_A$ . Since  $V_A/U_A \cong sl(U_A)$ , we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}_S}(V_A/U_A, U_A) &\cong H^0(S_A, sl(U_A) \otimes U_A) \\ &\cong H^0(S_A, U_A \oplus (S^3 U_A) \otimes L_A). \end{aligned}$$

Hence by Proposition 1.10,  $\bar{f}$  is zero and  $f$  is also zero. q.e.d.

Step VII.  $\delta$  is zero.

*Proof.* Let  $T$  be the cokernel of the natural injection  $\wedge^2 V_A \rightarrow \wedge^2 \mathcal{C}_0 \otimes \mathcal{O}_{S_A}$ . Since  $\delta \otimes 1$  belongs to  $\text{Hom}_{\mathcal{O}_S}(T, \mathcal{C}_0 \otimes \mathcal{O}_{S_A})$ , it suffices to show that  $\text{Hom}_{\mathcal{O}_S}(T, \mathcal{O}_{S_A})$  is zero. There is an exact sequence

$$0 \rightarrow V_A \otimes E_A \rightarrow T \rightarrow \bigwedge^2 E_A \rightarrow 0,$$

where  $E_A$  is the quotient bundle  $(\mathcal{C}_0 \otimes \mathcal{O}_{S_A})/V_A$  and isomorphic to  $U_A^\vee$  by the bilinear form  $b$  associated to  $q$ . By Step VI, we have  $\text{Hom}_{\mathcal{O}_S}(V_A \otimes E_A, \mathcal{O}_{S_A}) \cong \text{Hom}_{\mathcal{O}_S}(V_A, U_A) = 0$ . Since  $\wedge^2 E_A$  is an ample line bundle,  $\text{Hom}_{\mathcal{O}_S}(\wedge^2 E_A, \mathcal{O}_{S_A})$  is zero. Therefore, by the above exact sequence,  $\text{Hom}_{\mathcal{O}_S}(T, \mathcal{O}_{S_A})$  is zero. q.e.d.

By Step VII,  $1 \oplus \gamma$  is an automorphism of the Cayley algebra  $\mathcal{C}$ . The automorphism of  $X_{18} = G/P$  induced by  $1 \oplus \gamma$  maps  $S_A$  onto  $S_B$ . Hence we have (2.2) and, in particular, Theorem 0.2.

### §3. Generic K3 surfaces of genus 7, 8, and 9

The proof of Theorem 0.2 in the case  $g = 7, 8$ , and 9 is very similar to and rather easier than the case  $g = 10$  dealt in the previous sections. The  $(24 - 2g)$ -dimensional homogeneous spaces  $X = X_{2g-2} \subset \mathbf{P}^{22-g}$  ( $g = 7, 8$  and 9) are also generalized Grassmann variety as in the case  $g = 10$ . In the case  $g = 8$ ,  $X_{14} \subset \mathbf{P}^{14}$  is the Grassmann variety  $G(V, 2)$  of 2-dimensional quotient spaces of a 6-dimensional vector space  $V$  embedded into  $\mathbf{P}(\wedge^2 V)$  by the Plücker

coordinates. In the case  $g = 9$ ,  $X \subset \mathbf{P}^{13}$  is the Grassmann variety of 3-dimensional totally isotropic quotient spaces of a 6-dimensional vector space  $V$  with a nondegenerate skew-symmetric tensor  $\sigma \in \wedge^2 V$ , where a quotient  $f: V \rightarrow V'$  is totally isotropic with respect to  $\sigma$  if  $(f \otimes f)(\sigma)$  is zero in  $V' \otimes V'$ . The embedding  $X_{16} \subset \mathbf{P}^{13}$  is the linear hull of the composite of the natural embedding  $X \subset G(V, 3)$  and the Plücker embedding  $G(V, 3) \subset \mathbf{P}(\wedge^3 V)$ . In the case  $g = 7$ ,  $X \subset \mathbf{P}^{15}$  is a 10-dimensional spinor variety. Let  $V$  be a 10-dimensional vector space with a non-degenerate second symmetric tensor. The subset of  $G(V, 5)$  consisting of 5-dimensional totally isotropic quotient spaces of  $V$  has exactly two connected components, one of which is our spinor variety  $X$ . The pull-back of the tautological line bundle  $\mathcal{O}_{\mathbf{P}}(1)$  by the composite  $X \hookrightarrow G(V, 5) \hookrightarrow \mathbf{P}(\wedge^5 V)$  is twice the positive generator  $L$  of  $\text{Pic } X$  and the vector space  $H^0(X, L)$  is a half spinor representation of  $\text{Spin}(V)$ , the universal covering groups of  $\text{SO}(V)$ . In each case,  $X$  is a compact hermitian symmetric space and the anticanonical class of  $X$  is equal to  $\dim X - 2$  times the positive generator  $L$  of  $\text{Pic } X$  (Proposition 1.5 and [2] §16). Moreover, by Proposition 1.5 and an easy computation, we have that the embedded variety  $X \hookrightarrow \mathbf{P}(H^0(X, L))$  has degree  $2g - 2$ . Hence every smooth complete intersection of  $X$  and a linear subspace of codimension  $n - 2$  (resp.  $n - 3$ ,  $n - 1$ ) is the projective (resp. canonical, anticanonical) model of a K3 surface (resp. curve, Fano 3-fold) of genus  $g$ . Each homogeneous space  $X = X_{2g-2}$  has a natural homogeneous vector bundle  $\mathcal{E}$  on it. In the case  $g = 8$ , we have the exact sequence

$$(3.1) \quad 0 \rightarrow \mathcal{F} \rightarrow V \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{E} \rightarrow 0,$$

where  $\mathcal{E}$  (resp.  $\mathcal{F}$ ) is the universal quotient (resp. sub-) bundle and is of rank 2 (resp. 4). In the case  $g = 7$  (resp. 9), we have the exact sequence

$$(3.2) \quad 0 \rightarrow \mathcal{E}^\vee \rightarrow V \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{E} \rightarrow 0,$$

where  $\mathcal{E}$  is the universal maximally totally isotropic quotient bundle with respect to  $\sigma \otimes 1 \in V \otimes V \otimes \mathcal{O}_X$  and is of rank 5 (resp. 3).

Theorem 0.2 is a consequence of the openness of the stability condition and the following:

**Theorem 3.3.** *Let  $S$  and  $S'$  be two K3 surfaces which are complete intersections of  $X_{2g-2} \subset \mathbf{P}^{22-g}$  ( $g = 7, 8$  and  $9$ ) and linear subspaces  $R$  and  $R'$ , respectively. Then we have*

- (1) *if  $R$  is general, then the vector bundle  $\mathcal{E}|_S$  is stable with respect to  $\mathcal{O}_S(1)$ , the restriction of  $L = \mathcal{O}_X(1)$  to  $S$ , and*
- (2) *if  $\mathcal{E}|_S$  and  $\mathcal{E}|_{S'}$  are stable with respect to  $\mathcal{O}_S(1)$  and  $\mathcal{O}_{S'}(1)$  and if  $S \subset R$  and  $S' \subset R'$  are projectively equivalent, then  $R$  and  $R'$  are equivalent under the action of  $G$  on  $X$ .*

For the proof we need the following property of the vector bundle  $E = \mathcal{E}|_S$ .

is the Grassmann variety of 3-planes in a 6-dimensional vector space or  $\sigma \in \wedge^2 V$ , where a quotient map  $(f \otimes f)(\sigma)$  is zero in  $V' \otimes V'$ . The composite of the natural embedding  $G(V, 3) \subset \mathbf{P}(\wedge^3 V)$ . In the spinor variety. Let  $V$  be a 10-dimensional second symmetric tensor. The totally isotropic quotient spaces of which is our spinor variety  $\mathcal{O}_{\mathbf{P}}(1)$  by the composite  $X \hookrightarrow \text{Pic } X$  and the vector bundle  $L$  of  $\text{Spin}(V)$ , the universal covering hermitian symmetric space and 2 times the positive generator  $L$  by Proposition 1.5 and an easy map  $X \hookrightarrow \mathbf{P}(H^0(X, L))$  has degree of  $X$  and a linear subspace of the projective (resp. canonical, e, Fano 3-fold) of genus  $g$ . a natural homogeneous vector exact sequence

$$\mathcal{E} \longrightarrow 0,$$

sub-) bundle and is of rank 2 exact sequence

$$\mathcal{E} \longrightarrow 0,$$

ic quotient bundle with respect

of the stability condition and

s which are complete intersections  $R$  and  $R'$ , respectively

s stable with respect to  $\mathcal{O}_S(1)$ ,

$\mathcal{O}_S(1)$  and  $\mathcal{O}_{S'}(1)$  and if  $S \subset R$  then  $R$  and  $R'$  are equivalent

of the vector bundle  $E = \mathcal{E}|_S$ .

**Proposition 3.4.** Let  $S$  be a complete intersection of  $X = X_{2g-2} \subset \mathbf{P}^{22-g}$  and a  $g$ -dimensional linear subspace and  $E$  the restriction of  $\mathcal{E}$  to  $S$ . Then we have

- (1)  $H^i(S, \mathcal{E}) = 0$  for every  $i$ ,
- (2) the homomorphism  $H^0(\alpha): V \rightarrow H^0(S, E)$  is an isomorphism,
- (3) in the case  $g = 7$  (resp. 9), the kernel of the homomorphism  $H^0(S^2\alpha): S^2V \rightarrow H^0(S, S^2E)$  (resp.  $H^0(\wedge^2\alpha): \wedge^2V \rightarrow H^0(S, \wedge^2E)$ ) is 1-dimensional and generated by  $\sigma \otimes 1$ , and
- (4) in the case  $g = 7$  (resp. 8, resp. 9),  $E(-1)$ ,  $(\wedge^2 E)(-1)$ ,  $(\wedge^3 E)(-2)$  or  $(\wedge^4 E)(-2)$  (resp.  $E(-1)$ , resp.  $E(-1)$  or  $(\wedge^2 E)(-1)$ ) has no nonzero global sections.

We prove the proposition in the case  $g = 7$ . The other cases are similar. According to [4], we take  $\alpha_i = e_i - e_{i+1}$ ,  $1 \leq i \leq 4$ , and  $\alpha_5 = e_4 + e_5$  as a root basis of  $\text{SO}(10)$ . The positive roots are  $e_i \pm e_j$ ,  $i < j$  and the conjugacy class of the maximal parabolic subgroup  $P$  corresponds to  $\alpha_5$  (or  $\alpha_4$ ). The homogeneous vector bundles  $\mathcal{O}_X(1)$ ,  $\wedge^i \mathcal{E}$ ,  $\mathcal{E}$  and  $S^2 \mathcal{E}$  are induced by the irreducible representations of the reductive part of  $P$  with the highest weights  $\frac{1}{2}(e_1 + \dots + e_5)$ ,  $e_1 + \dots + e_i$ ,  $e_1 - e_5$  and  $2e_1$ , respectively. The half  $\rho$  of the sum of positive roots is equal to  $4e_1 + 3e_2 + 2e_3 + e_4$ . Applying Bott's theorem, we have

**Lemma 3.5.** ( $g = 7$ ) The cohomology groups of  $\mathcal{E}(n)$ ,  $(\wedge^2 \mathcal{E})(n)$ ,  $(\mathcal{E})(n)$  and  $(S^2 \mathcal{E})(n)$  vanish except for the following cases:

- (1)  $H^0(X, \mathcal{E}(n))$ ,  $H^0(X, (\wedge^2 \mathcal{E})(n))$ ,  $H^0(X, (S^2 \mathcal{E})(n))$  for  $n \geq 0$  and  $H^0(X, (\mathcal{E})(n))$  for  $n \geq 1$ ,
- (2)  $H^9(X, (\wedge^2 \mathcal{E})(-8))$ , and
- (3)  $H^{10}(X, \mathcal{E}(n))$ ,  $H^{10}(X, (\mathcal{E})(n))$  for  $n \leq -9$  and  $H^{10}(X, (\wedge^2 \mathcal{E})(m))$ ,  $H^{10}(X, (S^2 \mathcal{E})(m))$  for  $m \leq -10$ .

**Remark 3.6.** In the above case  $g = 7$ , the 10 roots  $e_i + e_j$ ,  $1 \leq i < j \leq 5$ , are complementary to  $P$ . Their sum is equal to  $4(e_1 + \dots + e_5)$  and this is 8 times the fundamental weight  $w$ . By Proposition 1.5, the self intersection number of  $\mathcal{O}_X(1)$  is equal to

$$10! \prod_{\beta \in R_P} \frac{(\beta, w)}{(\beta, \rho)} = 10! \prod_{0 \leq i < j \leq 4} (i + j)^{-1} = 12.$$

Hence  $X$  is a 10-dimensional variety of degree 12 in  $\mathbf{P}^{15}$  and the anticanonical class is 8 times the hyperplane section.

*Proof of Proposition 3.4* (in the case  $g = 7$ ):  $S$  is a complete intersection of 8 members of  $|\mathcal{O}_X(1)|$ . Hence, if  $\mathcal{A}$  is a vector bundle on  $X$  and  $H^{i+a}(X, \mathcal{A}(-a))$  vanishes for every  $0 \leq a \leq 8$ , then so does  $H^i(S, \mathcal{A}|_S)$ .

(1 and 4) (1) and the vanishings of  $H^0(S, E(-1))$  and  $H^0(S, (\wedge^2 E)(-1))$  follow immediately from Lemma 3.5. Since  $\wedge^5 \mathcal{E}$  is isomorphic to  $\mathcal{O}_X(2)$ ,  $\wedge^k \mathcal{E}$  is isomorphic to  $(\wedge^{5-k} \mathcal{E})^\vee \otimes \mathcal{O}_X(2)$ . Hence by the Serre duality and Lemma 3.5, we have

$$\begin{aligned} H^i(X, (\wedge^3 \mathcal{E})(-2-i)) &\cong H^{10-i}(X, (\wedge^3 \mathcal{E})^\vee(2+i-8))^\vee \\ &\cong H^{10-i}(X, (\wedge^2 \mathcal{E})(i-8))^\vee = 0 \end{aligned}$$

and

$$\begin{aligned} H^i(X, (\wedge^4 \mathcal{E})(-2-i)) &\cong H^{10-i}(X, (\wedge^4 \mathcal{E})^\vee(2+i-8))^\vee \\ &\cong H^{10-i}(X, \mathcal{E}(i-8))^\vee = 0, \end{aligned}$$

for every  $0 \leq i \leq 8$ . Therefore,  $(\wedge^3 E)(-2)$  or  $(\wedge^4 E)(-2)$  has no nonzero global sections.

(2) By the Serre duality,  $H^i(X, E^\vee(-i))$  and  $H^{i+1}(X, E^\vee(-i))$  are isomorphic to  $H^{10-i}(X, \mathcal{E}(i-8))^\vee$  and  $H^{9-i}(X, \mathcal{E}(i-8))^\vee$ , respectively and both are zero for every  $0 \leq i \leq 8$ , by Lemma 3.5. Hence both  $H^0(S, E^\vee)$  and  $H^1(S, E^\vee)$  vanish. Therefore, by the exact sequence (3.2), we have (2).

(3) Let  $\mathcal{K}$  be the kernel of the homomorphism  $S^2 \alpha : S^2 V \otimes \mathcal{O}_X \rightarrow S^2 \mathcal{E}$ . We have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow S^2 V \otimes \mathcal{O}_X \longrightarrow S^2 \mathcal{E} \longrightarrow 0.$$

The  $G$ -module  $S^2 V$  is isomorphic to the direct sum of an irreducible  $G$ -module of dimension 54 and a trivial  $G$ -module generated by  $\sigma$ . Hence the  $G$ -module  $H^0(X, \mathcal{K}) = \text{Ker } H^0(S^2 \alpha)$  is 1-dimensional and generated by  $\sigma$ . By Lemma 3.5 and the Kodaira vanishing theorem,  $H^{i-1}(X, (S^2 \mathcal{E})(-i))$  and  $H^i(X, \mathcal{O}_X(-i))$  are zero. Hence by the above exact sequence,  $H^i(X, \mathcal{K}(-i))$  vanishes for every  $1 \leq i \leq 8$ . By using the Koszul complex, we have that the restriction map  $H^0(X, \mathcal{K}) \rightarrow H^0(S, \mathcal{K}|_S)$  is surjective. Therefore, the kernel of  $H^0(S^2 \alpha|_S)$  is at most 1-dimensional. It is clear that the kernel contains  $\sigma \otimes 1$ . Hence we have (3). q.e.d.

*Proof of Theorem 3.3:* Let  $S$  (resp.  $S'$ ) be a K3 surface which is a complete intersection of  $X$  and a linear subspace  $P$  (resp.  $P'$ ) and  $E$  (resp.  $E'$ ) the restriction of  $\mathcal{E}$  to  $S$  (resp.  $S'$ ). If  $P$  is general, then  $\text{Pic } S$  is generated by  $\mathcal{O}_S(1)$  and, by (4) of Proposition 3.4,  $E$  is stable. Hence we have i). Assume that  $S$  and  $S'$  are isomorphic to each other as polarized surfaces and that  $E$  and  $E'$  are stable. By (1) of Proposition 3.4 and the same argument as Step I in §2,  $E$  and  $E'$  are isomorphic to each other. By (2) of Proposition 3.4, we

$E(-1)$  and  $H^0(S, (\wedge^2 E)(-1))$  is isomorphic to  $\mathcal{O}_X(2)$ ,  $\wedge^k \mathcal{E}$  is Serre duality and Lemma 3.5,

have an isomorphism  $\beta: V \xrightarrow{\sim} V'$  and a commutative diagram

$$\begin{array}{ccccc} V \otimes \mathcal{O}_S & \xrightarrow{\alpha|_S} & E & \longrightarrow & 0 \\ \beta \otimes 1 \downarrow \wr & & \downarrow \wr & & \\ V' \otimes \mathcal{O}_{S'} & \xrightarrow{\alpha|_{S'}} & E' & \longrightarrow & 0. \end{array}$$

Hence, in the case  $g = 8$ ,  $S$  and  $S'$  are equivalent under the action of  $\mathrm{GL}(V)$ . In the case  $g = 7$  or  $9$ , by (3) of Proposition 3.4,  $S^2\beta$  maps  $\sigma$  to  $a\sigma$  for a nonzero constant  $a$ . Hence, replacing  $\beta$  by  $a^{1/2}\beta$ , we may assume that  $S^2\beta$  preserves  $\sigma$ . Hence  $S$  and  $S'$  are equivalent under the action of  $\mathrm{SO}(V, \sigma)$  or  $\mathrm{Sp}(V, \sigma)$ . *q.e.d.*

#### §4. Generic K3 surface of genus 6

A K3 surface of genus 6 is obtained as a complete intersection in the Grassmann variety  $G(2, V^5)$  of 2-dimensional subspaces in a fixed 5-dimensional vector space  $V^5$ .  $G(2, V^5)$  is embedded into  $\mathbf{P}^9$  by Plücker coordinates and has degree 5. A smooth complete intersection  $X_5 \subset \mathbf{P}^6$  of  $G(2, V^5)$  and 3 hyperplanes in  $\mathbf{P}^9$  is a Fano 3-fold of index 2 and degree 5. A smooth complete intersection  $X_5$  and a quadratic hypersurface in  $\mathbf{P}^6$  is an anticanonical divisor of  $X_5$  and is a K3 surface of genus 6. The isomorphism class of  $X_5$  does not depend on the choice of 3 hyperplanes and  $X_5$  has an action of  $\mathrm{PGL}(2)$  (see below).

**Theorem 4.1.** *Let  $S$  and  $S'$  be two general smooth complete intersections of  $X_5$  and a quadratic hypersurface in  $\mathbf{P}^6$ . If  $S \subset \mathbf{P}^6$  and  $S' \subset \mathbf{P}^6$  are projectively equivalent, then they are equivalent under the action of  $\mathrm{PGL}(2)$  on  $X_5$ .*

All the Fano 3-folds of index 2 and degree 5 are unique up to isomorphism [5]. There are several ways to describe the Fano 3-folds. The following is most convenient for our purpose: Let  $V$  be a 2-dimensional vector space and  $f \in S^6 V$  an invariant polynomial of an octahedral subgroup of  $\mathrm{PGL}(V)$ .  $f$  is equal to  $xy(x^4 - y^4)$  for a suitable choice of a basis  $\{x, y\}$  of  $V$ . Then the closure  $X_5$  of the orbit  $\mathrm{PGL}(V) \cdot \bar{f}$  in  $\mathbf{P}_*(S^6 V) := (S^6 V - \{0\})/\mathbf{C}^*$  is a Fano 3-fold of index 2 and degree 5, [11].  $H^0(X_5, \mathcal{O}_X(2))$  is generated by  $H^0(X_5, \mathcal{O}_X(1)) = S^6 V$ , [5], and has dimension  $\frac{1}{2}(-K_X)^3 + 3 = 23$ . Hence the kernel  $A$  of the natural map  $S^2 H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(X, \mathcal{O}_X(2))$  is a 5-dimensional  $\mathrm{SL}(V)$ -invariant subspace. As an  $\mathrm{SL}(V)$ -module,  $S^2 H^0(X, \mathcal{O}_X(1))$  is isomorphic to  $S^2(S^6 V) \cong S^{12} V \oplus S^8 V \oplus S^4 V \oplus \mathbf{1}$ . Hence we have

**Proposition 4.2.** (1)  $H^0(X_5, \mathcal{O}_X(-K_X))$  is isomorphic to  $S^{12} V \oplus S^8 V \oplus \mathbf{1}$  as  $\mathrm{SL}(V)$ -module, and

(2) the vector space  $A$  of quadratic forms which vanish on  $X_5 \subset \mathbf{P}^6$  is isomorphic to  $S^4 V$  as  $\mathrm{SL}(V)$ -module.

$$\begin{aligned} & (\wedge^3 \mathcal{E})^\vee(2+i-8))^\vee \\ & (\wedge^2 \mathcal{E})(i-8))^\vee = 0 \end{aligned}$$

$$\begin{aligned} & (\wedge^4 \mathcal{E})^\vee(2+i-8))^\vee \\ & (\mathcal{E}(i-8))^\vee = 0, \end{aligned}$$

$E(-2)$  has no nonzero global

and  $H^{i+1}(X, E^\vee(-i))$  are isomorphic, respectively and both are both  $H^0(S, E^\vee)$  and  $H^1(S, E^\vee)$  we have (2).

on  $S^2\alpha: S^2V \otimes \mathcal{O}_X \rightarrow S^2\mathcal{E}$ . We

$$S^2\mathcal{E} \rightarrow 0.$$

sum of an irreducible  $G$ -module generated by  $\sigma$ . Hence the  $G$ -module  $S^2\mathcal{E}(-i)$  and  $H^i(X, \mathcal{O}_X(-i))$  vanishes for every  $i$ . We have that the restriction map  $\sigma$ , the kernel of  $H^0(S^2\alpha|_S)$  is at contains  $\sigma \otimes 1$ . Hence we have *q.e.d.*

K3 surface which is a complete intersection of  $S$  (resp.  $E$ ) and  $E'$  (resp.  $S'$ ) the Picard group of  $S$  is generated by  $\mathcal{O}_S(1)$ . Hence we have i). Assume  $S$  and  $S'$  are polarized surfaces and that  $E$  and  $E'$  are the same argument as Step 1. By (2) of Proposition 3.4, we



There is a non-empty open subset  $\Xi$  of  $| - K_X |$  and a natural morphism  $\Xi/\mathrm{PGL}(V) \rightarrow \mathcal{F}_6$ . Both the target and the source are of dimension 19 and the morphism is birational by the theorem. Hence by the proposition we have

**Corollary 4.3.** *The generic K3 surface of genus 6 can be embedded into  $X_5$  as an anticanonical divisor in a unique way up to the action of  $\mathrm{PGL}(V)$ . In particular, the moduli space  $\mathcal{F}_6$  is birationally equivalent to the orbit space  $(S^{12}V \oplus S^8V)/\mathrm{PGL}(V)$ .*

First we need to show that  $\mathrm{PGL}(V)$  is the full automorphism group of  $X_5$  :

**Proposition 4.4.** *The automorphism group  $\mathrm{Aut} X_5$  of  $X_5$  is connected and the natural homomorphism  $\mathrm{PGL}(V) \rightarrow \mathrm{Aut} X_5$  is an isomorphism.*

*Proof.* There is a 2-dimensional family of lines on  $X_5 \subset \mathbf{P}^6$  and a 1-dimensional subfamily of lines  $\ell$  of special type, i.e., lines such that  $N_{\ell/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$ . The union of all lines of special type is a surface and has singularities along a rational curve  $C$ .  $C$  is the image of the 6-th Veronese embedding of  $\mathbf{P}(V) \cong \mathbf{P}^1$  into  $\mathbf{P}(S^6V)$ .  $C$  is invariant under the action of  $\mathrm{Aut} X_5$ . Every automorphism of  $X_5$  induces an automorphism of  $C$ . Hence we have the homomorphism  $\alpha : \mathrm{Aut} X_5 \rightarrow \mathrm{Aut} C \cong \mathrm{PGL}(V)$ . Since  $\alpha|_{\mathrm{PGL}(V)}$  is an isomorphism,  $\mathrm{Aut} X_5$  is isomorphic to  $\mathrm{PGL}(V) \times \mathrm{Ker} \alpha$ . Let  $g$  be an automorphism of  $X_5$  which commutes with every element of  $\mathrm{SL}(V)$ . Since  $S^6V$  is an irreducible  $\mathrm{SL}(V)$ -module,  $g$  is the identity by Schur's lemma. q.e.d.

Next we construct an equivariant embedding of  $X_5$  into the Grassmann variety  $G(2, S^4V)$ . Let  $W$  be the 2-dimensional subspace of  $S^4V$  generated by  $x^4 + y^4$  and  $x^2y^2$  for some basis  $\{x, y\}$  of  $V$  and  $Y$  the closure of the orbit  $\mathrm{PGL}(V) \cdot [W]$  in  $G(2, S^4V)$ . Consider the morphism  $J : G(2, S^4V) \rightarrow \mathbf{P}_*(S^6V)$  for which

$$J([Cg + Ch]) = \det \begin{pmatrix} g_X & g_Y \\ h_X & h_Y \end{pmatrix},$$

where  $\{X, Y\}$  is the dual basis of  $\{x, y\}$ . Then  $J$  is a  $\mathrm{PGL}(V)$ -equivariant morphism and sends  $[W]$  to the point  $\bar{f}$ ,  $f = xy(x^4 - y^4)$ . Hence  $J$  maps  $Y$  onto  $X_5 \subset \mathbf{P}_*(S^6V)$ . Define two  $\mathrm{GL}(V)$ -homomorphisms  $\varphi : \wedge^2 S^4V \rightarrow S^2V \otimes (\det V)^3$  and  $j : \wedge^2 S^4V \rightarrow S^6V \otimes \det V$  by

$$\varphi(g \wedge h) = \sum_{i,j,k=\pm 1} ijk(D_i D_j D_k g)(D_{-i} D_{-j} D_{-k} h) \otimes (X \wedge Y)^{-3}$$

and

$$j(g \wedge h) = \det \begin{pmatrix} D_1(g) & D_{-1}(g) \\ D_1(h) & D_{-1}(h) \end{pmatrix} \otimes (X \wedge Y)^{-1},$$

$-K_X|$  and a natural morphism source are of dimension 19 and the by the proposition we have

enus 6 can be embedded into  $X_5$  up to the action of  $\mathrm{PGL}(V)$ . In ally equivalent to the orbit space

full automorphism group of  $X_5$  :

$\mathrm{Aut} X_5$  of  $X_5$  is connected and  $X_5$  is an isomorphism.

of lines on  $X_5 \subset \mathbf{P}^6$  and a 1-pe, i.e., lines such that  $N_{\ell/X} \cong$  type is a surface and has singu- of the 6-th Veronese embedding under the action of  $\mathrm{Aut} X_5$ . Ev- phism of  $C$ . Hence we have the  $V$ ). Since  $\alpha|_{\mathrm{PGL}(V)}$  is an isomor-  $\alpha$ . Let  $g$  be an automorphism of  $V$ ). Since  $S^6V$  is an irreducible ma. q.e.d.

ing of  $X_5$  into the Grassmann al subspace of  $S^4V$  generated and  $Y$  the closure of the orbit ism  $J : G(2, S^4V) \rightarrow \mathbf{P}_*(S^6V)$

$$\begin{pmatrix} g_V \\ h_V \end{pmatrix},$$

en  $J$  is a  $\mathrm{PGL}(V)$ -equivariant  $xy(x^4 - y^4)$ . Hence  $J$  maps omomorphisms  $\varphi : \wedge^2 S^4V \rightarrow$  by

$$D_{-j} D_{-k} h) \otimes (X \wedge Y)^{-3}$$

$$\otimes (X \wedge Y)^{-1},$$

where  $D_{\pm 1}$  are the derivations by  $X$  and  $Y$ . The  $\mathrm{GL}(V)$ -module  $\wedge^2 S^4V$  is decomposed into the direct sum of irreducible  $\mathrm{GL}(V)$ -submodules  $\mathrm{Ker} \varphi$  and  $\mathrm{Ker} j$ . Since  $\varphi(\wedge^2 W) = 0$ , the Plücker coordinates of  $W$  lies in the linear subspace  $P = \mathbf{P}_*(\mathrm{Ker} \varphi)$  of  $\mathbf{P}_*(\wedge^2 S^4V)$  and  $Y$  is contained in the intersection  $G(2, S^4V) \cap P$ . The morphism  $J$  is the composite of the Plücker embedding  $G(2, S^4V) \subset \mathbf{P}_*(\wedge^2 S^4V)$  and the projection  $\mathbf{P}_*(j) : \mathbf{P}_*(\wedge^2 S^4V) \rightarrow \mathbf{P}_*(S^6V)$  from the linear subspace  $\mathbf{P}_*(\mathrm{Ker} j)$ . Since the restriction of  $\mathbf{P}_*(j)$  to  $P$  is an isomorphism,  $J$  gives a  $\mathrm{PGL}(V)$ -equivariant isomorphism from the projective variety  $Y \subset P$  onto  $X_5 \subset \mathbf{P}_*(S^6V)$ .

**Lemma 4.5.**  $Y$  coincides with the intersection of  $G(2, S^4V)$  and  $P$  in  $\mathbf{P}_*(\wedge^2 S^4V)$ .

*Proof.* Let  $Y'$  be the intersection of  $G(2, S^4V)$  and  $P$  and  $B$  (resp.  $B'$ ) the vector space consisting of quadratic forms on  $P$  which vanish on  $Y$  (resp.  $Y'$ ). Both  $Y$  and  $Y'$  are intersections of quadratic hypersurfaces. Hence it suffices to show that  $B = B'$ . Since  $G(2, S^4V)$  does not contain  $P$ ,  $B'$  is not zero. On the other hand, since  $Y \subset P$  is isomorphic to  $X_5 \subset \mathbf{P}^6$ ,  $B$  is an irreducible  $\mathrm{SL}(V)$ -module by Proposition 4.2. As we saw above,  $Y$  is contained  $\mathrm{PGL}(V)$ -equivariantly in  $Y'$  and hence  $B'$  is an  $\mathrm{SL}(V)$ -submodule of  $B$ . Hence  $B'$  coincides with  $B$ . q.e.d.

So we have constructed a  $\mathrm{PGL}(V)$ -equivariant embedding of  $X_5$  into  $G(2, S^4V)$  and shown that  $X_5$  coincides with the intersection of its linear hull and  $G(2, S^4V)$ .

*Proof of Theorem 4.1.* There is a universal exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow S^4V \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$$

on  $G(2, S^4V)$ , where  $\mathcal{E}$  (resp.  $\mathcal{F}$ ) is the universal sub- (resp. quotient) bundle and has rank 2 (resp. 3). Let  $S$  and  $S'$  be two members of the anticanonical linear system  $|-K_X|$  on  $X_5$ . By the same arguments as in Sections 2 and 3, we have

(i)  $H^i(S, sl(\mathcal{E})|_S) = 0$  for every  $i$ ,

(ii) If  $S$  is general, then the vector bundle  $\mathcal{E}|_S$  is stable with respect to  $\mathcal{O}_S(1)$ , and

(iii) If  $\mathcal{E}|_S$  and  $\mathcal{E}|_{S'}$  are stable with respect to  $\mathcal{O}_S(1)$  and  $\mathcal{O}_{S'}(1)$ , respectively, and if  $S$  and  $S'$  are isomorphic as polarized surfaces, then there are isomorphisms  $\alpha : \mathcal{E}|_S \rightarrow \mathcal{E}|_{S'}$  and  $\beta \in \mathrm{GL}(S^4V)$  such that the diagram

$$\begin{array}{ccccc} 0 & \rightarrow & \mathcal{E}|_S & \rightarrow & S^4V \otimes \mathcal{O}_S \\ & & \alpha \downarrow & & \downarrow \beta \otimes 1 \\ 0 & \rightarrow & \mathcal{E}|_{S'} & \rightarrow & S^4V \otimes \mathcal{O}_{S'} \end{array}$$

is commutative. In particular, the automorphism  $\bar{\beta}$  of  $G(2, S^4V)$  induced by  $\beta$  maps  $S$  onto  $S'$  isomorphically.

Since  $X_5$  is the intersection of  $G(2, S^4V)$  and the linear span of  $S$  (resp.  $S'$ ), the automorphism  $\bar{\beta}$  maps  $X_5$  onto itself. Hence, by Proposition 4.4,  $S$  and  $S'$  are equivalent under the action of  $\text{PGL}(V)$  on  $X_5$ . q.e.d.

### §5. Fano 3-folds of genus 10

In this section we shall prove Theorem 0.9 in the case  $g = 10$ . The other cases  $g = 7, 8$  and  $9$  are very similar.

Let  $V$  and  $V'$  be Fano 3-folds which are complete intersections of  $X_{18} \subset \mathbf{P}^{13}$  and linear subspaces of codimension 2. By the Lefschetz theorem, both  $\text{Pic } V$  and  $\text{Pic } V'$  are generated by hyperplane sections. Let  $\mathcal{U}$  be the universal subbundle of  $\mathcal{C}_0 \otimes \mathcal{O}_{X_{18}}$  as in Section 1 and  $F$  and  $F'$  the restrictions of  $\mathcal{U}$  to  $V$  and  $V'$ , respectively.

**Proposition 5.1.** *Let  $\varphi : V \xrightarrow{\sim} V'$  be an isomorphism. Then  $\varphi^*(F')$  is isomorphic to  $F$ .*

*Proof.* Let  $S$  be the generic member of  $|-K_V|$  and put  $S' = \varphi(S)$ . The Picard group of  $S$  is generated by the hyperplane section. The restrictions  $E = F|_S$  and  $E' = F'|_{S'}$  are stable vector bundles as we saw in the proof of Proposition 2.1. Hence  $F$  and  $F'$  are also stable vector bundles. Put  $M = \text{Hom}_{\mathcal{O}_V}(F, \varphi^*F')$ . By Step I in Section 2, there is an isomorphism  $f_0 : E \xrightarrow{\sim} (\varphi|_S)^*E'$ . Hence the restriction of  $M$  to  $S$  is isomorphic to  $\text{End}_{\mathcal{O}_S}(E)$ . By Proposition 1.10, we have

$$H^1(S, M(n)|_S) \cong H^1(S, \mathcal{O}_S(n)) \oplus H^1(S, (sl E)(n)) = 0$$

for every integer  $n$ . Since  $H^1(V, M(n))$  is zero, if  $n$  is sufficiently negative, we have by induction on  $n$  that  $H^1(V, M(n))$  is zero for every  $n$ . In particular  $H^1(V, M(-1))$  vanishes and hence the restriction map  $H^0(V, M) \rightarrow H^0(S, M|_S)$  is surjective. It follows that there is a nonzero homomorphism  $f : F \rightarrow \varphi^*F'$  such that  $f|_S = f_0$ . Since  $f_0$  is an isomorphism, the cokernel of  $f$  has a support on a finite set. Since the Hilbert polynomials  $\chi(F(n))$  and  $\chi((\varphi^*F')(n))$  are same, the cokernel of  $f$  is zero and  $f$  is an isomorphism. q.e.d.

By Proposition 5.1 and similar arguments as Step II-VII in Section 2, we have an isomorphism  $\beta : F \xrightarrow{\sim} \varphi^*F'$  and an isomorphism  $\gamma : \mathcal{C}_0 \rightarrow \mathcal{C}_0$  such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\beta} & \varphi^*(F') \\ \cap & & \cap \\ \mathcal{C}_0 \otimes \mathcal{O}_V & \xrightarrow{\gamma \otimes 1} & \mathcal{C}_0 \otimes \mathcal{O}_V = \varphi^*(\mathcal{C}_0 \otimes \mathcal{O}_{V'}) \end{array}$$

morphism  $\bar{\beta}$  of  $G(2, S^4V)$  induced

and the linear span of  $S$  (resp.  $S'$ ),  
 i.e., by Proposition 4.4,  $S$  and  $S'$   
 $X_5$ . q.e.d.

in the case  $g = 10$ . The other

complete intersections of  $X_{18} \subset$   
 by the Lefschetz theorem, both  
 sections. Let  $\mathcal{U}$  be the universal  
 and  $F'$  the restrictions of  $\mathcal{U}$  to  $V$

morphism. Then  $\varphi^*(F')$  is iso-

$|K_V|$  and put  $S' = \varphi(S)$ . The  
 plane section. The restrictions  
 idles as we saw in the proof of  
 ble vector bundles. Put  $M =$   
 $\varphi$  is an isomorphism  $f_0 : E \xrightarrow{\sim}$   
 isomorphic to  $\mathcal{E}nd_{\mathcal{O}_S}(E)$ . By

$$H^1(S, (sl E)(n)) = 0$$

o, if  $n$  is sufficiently negative,  
 is zero for every  $n$ . In par-  
 restriction map  $H^0(V, M) \rightarrow$   
 $\varphi$  is a nonzero homomorphism  
 $\varphi$  is an isomorphism, the cokernel  
 bert polynomials  $\chi(F(n))$  and  
 and  $f$  is an isomorphism.

q.e.d.

Step II-VII in Section 2, we  
 omorphism  $\gamma : \mathcal{C}_0 \rightarrow \mathcal{C}_0$  such

$F')$

$$^*(\mathcal{C}_0 \otimes \mathcal{O}_{V'})$$

is commutative and such that  $1 \oplus \gamma$  is an automorphism of the Cayley algebra  $\mathcal{C}$ . Hence the automorphism of  $X_{18} = G/P$  induced by  $1 \oplus \gamma$  maps  $V$  onto  $V'$ , which shows Theorem 0.9 in the case  $g = 10$ .

## §6. Curves of genus $\leq 9$

In this section we shall show the following:

**Theorem 6.1.** *The generic curve of genus  $\leq 9$  lie on a K3 surface.*

In the case  $g \leq 6$ , the generic curve is realized as a plane curve  $C$  of degree  $d \leq 6$  with only ordinary double points. Take a general plane curve  $D$  of degree  $6-d$  and let  $S$  be the double covering of the plane with branch locus  $C \cup D$ . Then the minimal resolution  $\tilde{S}$  of  $S$  is a K3 surface and contains a curve isomorphic to  $C$ .

In the case  $6 \leq g \leq 9$ , we shall show that the generic curve  $C$  of genus  $g$  can be embedded into  $\mathbf{P}^5$  by the complete linear system of a line bundle  $L$  of degree  $g+4$  and that there is a K3 surface  $S$  which is a complete intersection of 3 quadratic hypersurfaces in  $\mathbf{P}^5$  and which contains the image of  $C$ .

Let  $C$  be a curve of genus  $6 \leq g \leq 9$  and  $D$  an effective divisor on  $C$  of degree  $g-6$ . Put  $L = \omega_C \otimes \mathcal{O}_C(-D)$ . Then  $L$  is a line bundle of degree  $g+4$ . If  $D$  is general, then  $\dim H^0(C, L) = 6$ . Since  $\deg L^{\otimes 2} > \deg \omega_C$ , we have  $\dim H^0(C, L^{\otimes 2}) = 2(g+4) + 1 - g = g+9$ .

**Proposition 6.2.** *If  $C$  and  $D$  are general, then we have*

- (1)  $L$  is very ample and  $\dim H^0(C, L) = 6$ ,
- (2) the natural map

$$S^2 H^0(C, L) \longrightarrow H^0(C, L^{\otimes 2})$$

*is surjective and its kernel  $V$  is of dimension  $12-g$ , and*

- (3) *there are 3 quadratic hypersurfaces  $Q_1, Q_2$  and  $Q_3$  in  $\mathbf{P}(H^0(C, L))$  which contains the image of  $C$  by  $\Phi_{|L|}$  and such that the intersection  $S = Q_1 \cap Q_2 \cap Q_3$  is a K3 surface.*

**Proof.** It suffices to show that there exists a pair of  $C$  and  $D$  which satisfies the conditions (1), (2) and (3). Let  $R$  be a smooth rational curve of degree  $g-4$  in  $\mathbf{P}^5$  whose linear span  $\langle R \rangle$  has dimension  $g-4$ . Since  $R$  is an intersection of quadratic hypersurfaces, the intersection of 3 general quadratic hypersurfaces  $Q_1, Q_2$  and  $Q_3$  which contain  $R$  is a smooth K3 surface. Let  $C_0$  be the intersection of  $S$  and a general hyperplane  $H$ . We show that the pair of the generic member  $C$  of the complete linear system  $|C_0 + R|$  on  $S$  and the divisor  $D = R|_C$  satisfies the conditions (1), (2) and (3).

The intersection number  $(C_0 \cdot R)$  is equal to  $\deg R = g-4 \geq 2$ . Hence the linear system  $|C_0 + R|$  has no base points. Therefore  $C$  is smooth and  $D$  is effective. The genus of  $C$  is equal to  $(C_0 + R)^2/2 + 1 = g$  and the degree of  $D$  is

equal to  $(C_0 + R.R) = g - 6$ . Since  $\omega_C$  is isomorphic to  $\mathcal{O}_C(C) \cong \mathcal{O}_C(C_0 + R)$ , the line bundle  $L = \omega_C(-D)$  is isomorphic to  $\mathcal{O}_C(C_0)$ , the restriction of the tautological line bundle of  $\mathbf{P}^5$  to  $C$ . There is a natural exact sequence

$$0 \longrightarrow \mathcal{O}_S(-R) \longrightarrow \mathcal{O}_S(C_0) \longrightarrow \mathcal{O}_C(C_0) \longrightarrow 0.$$

Since  $H^i(S, \mathcal{O}_S(-R)) = 0$  for  $i = 0$  and  $1$ , the restriction map  $H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}}(1)) \xrightarrow{\sim} H^0(S, \mathcal{O}_S(C_0)) \longrightarrow H^0(C, \mathcal{O}_C(C_0))$  is an isomorphism. Hence the morphism  $\Phi|_L$  is nothing but the inclusion map  $C \hookrightarrow \mathbf{P}^5$  and (1) and (3) are obvious by our construction of  $C$ .

*Claim.* Let  $V_0$  be the vector space of the quadratic forms on  $\mathbf{P}^5$  which are identically zero on  $C_0 \cup R$ . Then the dimension of  $V_0$  is at most  $12 - g$ .

Let  $F_i = 0$  be the defining equation of the quadratic hypersurface  $Q_i$  for  $i = 1, 2$  and  $3$  and  $G = 0$  that of the hyperplane  $H$ . Let  $F$  be any quadratic form on  $\mathbf{P}^5$  which is identically zero on  $C_0 \cup R$ . Since  $F$  is identically zero on  $C_0$ ,  $F$  is equal to  $a_1 F_1 + a_2 F_2 + a_3 F_3 + GG'$  for some constants  $a_1, a_2$  and  $a_3$  and linear form  $G'$ . Since  $F_1, F_2, F_3$  and  $F$  are identically zero on  $R$ , so is  $GG'$ . Hence  $G'$  is identically zero on  $R$ . Therefore, the vector space  $V_0$  is generated by  $F_1, F_2, F_3$  and  $GG', G'$  being all linears from vanishing on  $\langle R \rangle$ . Since  $\dim \langle R \rangle = g - 4$ , we have  $\dim V_0 \leq 3 + 5 - (g - 4) = 12 - g$ .

Since  $C$  is a general deformation of  $C_0 \cup R$ , we have, by the claim, that the dimension of  $V$  is also at most  $12 - g$ . Since

$$\dim S^2 H^0(C, L) - \dim H^0(C, L^{\otimes 2}) = 21 - (g + 9) = 12 - g,$$

$H^0(C, L^{\otimes 2})$  is generated by  $H^0(C, L)$  and  $V$  has exactly dimension  $12 - g$ .

*q.e.d.*

By the theorem and Corollaries 0.3 and 4.3, we have

**Corollary 6.3.** *The generic curve of genus  $3 \leq g \leq 9$  is a complete intersection in a homogeneous space.*

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phic to  $\mathcal{O}_C(C) \cong \mathcal{O}_C(C_0 + R)$ ,  
 $\mathcal{O}_C(C_0)$ , the restriction of the  
 natural exact sequence

$$\mathcal{O}_C(C_0) \longrightarrow 0.$$

restriction map  $H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}}(1))$   
 isomorphism. Hence the mor-  
 $\gamma \hookrightarrow \mathbf{P}^5$  and (1) and (3) are

latic forms on  $\mathbf{P}^5$  which are  
 of  $V_0$  is at most  $12 - g$ .

quadratic hypersurface  $Q_i$  for  
 $\in H$ . Let  $F$  be any quadratic  
 Since  $F$  is identically zero on  
 some constants  $a_1, a_2$  and  $a_3$   
 ntically zero on  $R$ , so is  $GG'$ .  
 vector space  $V_0$  is generated  
 vanishing on  $\langle R \rangle$ . Since  
 $-4) = 12 - g$ .

we have, by the claim, that the

$$-(g+9) = 12 - g,$$

exactly dimension  $12 - g$ .

*q.e.d.*

we have

$\leq 9$  is a complete intersection

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