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FANO 3-FOLDS. II

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Abstract. In this paper Fano 3-folds of the principal series V_{2g-2} in $\mathbf{P}^{g+1}$ are studied. A classification is given of trivial (i.e. containing a trigonal canonical curve) 3-folds of this kind. Among all Fano 3-folds of the principal series these are distinguished by the property that they are not the intersection of the quadrics containing them. It turns out that the genus g of such 3-folds does not exceed 10. Fano 3-folds of genus one (i.e. with Pic $V \simeq \mathbf{Z}$) containing a line are described. It is proved that they exist for $g \leqslant 10$ and $g = 12$. Their rationality for $g = 7$ and $g \geqslant 9$ is established by direct construction.

Bibliography: 21 titles.

Introduction

This is a continuation of our previous article, Part I [8]. In this part we will study Fano 3-folds of the principal series (that is, smooth 3-folds $V_{2g-2} \subset \mathbf{P}^{g+1}$ of degree $2g-2$) such that the anticanonical class $-K_V$ is the class of the hyperplane section. The ground field k is algebraically closed, and although this hypothesis is not essential in this article, we will assume that k has characteristic char $k = 0$; we mainly require this to be able to refer to Part I.

In §1 we give a new treatment of certain classical results relating to the elementary properties of Fano 3-folds of the principal series. We present some examples, and give a certain characterization of such varieties in terms of the curve sections $X = \mathbf{P}^{g-1} \cap V$; we list the Fano 3-folds which are complete intersections–these only occur for $g = 3$, 4 and 5. From the classical Noether-Enriques-Petri theorem on canonical curves (1.6) we deduce that a Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$ for $g \geqslant 5$ is an intersection of quadrics (that is, the intersection of all the quadrics containing it) provided that it is not trigonal (that is, if among its curve sections $X = \mathbf{P}^{g-1} \cap V$ there are no trigonal curves).

In §2 we study trigonal Fano 3-folds; the main result is Theorem (2.5), which gives a complete classification; trigonal Fano 3-folds only exist for $g \leqslant 7$ and $g = 10$.

In §3 we study the family of lines on a Fano 3-fold V_{2g-2}. The main result is Theorem (3.4), which asserts that under the hypothesis Pic $V_{2g-2} \simeq \mathbf{Z}$ the lines on V_{2g-2}, provided that some exist, form a 1-dimensional family parametrized by a certain curve Γ with only double points as singularities. For $g \geqslant 4$, Γ is reduced at the generic point of each component; for $g = 3$ this is not always the case (see [14], and Remark (3.5), (ii)). We prove

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that for $g \geqslant 4$ and Pic $V_{2g-2} \simeq \mathbf{Z}$ every line on V_{2g-2} meets only a finite number of other lines. The results of §§3 and 4 are used in an essential way in §6 to prove the bound $g \leqslant 12$ (see (6.1)).

In §4 we study the family of conics on a Fano 3-fold of the principal series. The main result is Theorem (4.4), in which it is proved that on a V_{2g-2} satisfying Pic $V_{2g-2} \simeq \mathbf{Z}$ the conics, provided that some exist, form a 2-dimensional family Δ, reduced at a general point of each component, and such that through a general point $v \in V_{2g-2}$ there pass a finite number of conics. The results of §§3 and 4 were known in the classical literature ([4], [12], and also [15]) for varieties "in general position". However, even in this situation complete proofs of these assertions were in fact lacking.

The most important results of the present article are contained in §§5 and 6, where we study Fano 3-folds V with Pic $V \simeq \mathbf{Z}$. Following the classical terminology [4], we will call these varieties *of the first species.*

The most interesting are the varieties of index 1; that is, Pic $V = \mathbf{Z} \cdot K_V$. In [4] Fano asserts that such varieties (when anticanonically embedded in $\mathbf{P}^{g+1}$) always contain lines. This assertion–Fano's Conjecture I (see (3.6))–remains unproved.(1) Based on this conjecture, Roth asserts in [11] (see also [12]) that Fano varieties of the first species V_{2g-2} of index 1 only exist for $g \leqslant 10$. It turns out that this is not quite true. In Theorem (6.1) we prove that such varieties exist not only for $g \leqslant 10$, but also for $g = 12$. The proof is based on the same method used by Fano [5] and Roth [11], the method of double projection from a line. This method is entirely constructive, and on the way leads to a proof of Fano's assertion [5] that for $g = 7$ and $g \geqslant 9$ these varieties are rational.

At the end of the article we give a table of all known types of Fano 3-folds of the first species.

We note that one of the basic assertions of Fano's theory remains unproved:(1)

Fano's conjecture II. The degree of a Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$ is bounded above by the absolute constant 72; that is, for $g > 37$ there do not exist any such varieties.

The known varieties for which this bound is reached (see [4] and [12]) have singularities. It seems very plausible that for smooth Fano 3-folds the degree is bounded by 64 (the degree of the Veronese image of $\mathbf{P}^3$ under its anticanonical embedding).

Finally note that Bogomolov (see [10]) has proved the stability of the tangent bundle to a Fano 3-fold of the first species with index 1, and for these obtains the bound deg $V \leqslant 72$ without assuming the existence of lines.

In referring to [8] we will indicate the section number, adding "Part I"; example: (4.2, Part I).

EPILOGUE.* The two main conjectures on which the author's classification rests, namely Hypothesis (1.14, Part I) and Fano's Conjecture I (3.6) on the existence of lines, have been proved recently by V. V. Šokurov [19], [20]. In this translation references to these conjectures have been replaced by references to [19] and [20] througout.

In [21] and in the forthcoming Part III the classification of Fano 3-folds in a rather complete form is given; in particular the above conjecture $-K_V^3 \leqslant 64$ is proved.

(1) See the end of this Introduction.

*Added in translation.

§3 of Part I contains one error, leading to the omission of one class of variety, namely the direct product $\mathbf{P}^1 \times F$ of a del Pezzo surface F of degree 1 with $\mathbf{P}^1$. The mistake occurs in step 5) of the proof of Theorem 3.3, where it is asserted that the normal bundle $\mathcal{N}_Z$ to the smooth rational curve Z, which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-2) \longrightarrow \mathcal{N}_Z \longrightarrow \mathcal{O}_Z(m-2) \longrightarrow 0,$$

is necessarily of the form (a) $\mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(m-3)$ or (b) $\mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(m-2)$. In fact, *a priori* any possibility $\mathcal{N}_Z = \mathcal{O}_Z(d_1) \oplus \mathcal{O}_Z(d_2)$ with $-2 \leqslant d_1 \leqslant d_2$ and $d_1 + d_2 = m - 4$ can occur.

However, an argument identical to that of Lemma 3.4 proves that only two cases occur (in the notation of this lemma): $m = 3$ and $Z' = \mathbf{F}_1$, or $m = 4$ and $Z' = \mathbf{F}_0$. The second case leads to the omitted class of varieties $\mathbf{P}^1 \times F$, and there are no others.

§1. Fano 3-folds of the principal series

(1.1) DEFINITION. A smooth complete irreducible 3-fold V over a field k will be called a *Fano 3-fold of the principal series* if the antocanonical invertible sheaf $\mathcal{K}_V^{-1}$ is very ample.

From results obtained in Part I and from [19] it follows that all Fano 3-folds (Definition 1.1, Part I) are Fano 3-folds of the principal series, with the exceptions of

a) 3-folds of type (3.1 (a) and (b), Part I), and

b) hyperelliptic Fano 3-folds (§7, Part I).

Fano 3-folds of the principal series have the following properties:

(i) Under the anticanonical embedding $\varphi_{|-K_V|}\colon V \hookrightarrow \mathbf{P}(H^0(V, \mathcal{K}_V^{-1}))$ the image is a variety $V_{2g-2} \subset \mathbf{P}^{g+1}$ of degree $2g-2$, with $g = g(V) \geqslant 3$ being the genus of V.

(ii) Each nonsingular hyperplane section H_{2g-2} of V_{2g-2} is a $K3$ surface.

(iii) Each nonsingular section X_{2g-2} of V_{2g-2} by a linear subspace of codimension 2 of $\mathbf{P}^{g+1}$ is a canonical curve of genus g.

For the proofs, see 1.5, 1.6 and 1.7 in Part I.

(1.2) PROPOSITION. *Each nonsingular 3-fold $V \subset \mathbf{P}^{g+1}$ of degree $2g-2$ (not lying in a hyperplane) satisfying the two following conditions** (i) *and* (ii) *is a Fano 3-fold of the principal series, embedded in $\mathbf{P}^{g+1}$ by means of its anticanonical sheaf $\mathcal{K}_V^{-1}$.*

(i) *The curve sections $X_{2g-2} = V \cap \mathbf{P}^{g-1}$ are canonical curves of genus g.*

(ii) *$H^2(H, \mathcal{O}_H) \neq 0$ and $H^2(V, \mathcal{O}_V) = 0$, where H is a hyperplane section of V.*

PROOF. It is enough to show that $\mathcal{O}_V(1) \simeq \mathcal{K}_V^{-1}$; that is, $\mathcal{O}_V(H + K_V) \simeq \mathcal{O}_V$. Indeed, then by (1.6, Part I) $h^0(\mathcal{O}_V(1)) = h^0(\mathcal{K}_V^{-1}) = -K_V^3/2 + 2 = g + 2$; that is, V is embedded in $\mathbf{P}^{g+1}$ by the complete linear system $|-K_V|$.

Using (i) and the adjunction formula $K_X = (X \cdot 2H + K_V)$ we get that $(X \cdot H + K_V) = 0$. It is therefore enough to show that $H^0(\mathcal{O}_V(H + K_V)) \neq 0$. Then $\mathcal{O}_V(H + K_V) \simeq \mathcal{O}_V$, for otherwise $D \in |H + K_V|$ would be an effective divisor with $X \cdot D = (X \cdot H + K_V) = 0$; this is impossible, since the curves X sweep out the whole of V.

By duality $h^0(\mathcal{O}_V(H + K_V)) = h^3(\mathcal{O}_V(-H))$. From the cohomology long exact sequence associated to the short exact sequence of sheaves $0 \longrightarrow \mathcal{O}_V(-H) \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_H$

**Translator's note.* It is easy to see that (i) implies (ii).

$\longrightarrow 0$ and from (ii), we get $0 \longrightarrow H^2(H, \mathcal{O}_H) \longrightarrow H^3(\mathcal{O}_V(-H))$; hence by (ii) $h^3(\mathcal{O}_V(-H)) \neq 0$, and hence $h^0(\mathcal{O}_V(H+K_V)) \neq 0$. This proves the proposition.

Let us agree that in future by a variety $V_{2g-2} \subset \mathbf{P}^{g+1}$ we will mean a Fano 3-fold of the principal series in its anticanonical embedding.

(1.3) PROPOSITION. *A Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$ is a complete intersection only for $g = 3, 4$ or 5, and we have that*

$V_4 \subset \mathbf{P}^4$ is a quartic hypersurface,

$V_6 = V_{2\cdot 3}$ is an intersection of a quadric and a cubic in $\mathbf{P}^5$,

$V_8 = V_{2\cdot 2\cdot 2}$ is an intersection of 3 quadrics in $\mathbf{P}^6$.

Conversely, each smooth complete intersection of the types indicated is a Fano 3-fold of the principal series.

PROOF. Let $V_{2g-2} = F_{n_1} \cap \cdots \cap F_{n_{g-2}} \subset \mathbf{P}^{g+1}$ be a complete intersection of hypersurfaces F_{n_i} for $i = 1, \ldots, g-2$, with $\deg F_{n_i} = n_i \geqslant 2$, and $n_1 \geqslant \cdots \geqslant n_{g-2}$. Then

$$2g-2 = \prod_{i=1}^{g-2} n_i \geqslant 2^{g-2}.$$

Hence $g = 3, 4$ or 5, and if $g = 3$ then $V_4 = F_4$; if $g = 4$ then $V_6 = F_2 \cap F_3$; and if $g = 5$ then $V_8 = F_2' \cap F_2'' \cap F_2'''$. The converse assertion follows from the adjunction formula

$$K_V \sim \left(\sum_{i=1}^{g-2} n_i - g - 2\right) H,$$

with H the hyperplane section of V, giving the canonical class of a complete intersection, and from the equality $h^0(\mathcal{O}_V(1)) = g + 2$, which is an easy consequence of the formulas for the cohomology $H^i(\mathcal{O}_V(n))$ of complete intersections. The proposition is proved.

(1.4) EXAMPLES. Let us give some examples of Fano 3-folds $V_{2g-2} \subset \mathbf{P}^{g+1}$ which are complete intersections inside some rather simple types of varieties.

(i) $g = 6$; V_{10} is a section of the Grassmannian $G(1, 4)$ of lines in $\mathbf{P}^4$ by two hyperplanes and a quadric in the natural Plücker embedding $G(1, 4) \subset \mathbf{P}^9$. Since $\operatorname{Pic} G(1, 4) \simeq \mathbf{Z}$, with a generator provided by the hyperplane section, by Lefschetz' theorem $\operatorname{Pic} V_{10} = \mathbf{Z} \cdot H$, with $H \sim -K_V$ the class of a hyperplane section (see [15], Lecture 4).

Note that since the Grassmannian $G(1, 3)$ of lines in $\mathbf{P}^3$ in its natural projective embedding is a quadric of $\mathbf{P}^5$, we can consider $V_6 \subset \mathbf{P}^5$ (with $g = 4$) as sections of $G(1, 3)$ by some cubic—in the classical terminology this is called a "cubic complex" of lines in $\mathbf{P}^3$.

(ii) $g = 7$; let $W = \mathbf{P}^2 \times \mathbf{P}^2$, embedded in $\mathbf{P}^8$ by the Segre embedding, and let Q be a sufficiently general quadric of $\mathbf{P}^8$. Then $V_{12} = W \cap Q$ is a Fano 3-fold of index 1 with $\operatorname{Pic} V_{12} \simeq \mathbf{Z} \oplus \mathbf{Z}$. Indeed, since $\mathcal{K}_W \simeq \mathcal{O}_W(-3)$ we get from the adjunction formula that $\mathcal{K}_{V_{12}}^{-1} \simeq \mathcal{O}_{V_{12}}(1)$. By the Lefschetz theorem $\operatorname{Pic} V_{12} \simeq \operatorname{Pic} W \simeq \mathbf{Z} \oplus \mathbf{Z}$ is generated by the classes $\mathrm{pr}_i^* \mathcal{O}_{\mathbf{P}^2}(1)$, where $\mathrm{pr}_i\colon \mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^2$ for $i = 1$ and 2 are the projections onto the ith factor. Since $\deg V_{12} = 12$ is not divisible by any cube greater than 1, the index of V_{12} cannot be greater than 1. Other examples of Fano 3-folds $V_{12} \subset \mathbf{P}^8$ with $\operatorname{Pic} V_{12} \simeq \mathbf{Z}$ will be considered in §6.

(iii) $g = 8$; the intersection of the Grassmannian $G(1, 5) \subset \mathbf{P}^{14}$ by five hyperplanes in general position is a Fano 3-fold $V_{14} \subset \mathbf{P}^9$. By the Lefschetz theorem Pic $V_{14} = \mathbf{Z} \cdot H$, where H is the hyperplane section (see Lecture 4 of [15]).

Another type of example is given by the anticanonical models of Fano 3-folds of index $r \geqslant 2$. Some of these are complete intersections inside Veronese varieties; that is, images of $\mathbf{P}^d$ in $\mathbf{P}^N$ under embeddings corresponding to $\mathcal{O}_{\mathbf{P}^d}(n)$:

(iv) $g = 4d + 1, d = 2, 3, \ldots, 7$; $V_{2g-2} \subset \mathbf{P}^{g+1}$ are Fano 3-folds of index 2 (see 4.2, Part I).

(v) $g = 28$; $V_{54} \subset \mathbf{P}^{29}$ is a Fano 3-fold of index 3–the anticanonical model of a quadric 3-fold.

(vi) $g = 33$; $V_{64} \subset \mathbf{P}^{34}$ is a Fano 3-fold of index 4–the 4-fold Veronese embedding of projective space $\mathbf{P}^3$.

(1.5) REMARK. One can show that every Fano 3-fold $V_{10} \subset \mathbf{P}^7$ with Pic $V_{10} \simeq \mathbf{Z}$ is a section of the Grassmannian $G(1, 4) \subset \mathbf{P}^9$; it seems plausible that the analogous statement for $V_{14} \subset \mathbf{P}^9$ is also true.

In the sequel we will use the following result.

(1.6) LEMMA (the Noether-Enriques-Petri theorem; see for example [18]). *Let $X \subset \mathbf{P}^{g-1}$ be a smooth canonical curve of genus $g \geqslant 3$. Then the following assetions are true*:

(i) *X is projectively normal in $\mathbf{P}^{g-1}$.*

(ii) *If $g = 3$, then X is a plane curve of genus 4. If $g \geqslant 4$, then X is the intersection of quadrics and cubics of $\mathbf{P}^{g-1}$ passing through X.*

(iii) *X fails to be an intersection of quadrics only in the following cases*:

a) *X is trigonal (that is, it has a 1-dimensional linear system g^1_3 of degree 3)*;

b) *X is a curve of genus 6 isomorphic to a plane curve of degree 5.*

(iv) *In cases* (iii) *the quadrics of $\mathbf{P}^{g-1}$ through X intersect in a surface F, which is one of the following*:

a) *a quadric in $\mathbf{P}^3$ (possibly singular) if $g = 4$*;

b) *a nonsingular rational normal scroll of degree $g - 2$ in $\mathbf{P}^{g-1}$ in case* (iii, a) *for $g \geqslant 5$, with the system g^1_3 cut out on X by the ruling of F*;

c) *the Veronese surface $F_4 \subset \mathbf{P}^5$.*

From this result and from Lemmas (2.9 and 2.10, Part I) we at once get

(1.7) PROPOSITION (compare 4.4, Part I). *Let $V = V_{2g-2} \subset \mathbf{P}^{g+1}$ be a Fano 3-fold, and suppose $g \geqslant 4$. Then the following assertions are true*:

(i) *V is projectively normal in $\mathbf{P}^{g+1}$.*

(ii) *V is an intersection of quadrics and cubics.*

(iii) *V is an intersection of quadrics if and only if it does not have a smooth canonical curve section which is trigonal.*

PROOF. Only (iii) requires proof. If V is an intersection of quadrics in $\mathbf{P}^{g+1}$ then any canonical curve section $X = V \cap \mathbf{P}^{g-1}$ is an intersection of quadrics in $\mathbf{P}^{g-1}$. Hence according to (1.6, iii) X cannot be trigonal. Conversely, if a smooth canonical curve section $X \subset V$ is not trigonal, and is not a curve of type (1.6, iii, b), then according to (1.6, iii), to (i) of the proposition, and to Lemma (2.10, Part I) V is an intersection of quadrics. It remains to prove that on a smooth V a canonical curve section X cannot be of type (1.6, iii, b). Passing to the hyperplane section it is enough to prove that a smooth $K3$ surface $H \subset \mathbf{P}^g$ cannot have such a curve as section. This is proved in [13], §(7.12). The proof is complete.

§2. Trigonal Fano 3-folds

(2.1) DEFINITION. A Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$ will be called *trigonal* if it contains a smooth trigonal canonical curve section X_{2g-2}.

(2.2) LEMMA. *On a trigonal* 3*-fold V every smooth canonical curve section is trigonal.*

PROOF. As we showed at the end of the proof of Proposition (1.7), V cannot contain smooth curves of type (1.6, iii, b). According to (1.7) V is not an intersection of quadrics, and hence according to Lemma (2.10, Part I) every smooth canonical curve X fails to be an intersection of quadrics. By (1.6, iii) X must be a trigonal curve, since case (1.6, iii, b) was excluded at the end of the proof of Proposition (1.7). The lemma is proved.

(2.3) PROPOSITION. *Let $V_{2g-2} \subset \mathbf{P}^{g+1}$ be a trigonal* 3*-fold with $g \geqslant 5$. Let W denote the closed subscheme of $\mathbf{P}^{g+1}$ which is the intersection of all quadrics of $\mathbf{P}^{g+1}$ containing V_{2g-2}. Then $W = W_{g-2} \subset \mathbf{P}^{g+1}$ is a nonsingular rational scroll of dimension* 4 *(see Definition* 2.7, *Part* I*).*

PROOF. Let $\mathbf{P}^{g-1} \subset \mathbf{P}^{g+1}$ be a sufficiently general linear subspace such that $X = V_{2g-2} \cap \mathbf{P}^{g-1}$ is a smooth irreducible curve. Then $F = W \cap \mathbf{P}^{g-1}$ is cut out by the quadrics of $\mathbf{P}^{g-1}$ containing X, and by (1.6, iv, b) F is a smooth rational scroll, and is a surface of degree $g-2$ in $\mathbf{P}^{g-1}$. Hence there is a reduced irreducible component W^0 of W containing V_{2g-2} and such that $\dim W^0 = 4$ and $\deg W^0 = g-2$. Then W^0 satisfies the condition

$$\deg W^0 = \operatorname{codim} W^0 + 1.$$

According to the classification in (2.8 and 2.5, Part I) each such variety is an intersection of quadrics. As is well known (see for example [18]), the number of linearly independent quadrics containing $F \subset \mathbf{P}^{g-1}$ is $(g-2)(g-3)/2$, and by Lemma (2.10, Part I) the same number contain W^0, and at least this number contain W, since their restrictions to $\mathbf{P}^{g-1}$ cut out the surface F. Since $W^0 \subset W$, it follows that $W^0 = W$.

It remains to show that W is nonsingular. Suppose that $w \in W$ is a singular point. We can choose a hyperplane $\mathbf{P}^g \subset \mathbf{P}^{g+1}$ through w such that the hyperplane section $H = V_{2g-2} \cap \mathbf{P}^g$ is nonsingular. Indeed, by Bertini's theorem, the general element H of the linear system on V_{2g-2} cut out by hyperplanes through w can only have singularities at the base point w, and only then if $w \in V_{2g-2}$. However, since V_{2g-2} is nonsingular, through any point $v \in V_{2g-2}$ one can pass a hyperplane section which is smooth at this point. This shows that there exists a smooth surface section $H = V_{2g-2} \cap \mathbf{P}^g$ with $w \in W \cap \mathbf{P}^g$. Since $w \in W$ is singular, it remains a singular point on the section $\overline{W} = W \cap \mathbf{P}^g$.

Repeating the above argument for $\overline{W}$ and H, we prove that there exists a $\mathbf{P}^{g-1} \subset \mathbf{P}^g$ containing w and such that $X = V_{2g-2} \cap \mathbf{P}^{g-1}$ is a smooth curve and $F = W \cap \mathbf{P}^{g-1}$ is a singular surface, cut out by quadrics of $\mathbf{P}^{g-1}$ containing X. This contradicts (1.6, iv, b). Hence W is nonsingular, and this completes the proof of the proposition.

(2.4) According to (2.4–2.6, Part I) there exist integers $d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 > 0$ such that $W = \varphi_{\mathcal{M}}(\mathbf{P}(\mathscr{E}))$, where

$$\mathscr{E} = \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(d_4)$$

and $\mathscr{M} = \mathcal{O}_{\mathbf{P}(\mathscr{E})/\mathbf{P}^1}(1)$ is the tautological invertible sheaf. By Lemma (2.5, Part I) $\mathscr{M}$ is very ample, since $d_i > 0$ for $i = 1, \ldots, 4$. Hence $\varphi_{\mathscr{M}}: \mathbf{P}(\mathscr{E}) \xrightarrow{\sim} W$ is an isomorphism. Let M and L be the divisor classes on W corresponding under the isomorphism $\varphi_{\mathscr{M}}$ to the invertible sheaves $\mathscr{M}$ and $\mathscr{L} = f^*\mathcal{O}_{\mathbf{P}^1}(1)$, where $f: \mathbf{P}(\mathscr{E}) \longrightarrow \mathbf{P}^1$ is the natural morphism. Then from the adjunction formula $K_V \sim V \cdot (V + K_W)$ and the formulas for the canonical classes

$$K_V \sim -V \cdot M, \quad K_W \sim -4M - \left(2 - \sum_{i=1}^{4} d_i\right) L$$

(see 7.5, Part I) we get at once

$$V \sim 3M + \left(2 - \sum_{i=1}^{4} d_i\right) L, \tag{2.4.1}$$

where $V = V_{2g-2} \subset W$ is the trigonal Fano 3-fold. Thus to describe the trigonal 3-folds with $g \geqslant 5$ it suffices to find all possible W (or all possible values of the integers $d_1 \geqslant \cdots \geqslant d_4 > 0$) for which the linear system

$$\left| 3M + \left(2 - \sum_{i=1}^{4} d_i\right) L \right|$$

contains a smooth divisor V. Recall (2.5, Part I) that

$$g = \sum_{i=1}^{4} d_i + 2 \tag{2.4.2}$$

and hence

$$\deg V = 2g - 2 = 2\sum_{i=1}^{4} d_i + 2,$$

$$\deg W = g - 2 = \sum_{i=1}^{4} d_i. \tag{2.4.3}$$

(2.5) THEOREM. *Trigonal Fano 3-folds V_{2g-2} with $g \geqslant 5$ only exist for the following values of the invariants*:

N	d_1	d_2	d_3	d_4	g	
1	1	1	1	1	6	*given by V: $t_0 F_3 + t_1 G_3 = 0$, with (t_0, t_1) homogeneous coordinates on $\mathbf{P}^1$ and F_3, G_3 cubic forms on $\mathbf{P}^3$.*
2	2	1	1	1	7	*V is the blow-up of a smooth cubic of $\mathbf{P}^4$ in a plane cubic curve.*
3	2	2	2	2	10	*$V = S_3 \times \mathbf{P}^1$, with $S_3 \subset \mathbf{P}^3$ a smooth cubic surface*

(2.5.1)

PROOF. Inside $W \simeq \mathbf{P}(\mathscr{E})$ we have the distinguished subvarieties Y_{d_i} for $i = 2, 3, 4$ which are the images of the natural embeddings

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(d_i) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(d_4)) \to \mathbf{P}(\mathscr{E}), \text{ for } i = 2, 3, 4.$$

By Lemma (7.4, Part I), every divisor in the nonempty linear system $|aM + bL|$ has multiplicity $\geqslant q$ along Y_{d_i} if and only if the inequality

$$ad_i + b + (d_1 - d_i)(q - 1) < 0 \tag{2.5.2}$$

holds. The irreducibility and nonsingularity of $V \in |3M + (2 - \Sigma_1^4 d_i)L|$ leads to the following two conditions: Y_{d_2} is not contained in V with multiplicity $\geqslant 1$, and Y_{d_4} is not contained in V with multiplicity $\geqslant 2$.

Using (2.5.2) we get the system of inequalities

$$\begin{gathered} 2d_2 + 2 - d_1 - d_3 - d_4 \geqslant 0, \\ d_4 + 2 - d_2 - d_3 \geqslant 0. \end{gathered} \tag{2.5.3}$$

It is elementary to check that the following values are the only solutions to the inequalities (2.5.3):

Case	d_1	d_2	d_3	d_4
1	1	1	1	1
2	2	1	1	1
3	2	2	2	2
4	2	2	1	1
5	3	2	1	1
6	4	2	1	1

(2.5.4)

Now let us show that the first 3 cases of (2.5.4) actually occur; that is, there exists a smooth $V \in |3M + (2 - \Sigma_1^4 d_i)L|$, and that in Cases 4, 5 and 6 the corresponding V always has singularities.

Cases 1 *and* 3. Here $W \simeq \mathbf{P}^3 \times \mathbf{P}^1$, and in Case 1 $\mathscr{M} \simeq p_1^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(1)$, while in Case 3 $\mathscr{M} \simeq p_1^* \mathcal{O}_{\mathbf{P}^3}(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(2)$. Let M_0 be the divisor class of the sheaf $p_1^* \mathcal{O}_{\mathbf{P}^3}(1)$. Then $V \in |3M - 2L| = |3M_0 + L|$ in Case 1, and $V \in |3M - 6L| = |3M_0|$ in Case 3. In either case the linear system $|3M + (2 - \Sigma_1^4 d_i)L|$ is obviously without fixed components and base points, so that the existence of a smooth divisor V follows from Bertini's theorem. Actually, one can check this readily by writing down the general equation for V. In each of these two cases V is of the form shown in the corresponding place in Table (2.5.1). From this one sees in particular that V is a rational variety.

Case 2. Set $\mathscr{M}_0 = \mathscr{M} \otimes \mathscr{L}^{-1}$, and let M_0 be the divisor class corresponding to $\mathscr{M}_0$. Then $V \in |3M_0|$. Let $f\colon \mathbf{P}(\mathscr{E}) = W \longrightarrow \mathbf{P}^1$ be the natural morphism, so that $f_* \mathscr{M}_0 = \mathscr{E}_0 = \mathscr{E} \otimes \mathcal{O}_{\mathbf{P}^1}(-1)$. From this one sees that $\mathscr{M}_0$ is spanned by its global sections, and defines a birational morphism $\varphi_{\mathscr{M}_0}\colon W \longrightarrow \mathbf{P}^4$, which is just the blow-up of the plane $\mathbf{P}^2 = \varphi_{\mathscr{M}_0}(\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1})) \subset \mathbf{P}^4$ (see (1.6, Part I)). The linear system $|3M_0|$ is the inverse image under $\varphi_{\mathscr{M}_0}$ of the system of cubics $|\mathcal{O}_{\mathbf{P}^4}(3)|$ in $\mathbf{P}^4$. Hence in this case $\varphi_{\mathscr{M}_0}|V\colon V \longrightarrow V' \subset \mathbf{P}^4$ is the blow-up of a smooth cubic 3-fold $V' \subset \mathbf{P}^4$ with center in a smooth cubic curve $C = V' \cap \mathbf{P}^2$. This shows that Case 2 occurs.

The theorem will be proved if we show that in Cases 4–6 in Table (2.5.4) V must be singular.

Cases 4–6. Here $d_3 = d_4 = 1$ and every divisor $V \in |3M + (2 - \Sigma_1^4 d_i)L|$ contains the quadric $Y_{d_3} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)) \subset W$ with multiplicity 1. Indeed, setting $i = 3$, $q = 1$, $d_3 = d_4 = 1$, $a = 3$ and $b = (2 - \Sigma_1^4 d_i)$ in (2.5.2), we get the inequality $3 - d_1 - d_3 < 0$, which holds in each of the Cases 4–6. On the other hand, if we set $q = 2$ under the same conditions, (2.5.2) does not hold. Hence the general element

$$V \in \left| 3M + \left(2 - \sum_{i=1}^{4} d_i\right) L \right|$$

contains Y_{d_3} with multiplicity 1. It is not difficult to calculate the dimension of the linear system $|3M + (2 - \Sigma_1^4 d_i)L|$, using the isomorphism induced by the natural map $f\colon W \longrightarrow \mathbf{P}^1$:

$$h^0\left(\mathcal{O}_W\left(3M + \left(2 - \sum_{i=1}^{4} d_i\right)L\right)\right) = h^0\left(\mathbf{P}^1, S^3\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}\left(2 - \sum_{i=1}^{4} d_i\right)\right)$$

and to check that in any of the Cases 4–6 it is nonempty.

Now let us show that V cannot be nonsingular. By the Lefschetz theorem the embedding $i\colon V \hookrightarrow W$ induces an isomorphism $i^*\colon \operatorname{Pic} W \xrightarrow{\sim} \operatorname{Pic} V$. Hence as generators of Pic V we can take $i^*\mathcal{M}$ and $i^*\mathcal{L}$. Since V is nonsingular, Y_{d_3} is a Cartier divisor on V; hence there exist integers α and β such that

$$Y_{d_3} \sim V \cdot (\alpha M + \beta L) \sim \left(3M + \left(2 - \sum_{i=1}^{4} d_i\right) L \cdot \alpha M + \beta L\right)_W .$$

Restricting this relation to the fiber L:

$$Y_{d_3} \cdot L \sim 3\alpha M^2 \cdot L$$

and intersecting with M, we get the equality $1 = 3\alpha$, which is impossible for integral α. Hence V has singularities on Y_{d_3}, and Y_{d_3} is not a Cartier divisor on V. This completes the proof of the theorem.

(2.6) Corollary. *Every trigonal Fano* 3*-fold* V *has a pencil of cubic surfaces, the restriction to* V *of the pencil* $|L|$ *on* W; *and* Pic $V \simeq$ Pic $W \simeq \mathbf{Z} \oplus \mathbf{Z}$.

(2.7) Remark. The method of proof of Theorem (2.5) can also be applied to trigonal Fano 3-folds having only isolated double points as singularities. Simple computations show that the degree of such varieties is also bounded: $g \leqslant 10$.

§3. The family of lines on a Fano 3-fold $V_{2g-2} \subset \mathbf{P}^{g+1}$

(3.1) Let $V = V_{2g-2} \subset \mathbf{P}^{g+1}$ be a Fano 3-fold with $g \geqslant 3$, and let $G(1, g+1)$ be the Grassmannian of lines of $\mathbf{P}^{g+1}$; let $\Gamma = \Gamma(V)$ be the closed subscheme of $G(1, g+1)$ parametrizing lines lying on V, and let $S = S(V)$ be the family of lines of V (the restriction to Γ of the universal family of lines on $G(1, g+1)$):

$$\begin{array}{ccc} S & \xrightarrow{p} & V \\ {\scriptstyle \pi}\downarrow & & \\ \Gamma & & \end{array}$$

is the diagram of natural morphisms, and write $R = R(V)$ for the image $p(S) \subset V$. A line of V and the corresponding fiber of S will be denoted by one and the same letter Z. $\mathscr{N}_{Z/V}$ will denote the normal sheaf to Z in V, and $\mathscr{N}_{Z/S}$ the normal sheaf to Z in S. Since π: $S \longrightarrow \Gamma$ is a locally trivial fibration, $\mathscr{N}_{Z/S}$ is a free sheaf of rank equal to the dimension of the Zariski tangent space to Γ at the corresponding point $\gamma = \pi(Z)$.

(3.2) LEMMA. *Suppose that $S \neq \varnothing$. Let $Z \subset V$ be a line, and $\gamma \in \Gamma$ the corresponding point. Then the following assertions are true*:

(i) *There are just two possibilities for the normal sheaf* $\mathscr{N}_{Z/V}$:

a) $\mathscr{N}_{Z/V} \simeq \mathscr{O}_Z(-1) \oplus \mathscr{O}_Z$, *or*

b) $\mathscr{N}_{Z/V} \simeq \mathscr{O}_Z(-2) \oplus \mathscr{O}_Z(1)$.

(ii) *Let m_γ be the maximal ideal in the local ring $\mathscr{O}_\gamma$ of the point $\gamma \in \Gamma$; then*

$$\dim m_\gamma/m_\gamma^2 = h^0(\mathscr{N}_{Z/V}) = \begin{cases} 1, & \textit{in case a)}, \\ 2, & \textit{in case b)} \end{cases}$$

for the Zariski tangent space at γ, and, in a neighborhood of γ,

$$2 \geqslant h^0(\mathscr{N}_{Z/V}) \geqslant \dim \Gamma \geqslant h^0(\mathscr{N}_{Z/V}) - h^1(\mathscr{N}_{Z/V}) = 1.$$

PROOF. (i) Let us first show that through any line $Z \subset V$ we can pass a nonsingular hyperplane section. Consider the linear system $|H - Z|$ of hyperplane sections through Z. It is clear that this linear system has base locus just Z, since this is the case for the linear system of hyperplanes through Z in the ambient $\mathbf{P}^{g+1}$. By Bertini's theorem almost all surfaces of $|H - Z|$ have no singularities outside Z. Let us show that there exists a suface in $|H - Z|$ which is nonsingular along Z. The projection π_Z: $\mathbf{P}^{g+1} \longrightarrow \mathbf{P}^{g-1}$ from Z is given by the linear system of hyperplanes through Z. Let σ_Z: $P' \longrightarrow \mathbf{P}^{g+1}$ be the blow-up of Z. Then we have a diagram

$$\begin{array}{ccccc} V' & \subset & P' & & \\ \downarrow{\scriptstyle \sigma_Z} & & \downarrow{\scriptstyle \sigma_Z} & \searrow{\scriptstyle f} & \\ V & \subset & \mathbf{P}^{g+1} & \xrightarrow{\pi_Z} & \mathbf{P}^{g-1} \end{array} \tag{3.2.1}$$

where $f = \pi_Z \circ \sigma_Z$ is a morphism, and σ_Z: $V' \longrightarrow V$ is the blow-up of Z in V. V' is nonsingular and the morphism f: $V' \longrightarrow \mathbf{P}^{g-1}$ is given by the linear system $|\sigma_Z^* H - \sigma_Z^{-1}(Z)|$. This linear system is without fixed components and base points, since it is the restriction to V' of a linear system on P' with this property. The restriction of $|\sigma_Z^* H - \sigma_Z^{-1}(Z)|$ to the ruled surface $Z' = \sigma_Z^{-1}(Z)$ is also without fixed components and base points, and is a certain linear system consisting of sections of Z' over Z. Hence there exists a surface $H' \in |\sigma_Z^* H - Z'|$ such that the curve $Z' \cap H'$ is an irreducible (and hence nonsingular) section of Z'. Under these circumstances the morphism σ_Z: $H' \longrightarrow \sigma_Z(H')$ is an isomorphism, and since H' has no singular points on $Z' \cap H'$, $\sigma_Z(H')$ is a surface with no singularities on Z.

Thus there exists a smooth hyperplane section H of V through Z. Since H is a $K3$ surface, $(Z \cdot Z)_H = -2$. Hence we have an exact sequence:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}_Z(-2) & \to & \mathcal{N}_{Z/V} & \to & \mathcal{O}_Z(1) & \to 0 \\ & \wr\| & & & & \wr\| & \\ & \mathcal{N}_{Z/H} & & & & \mathcal{N}_{H/V}|Z & \end{array} \tag{3.2.2}$$

From this one sees that the invertible sheaf of maximal degree in $\mathcal{N}_{Z/V}$ has degree 0 or 1; these lead to cases a) and b) respectively.

(ii) These assertions are a direct consequence of infinitesimal deformation theory [7], and the trivial computations of $h^i(\mathcal{N}_{Z/V})$ in cases a) and b) of (i). The lemma is proved.

(3.3) PROPOSITION. *Suppose that $S \neq \varnothing$. Let Γ^0 be an irreducible closed component of the scheme Γ, let S^0 be the corresponding family of lines over Γ^0, and let $R^0 = p(S^0)$. Then the following assertions are true*:

(i) *If* $\dim \Gamma^0 = 2$ *then R^0 is a projective plane lying on V, and $\Gamma^0 \simeq \mathbf{P}^2$; conversely, if V contains a plane then there exists a 2-dimensional component $\Gamma^0 \subset \Gamma$ with $\Gamma^0 \simeq \mathbf{P}^2$.*

(ii) *If* $\dim \Gamma^0 = 1$ *and Γ^0 is generically reduced then the scheme Γ^0 can have at most double points as singularities, and possibly 0-dimensional embedded components; R^0_{red} is a ruled surface on V, and the morphism p: $S^0_{\mathrm{red}} \longrightarrow R^0_{\mathrm{red}}$ is birational.*

(iii) *Γ^0 is nonreduced at the generic point if and only if R^0_{red} is a 2-dimensional cone contained in V.*

PROOF. (i) If $\dim \Gamma^0 = 2$ then, according to (3.2, ii), $\dim m_\gamma/m_\gamma^2 = \dim \Gamma^0 = 2$ for all $\gamma \in \Gamma^0$; hence Γ^0 is a smooth surface. In this case for any line Z with $\pi(Z) \in \Gamma^0$ we have $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$, since $h^0(\mathcal{N}_{Z/V}) = \dim \Gamma^0 = 2$ (see (3.2)). The image R^0 of the morphism p: $S^0 \longrightarrow V$ obviously has dimension $\geqslant 2$ (if S^0 is a single line then $\dim \Gamma^0 = 0$, contradicting (3.2, ii)). Suppose that $\dim R^0 = 3$; then since the morphism p: $S^0 \longrightarrow V$ is proper (S^0 is complete), $R^0 = V$; and since $\dim S^0 = \dim V$, p is generically finite. Consider the differential dp, restricted to the line $Z \subset S_0$. It is obvious that, on the tangent bundle T_Z to Z, dp is the identity isomorphism. We have a homomorphism of the normal sheaves defined:

$$\begin{array}{ccc} dp: \mathcal{N}_{Z/S^0} & \to & \mathcal{N}_{Z/V} \\ \wr\| & & \wr\| \\ \mathcal{O}_Z \oplus \mathcal{O}_Z & \to & \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1). \end{array} \tag{3.3.1}$$

Since p is generically finite, for a sufficiently general line Z, by Sard's theorem the morphism dp has rank 2 at the general point of Z. But the lower arrow in (3.3.1) shows that no such morphisms exist. Indeed, each of the summands $\mathcal{O}_Z$ can only have a nonzero homomorphism into the second summand of $\mathcal{N}_{Z/V}$, so that dp has at least a 1-dimensional kernel. Hence our hypothesis that $\dim R^0 = 3$ is invalid, and there remains the single possibility $\dim R^0 = 2$.

The surface R^0 has a 2-dimensional family of lines. This property is enjoyed only by a plane $\mathbf{P}^2 \subset V$. In this case it is clear that $\Gamma^0 \simeq \mathbf{P}^2$. The converse assertion in (i) is obvious.

(ii) The first assertion follows from Lemma (3.2, ii). Furthermore $\dim S^0_{\mathrm{red}} = 2$, and, as pointed out above, $\dim R^0 \neq 1$. Hence R^0_{red} is a ruled surface, and the morphism p: $S^0_{\mathrm{red}} \longrightarrow R^0_{\mathrm{red}}$ is generically finite. p cannot have degree greater than 1, since otherwise through almost all points $z \in R^0_{\mathrm{red}}$ there would pass more than 1 line from one and the same irreducible family S^0. This is impossible, since $\dim \Gamma^0 = 1$. Hence $p\colon S^0_{\text{red}} \to R^0_{\text{red}}$ is birational.

(iii) As we have just shown, $p\colon S^0_{\text{red}} \to V$ maps S^0_{red} birationally to R^0_{red}. If Γ^0 is nonreduced at the generic point, then for almost all lines $Z \subset S^0$ the normal sheaf $\mathcal{N}_{Z/V}$ is isomorphic to $\mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$ (see 3.2). The restriction of the differential dp_{red} to the normal sheaf of a sufficiently general line $Z \subset S^0_{\text{red}}$ defines a nonzero homomorphism of sheaves:

$$\begin{array}{cccc} dp_{\text{red}}\colon & \mathcal{N}_{Z/S^0_{\text{red}}} & \to & \mathcal{N}_{Z/V} \\ & \wr\| & & \wr\| \\ & \mathcal{O}_Z & \to & \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1) \end{array} \tag{3.3.2}$$

There does not exist a nonzero homomorphism of $\mathcal{O}_Z$ into $\mathcal{O}_Z(-2)$. A (nonzero) homomorphism $\mathcal{O}_Z \to \mathcal{O}_Z(1)$ has cokernel k_z supported on a single point $z \in Z$. This means that the general line $Z \subset S^0_{\text{red}}$ intersects the closed set $D \subset S^0_{\text{red}}$, the degeneracy locus of p_{red}, in a single point. Let $D^0 \subset D$ be the irreducible component of D meeting each Z, so that D^0 is a section of the morphism $\pi_{\text{red}}\colon S^0_{\text{red}} \to \Gamma^0_{\text{red}}$. Since $p_{\text{red}}\colon S^0_{\text{red}} \to R^0_{\text{red}}$ is a birational morphism and dp_{red} degenerates in a direction normal to Z (that is, tangent to D^0), this means that p_{red} contracts D^0 to a point of R^0_{red}. Hence $R^0_{\text{red}} \subset V$ is a cone.

It remains to show that R^0_{red} cannot be a cone if Γ^0 is reduced at its general point. In this case for all but a finite number of lines $Z \subset S^0$ we have $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$ and $\mathcal{N}_{Z/S^0} \simeq \mathcal{O}_Z$ (see 3.2, ii), and the homomorphism $dp\colon \mathcal{N}_{Z/S^0} \to \mathcal{N}_{Z/V}$ is nonzero. There is only one possibility, $dp\colon \mathcal{N}_{Z/S^0} \xrightarrow{\sim} \mathcal{O}_Z$, the projection onto the second summand of $\mathcal{N}_{Z/V}$. Hence the birational morphism $p\colon S^0 \to R^0$ is an immersion in a neighborhood of $Z \subset S^0$, and so R^0_{red} cannot be a cone. The proposition is proved.

(3.4) THEOREM. *In the notation of* (3.1) *suppose that* Pic $V = \mathbf{Z} \cdot H$, *where H is the hyperplane class, and suppose that V contains a line Z. Then the following assertions are true*:

(i) *Any irreducible component $\Gamma^0 \subset \Gamma$ is 1-dimensional.*

(ii) *If $g \geqslant 4$, Γ^0 is reduced at a generic point (that is, V does not contain any 2-dimensional cones, according to* (3.3, iii)).

(iii) *If $g \geqslant 4$, then every line $Z \subset V$ meets only a finite number of other lines.*

(iv) *If d is the integer such that $R \sim dH$, then for $g \geqslant 4$ every line Z not contained in the singular locus of R meets $d + 1$ other lines of V (counted with multiplicities).*

PROOF. (i) If $\dim \Gamma^0 = 2$ then according to (3.3, i) V contains a plane $\mathbf{P}^2$, contradicting the condition Pic $V = \mathbf{Z} \cdot H$.

(ii) According to (3.3, iii) we have to show that under the stated hypotheses V does not contain any 2-dimensional cones. Suppose that $R^0_{\text{red}} \subset V$ is a cone with vertex $v \in V$. Then every hyperplane of $\mathbf{P}^{g+1}$ tangent to V at v contains R^0_{red}. Such hyperplanes cut out on V the linear system $|H - 2v|$ which contains R^0_{red} as a fixed component, and having dimension $\geqslant g - 3$. Since Pic $V = \mathbf{Z} \cdot H$, every hyperplane section H is irreducible. Hence for $g \geqslant 4$ we obtain a contradiction to the assumption that R^0_{red} is a cone. Note that for $g = 4$ a cone R^0_{red} can only be a hyperplane section.

(iii) Suppose that the line Z meets infinitely many other lines of V. According to (ii) V does not contain any cones, so that at least one line of V must pass through each point

$z \in Z$. Let G be an irreducible component of the ruled surface R swept out by lines of V meeting Z. Let $m \geqslant 1$ be the integer such that $G \sim mH$, and let r be the multiplicity of G in a general point of Z. The projection $\pi_Z \colon V \longrightarrow \mathbf{P}^{g-1}$ from Z contracts to a point every line meeting Z, and hence contracts G onto some curve (G cannot be mapped to a point, since otherwise $G \subset V$ would have to be a plane, contradicting (i)). Let us show that $r = m + 1$ or $m + 2$. Any plane of $\mathbf{P}^{g+1}$ containing Z can contain only 2 further lines of V, and the case of 2 occurs only if $g = 4$. Indeed, if $g > 4$ then, since Pic $V \simeq \mathbf{Z}$, V is not trigonal, and according to (1.7, ii) and (1.3) V is an intersection of quadrics. Hence $V \cap \mathbf{P}^2$ can only be a curve of degree $\leqslant 2$, since V does not contain any planes. If $g = 4$ then the unique quadric containing V (see (1.3)) could contain $\mathbf{P}^2$, in which case $V \cap \mathbf{P}^2$ is a curve of order 3.

Let us choose a sufficiently general hyperplane section H containing Z (see the beginning of the proof of Lemma 3.2). The $G \cap H = rZ + \Sigma_1^N Z_i$, where $Z_i \subset G$ are certain lines. We have

$$m = (Z_i \cdot mH)_V = (Z_i \cdot G)_V = \left(Z_i \cdot rZ + \sum_{i=1}^{N} Z_i\right)_H$$
$$= r + (Z_i \cdot Z_i)_H + \left(Z_i \cdot \sum_{j \neq i} Z_j\right)_H = r - 2 + \delta,$$

where $\delta = 0$ or 1, since every line of G meeting Z_i lies in the plane $\mathbf{P}^2$ spanned by the lines Z and Z_i. Hence $r = m + 2 - \delta$.

Let $\sigma \colon V' \longrightarrow V$ be the blow-up of Z, let $Z' = \sigma^{-1}(Z)$ be the exceptional ruled surface, and let G' be the proper transform on V' of G. The linear system $|\sigma^* H - Z'|$ defines the morphism $f_Z \colon V' \longrightarrow \mathbf{P}^{g-1}$, the resolution of the projection π_Z. Since G' is contracted by f_Z, we have

$$(\sigma^* H - Z')^2 \cdot G' = 0.$$

Hence

$$m(\sigma^* H - Z')^3 = (\sigma^* H - Z')^2 \cdot (m\sigma^* H - rZ' + (2-\delta)Z')$$
$$= (\sigma^* H - Z')^2 \cdot G' + (2-\delta)(\sigma^* H - Z')^2 \cdot Z'. \tag{3.4.1}$$

Now we use the multiplication table in the Chow ring $A(V')$ (see 2.11, Part I).

Simple computations give

$$(\sigma^* H - Z')^3 = 2g - 6, \quad (\sigma^* H - Z')^2 \cdot Z' = 3.$$

Substituting in (3.4.1), we get

$$m(2g - 6) = 3(2 - \delta). \tag{3.4.2}$$

Since $(2 - \delta) = 1$ or 2, (3.4.2) is only possible for $g = 4$ and $m = 3$.

It remains to exclude this case. Here V is a complete intersection of a quadric and a cubic in $\mathbf{P}^5$, and the morphism $f_Z \colon V' \longrightarrow \mathbf{P}^3$ has degree 2 (generically), with ramification locus a surface $D \subset \mathbf{P}^3$ of degree 6. The curve $f_Z(G')$ is obviously contained in D, and has degree equal to the number of lines Z_i, $Z_i \neq Z$, in the intersection $G \cap H$; that is, $6m - r = 13$. Let D_1 denote the surface $f_Z(Z')$. We have

$$\deg D_1 = (\sigma^* H - Z')^2 \cdot Z' = 3.$$

Let us show that $D_1 \subset D$. If Z' were not contained in the branch locus of the morphism f_Z then the involution interchanging the leaves of the double cover $f_Z: V' \longrightarrow \mathbf{P}^3$ would take Z' into some surface Z'', and then

$$f_Z^{-1}(D_1) = Z' \cup Z'' \cup G' \sim 3(\sigma^* H - Z').$$

But since $G' \sim 3\sigma^* H - 5Z'$ we have $Z'' \sim Z'$. But this is only possible if $Z' = Z''$, since Z' is an exceptional surface, and Z'' is irreducible. Hence $D_1 \subset D$. Let D_2 be the complementary cubic surface; that is, $D = D_1 \cup D_2$. The curve $C = D_1 \cup D_2$ is contained in the singular locus of the branch divisor D; hence $f_Z^{-1}(C)$ is a surface on V' such that the image $\sigma(f_Z^{-1}(C))$ on V is swept out by lines meeting Z or conics intersecting Z twice. However, it is not difficult to see that on a smooth $V = V_{2\cdot 3}$ every line can only intersect twice with 2 conics: these lie in the two planes contained in the quadric through V, and intersect along this line. Hence $\sigma(f_Z^{-1}(C))$ is a surface swept out by lines meeting Z. Now note that

$$\deg \sigma\, q\, (f_Z^{-1}(C)) < \deg G,$$

since $\deg C = 9 < \deg f_Z(G') = 13$. But this contradicts (3.4.2). The assertion (iii) is proved.

(iv) A proof of this assertion going back to Fano [4] was reproduced in [15], Lecture 4, §2 (proof of Lemma 5). We will not repeat the arguments given in [15], noting only that these work under the assumption that the given line meets only a finite number of other lines, and that the line is not singular for the ruled surface R. Both of these conditions hold in our case: the first because of (iii), and the second by hypothesis. The proof of Theorem (3.4) is complete.

(3.5) REMARKS. (i) It is not known if there exist (smooth) Fano 3-folds V_{2g-2} containing a plane. If Pic $V_{2g-2} \simeq \mathbf{Z}$ then these do not exist by the Lefschetz theorem.

(ii) For $g = 3$ the assertion (3.4, ii) is no longer true. A corresponding example is considered in [14]: this is the diagonal quartic $\Sigma_0^4 x_i^4 = 0$, which contains 40 cones, cut out by the hyperplanes $x_i = \epsilon x_j$ $(i \neq j)$, where ϵ runs through the primitive 8th roots of 1. Each of the cones appears in the surface $R = p(S)$ with multiplicity 2.

(iii) Note that if a $V_{2g-2} \subset \mathbf{P}^{g+1}$ with $g > 3$ does contain a cone R^0, then $\deg R^0 \leqslant 3$. Indeed, R^0 is contained in the intersection $V \cap \mathbf{P}^3$ of V with the tangent space to V at the vertex of the cone. If V is not trigonal and $g \geqslant 5$, then, according to (1.7, iii), V_{2g-2} is an intersection of quadrics; hence $\deg R^0 \leqslant 2$. In the trigonal case, $\deg R^0 \leqslant 3$ according to (1.7, ii).

(iv) Starting from (3.4, i) one shows using elementary methods involving a count of constants that on a V_{2g-2} which is a complete intersection (see (1.3)) or a Grassmannian section (see (1.4)) there do exist lines, and these form a 1-dimensional family (for $g = 3$ see [14]). In [4] Fano asserts that lines exist on every 3-fold V_{2g-2} with Pic $V_{2g-2} = \mathbf{Z} \cdot H$. However, his arguments are not convincing, and until recently a new proof had not been found; in this connection we had the following open problem:

(3.6) FANO'S CONJECTURE I. *Every Fano 3-fold* $V_{2g-2} \subset \mathbf{P}^{g+1}$ *with* Pic $V_{2g-2} = \mathbf{Z} \cdot H$ *contains a line.*

This has recently been proved by V. V. Šokurov [20].

In §5 during the proof of the main Theorem (5.1) we will need some facts on lines of V in general position, and we find it convenient to set forth these results here.

(3.7) LEMMA. *Under the hypotheses of Theorem* (3.4), *on a Fano 3-fold* V_{2g-2} *with* $g \geqslant 4$ *there exists a dense open set* $\mathrm{B} \subset \Gamma$ *in the family of lines* Γ *such that every line* Z *in* B *satisfies the following conditions*:

(i) $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$.

(ii) *The* $d+1$ *lines* $Z_1, \ldots, Z_{d+1}$ *meeting* Z (*see*(4.3, iv)) *are distinct, and for each of them* (i) *holds.*

(iii) *Let* $\sigma\colon V' \longrightarrow V$ *be the blow-up of* Z, *let* $Z' = \sigma^{-1}(Z)$ *be the exceptional ruled surface, and let* Z_i^0 *be the proper transforms of the lines* Z_i (*for* $i = 1, \ldots, d+1$). *Then for all* $i = 1, \ldots, d+1$

$$\mathcal{N}_{Z_i^0/V'} \simeq \mathcal{O}_{Z_i^0}(-1) \oplus \mathcal{O}_{Z_i^0}(-1)$$

and the point of intersection $z_i = Z' \cap Z_i^0$ *does not lie on the negative section of the ruled surface* Z'.

(iv) *For* $g \geqslant 5$ *not more than* 4 *lines pass through every point* $v \in V$, *and no* 3 *of these can lie in a common plane*; *for* $g = 4$ *not more than* 6 *lines pass through every point* $v \in V$, *and no* 4 *of these can lie in a common plane.*

PROOF. According to (3.4, ii) and (3.3, ii), lines with property (i) form an open subset of Γ. The finite number of lines not satisfying (i) only meet a finite number of other lines according to (3.4, iii). Hence all but a finite number of lines of R satisfy (i) and (ii); in the parameter curve Γ they form an open set, which we denote by B. Let us show that for lines in B (iii) is also satisfied.

Let $T = \pi^{-1}(\mathrm{B})$. Then from (i) we deduce, as at the end of the proof of Proposition (3.3), that the natural morphism $p\colon T \longrightarrow V$ is an immersion. Furthermore, two distinct irreducible components of the family S cannot map to one and the same component of the surface R, for otherwise (3.4, ii) would fail. From this and from (3.3, ii) it follows that the morphism $p\colon S \longrightarrow R$ is birational, and hence $p\colon T \longrightarrow V$ is a birational immersion. Hence if $t_1, t_2 \in T$, $t_1 \neq t_2$, are points such that $p(t_1) = p(t_2)$, then the point $v = p(t_1) = p(t_2)$ is a singular point of R, and the images of the tangent spaces to t_1 and t_2 span the whole 3-dimensional tangent space to V at v. Let Z_1 and Z_2 be lines through t_1 and t_2, and let U_1 and U_2 be normal neighborhoods of Z_1 and Z_2 respectively. Then from what we have said above it follows that the image of Z_1 meets $p(U_2)$ transversally at v, and similarly for Z_2 and $p(U_1)$. Hence on performing the blow-up $\sigma_1\colon V_1 \longrightarrow V$ of the line Z_1 on V the normal component $\mathcal{O}_{Z_2} \simeq \mathcal{N}_{Z_2/U_2}$ of the normal sheaf $\mathcal{N}_{Z_2/V}$ is replaced by $\mathcal{O}_{Z_2^0}(-1)$, where $Z_2^0 \subset V_1$ is the proper transform of Z_2. Since $K_{V_1} = \sigma_1^* K_V + \sigma_1^{-1}(Z_1)$, one computes easily that $\det \mathcal{N}_{Z_2^0/V_1} \simeq \mathcal{O}_{Z_2^0}(-2)$. Thus

$$\mathcal{N}_{Z_2^0/V_1} \simeq \mathcal{O}_{Z_2^0}(-1) \oplus \mathcal{O}_{Z_2^0}(-1).$$

An analogous thing happens on blowing up Z_2. This proves the first assertion of (iii). The second assertion follows from the fact that the negative section of the ruled surface Z'

corresponds to the second summand $\mathcal{O}_Z$ in the normal sheaf $\mathcal{N}_{Z/V} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$, and from the previous arguments which show that the direction of each of the $d+1$ lines Z_i is transversal to this normal component.

Let us prove (iv). If $g \geqslant 5$, since Pic $V \simeq \mathbf{Z}$ it follows that V is not trigonal, and is hence an intersection of quadrics in $\mathbf{P}^{g+1}$ (see (1.7)). Let $\mathbf{P}^3_v$ be the projective tangent space to V at v. Then the lines through v are contained in $V \cap \mathbf{P}^3_v$, and since $V \cap \mathbf{P}^3_v$ is cut out by quadrics of $\mathbf{P}^3$ there can be at most 4 of these lines. For the same reason in a plane $\mathbf{P}^2$ there can be at most 2 lines of V, since otherwise $V \cap \mathbf{P}^2$ could not be an intersection of conics of $\mathbf{P}^2$.

In the case $g = 4$, $V = V_{2\cdot 3} \subset \mathbf{P}^5$ is a complete intersection of a quadric and a cubic. Hence $V \cap \mathbf{P}^3_v$ is a curve of degree 6 and at worst can break up into 6 lines. If in one plane $\mathbf{P}^2$ there are more than 2 lines of V, then this $\mathbf{P}^2$ must be entirely contained in the quadric through V, and the cubic through V will cut out on $\mathbf{P}^2$ a curve of degree 3. Here and above we are using the fact that V does not contain a plane (see (3.4)). The proof of the lemma is complete.

§4. The family of conics on a Fano 3-fold of the principal series

(4.1) Let $V = V_{2g-2} \subset \mathbf{P}^{g+1}$ be a Fano 3-fold, and let $\Delta = \Delta(V)$ be the scheme parametrizing the conics on V; this is a closed subscheme of the Hilbert scheme of closed subschemes of $\mathbf{P}^{g+1}$ with Hilbert polynomial $2n+1$. Let $T = T(V)$ denote the universal family of conics over Δ, and let

$$\begin{array}{ccc} T & \xrightarrow{q} & V \\ {\scriptstyle p}\downarrow & & \\ \Delta & & \end{array}$$

be the diagram of natural morphisms. Set $Q = q(T)$. A conic on V (possibly reducible or nonreduced) and the corresponding fiber of T will be denoted by one and the same letter C. As usual, $\mathcal{N}_{C/V}$ will denote the normal sheaf. On decomposing $\mathcal{N}_{C/V}$ into a direct sum, or in representing it as an extension of invertible sheaves, the symbol $\mathcal{O}_C(d)$ will denote the invertible sheaf of degree d on C.

(4.2) LEMMA. *Suppose that on V there exists a smooth conic C. Then there are only the following 4 possibilities for the normal sheaf $\mathcal{N}_{C/V}$:*

a) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$,

b) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$,

c) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2)$,

d) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-4) \oplus \mathcal{O}_C(4)$.

PROOF. If we can pass a smooth hyperplane section H through C, then we have an extension

$$0 \to \mathcal{O}_C(-2) \to \mathcal{N}_{C/V} \to \mathcal{O}_C(2) \to 0, \tag{4.2.1}$$

since $\mathcal{N}_{C/H} \simeq \mathcal{O}_C(-2)$, since C is a curve of genus 0 on the $K3$ surface H, and $(C \cdot C) = -2$. As in (3.2.2) we get from this the first 3 possibilities a), b) and c) for $\mathcal{N}_{C/V}$.

Let $P = P(C)$ be the plane of a conic C in $\mathbf{P}^{g+1}$. If $P \subset V$, then $(C \cdot C)_P = 4$ and we have the exact sequence

$$0 \to \mathcal{O}_C(4) \to \mathcal{N}_{C/V} \to \mathcal{O}_C(-4) \to 0, \tag{4.2.2}$$

which obviously splits as a direct sum. This gives the remaining possibility d).

It remains to show that there are no other possibilities for the decomposition of $\mathcal{N}_{C/V}$ into a direct sum. As in the beginning of the proof of Lemma (3.2), it is not difficult to show that there exists a smooth surface $F \in |2H - C|$ cut out on V by some quadric of $\mathbf{P}^{g+1}$ through C. Since $K_F \sim F \cdot H$ it follows at once from the formula for the genus of C that $(C \cdot C)_F = -4$. From this we get an exact sequence

$$0 \to \mathcal{O}_C(-4) \to \mathcal{N}_{C/V} \to \mathcal{O}_C(4) \to 0. \tag{4.2.3}$$

This sequence shows that apart from cases a), b), c) and d) only one other case is possible:

e) $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-3) \oplus \mathcal{O}_C(3)$.

We will prove that in fact this case does not occur.

Let $D = V \cap P$ be the closed subscheme of V cut out by the plane P. The case $D = P$ has already been considered. Thus we can assume that $\dim D = 1$. Clearly $C \subset D$; let $\mu_C(D)$ be the multiplicity of D in the general point of C.

Let us show that $\mu_C(D) = 1$ or 2, with $\mu_C(D) = 2$ only if $V = V_4 \subset \mathbf{P}^4$, and the plane P touches V along C. According to (1.7, ii), if $g \geqslant 4$ then V is an intersection of quadrics and cubics. Hence $D \subset P$ is also cut out by conics and cubics. It follows that $\mu_C(D) = 1$ for $g \geqslant 4$. If $g = 3$ then $D \subset P$ is a curve of degree 4. If $D = C \cup C^0$, with C^0 a conic distinct from C, then obviously $\mu_C(D) = 1$. There remains only the case where $C = C^0$ and D is a double conic of P; that is, $\mu_C(D) = 2$.

Consider the linear system $|H - C|$ on V. Obviously D and only D is the base scheme of this linear system. Let $\sigma\colon V' \longrightarrow V$ be the blow-up of C, and let $C' = \sigma^{-1}(C)$ be the exceptional ruled surface, having s and f the class of the negative section and the fiber of C' respectively. We have

$$(\sigma^*H - C') \cdot C' = -2\sigma^*H \cdot C'^2 + C'^3 = 2(H \cdot C)_V - \deg\det \mathcal{N}_{C/V} = 4 \tag{4.2.4}$$

(see (2.11), Part I). The linear system $|\sigma^*H - C'|$ cuts out on C' a certain linear system of sections of the ruling; that is, $(\sigma^*H - C') \cdot C' \sim s + \alpha f$ for some integer α. If $\mu_C(D) = 1$, then this linear system is without fixed components; and, conversely, it has a fixed component if $\mu_C(D) = 2$. From (4.2.3) we get

$$4 = (s + \alpha f \cdot s + \alpha f) = s^2 + 2\alpha. \tag{4.2.5}$$

For $\mu_C(D) = 1$ the negative section of C' is not a fixed component, and hence

$$(s \cdot s + \alpha f) \geqslant 0; \text{ that is, } s^2 + \alpha \geqslant 0.$$

Using (4.2.5) we get from this that $\alpha \leqslant 4$ and $s^2 \geqslant -4$. Since $C' = \mathbf{P}(\check{\mathcal{N}}_{C/V})$, with $\check{\mathcal{N}}_{C/V}$ the conormal sheaf, it follows that $\check{\mathcal{N}}_{C/V} \simeq \mathcal{O}_C(-d) \oplus \mathcal{O}_C(d)$ with $d = 0$, 1 or 2, and the same is true of $\mathcal{N}_{C/V}$. Hence in this case only the possibilities a), b) and c) can occur.

Now let $\mu_C(D) = 2$. In this case a hyperplane section H of $V = V_4$ through C is a quartic surface of $\mathbf{P}^3$ which touches P along C. If $x_0 = 0$ is an equation for $P \subset \mathbf{P}^3$ and $(x_0, \dots, x_3)$ are homogeneous coordinates, then $C \subset P$ has equation $Q_2(x_1, x_2, x_3) = 0$, and H is given by an equation of the form

$$Q_2(x_1, x_2, x_3)^2 + x_0F_3(x_0, \ldots, x_3) = 0,$$

where $F_3(x_0, \ldots, x_3)$ is some cubic form. The general hyperplane H is nonsingular at the general point of C, and hence the cubic $F_3(x_0, \ldots, x_3) = 0$ does not contain C. Simple computations show that H has 6 singular points on C—the intersections of C with the cubic $F_3(x_0, \ldots, x_3) = 0$. Hence the linear system $|\sigma^*H - C'|$ cuts out on C' curves in the class $s + \beta f$, with $\beta \geqslant 6$. Since $(s + \beta f \cdot s + \beta f) = 4$ (see 4.2.5), $\beta \geqslant 6$ is only possible if $s^2 \leqslant -8$. Since $s^2 = -2d$, there remains the unique possibility $\beta = 6$ and $d = 4$; that is, case d). The lemma is proved.

(4.3) PROPOSITION. *Suppose that $T \neq \emptyset$, and let T^0 be an irreducible component of the scheme T, Δ^0 the corresponding component of the scheme Δ, and $Q^0 = q(T^0)$ a component of Q. Then the following assertions are true*:

(i) *If for the general conic $C \subset Q^0$ the normal sheaf $\mathcal{N}_{C/V}$ is of type* a) *of Lemma* (4.2), *then Δ^0 is nonsingular at its general point*, $\dim \Delta^0 = 2$ *and $Q^0 = V$; that is, the morphism $q\colon T^0 \longrightarrow V$ is generically finite.*

(ii) *If, for the general conic $C \subset Q^0$, $\mathcal{N}_{C/V}$ is of type* b), *then Δ^0 is nonsingular at its general point*, $\dim \Delta^0 = 2$, $\dim Q^0 = 2$, *and Q^0 is either the Veronese surface in* $\mathbf{P}^5$, *or one of its projections into a lower-dimensional space, but not a plane* $\mathbf{P}^2$ *or a quadric of* $\mathbf{P}^3$.

(iii) *If, for the general conic $C \subset Q^0$, $\mathcal{N}_{C/V}$ is of type* c), *then Δ^0 is nonsingular at the general point*, $\dim \Delta^0 = 3$, *and Q^0 is a quadric surface in V.*

(iv) *If, for the general $C \subset Q^0$, $\mathcal{N}_{C/V}$ is of type* d), *then Q^0 is a plane $P \subset V$.*

PROOF. (i) The smoothness of Δ^0 at a general point and the equality $\dim \Delta^0 = 2$ follow from general deformation theory [7]. Clearly $\dim Q^0 \geqslant 2$. If $\dim Q^0 = 2$ then Q^0 is a surface containing a 2-dimensional family of conics. According to a classical result of Bertini [2], Q^0 is either the Veronese surface in $\mathbf{P}^5$, or one of its projections in a lower dimensional space. All of these surfaces are well known (see, for example [17]):

$$\begin{array}{ccccc} F_4 & & & \text{the Veronese in } \mathbf{P}^5 & \\ \downarrow & \searrow & & & \\ F_4',\ F_4'' & R_3 & & \text{in } \mathbf{P}^4 & \\ \downarrow & \downarrow & \searrow & & \\ S_4 & R_3' & Q_2 & \text{in } \mathbf{P}^3 & \end{array} \tag{4.3.1}$$

and the plane $\mathbf{P}^2$, where S_4 is the Steiner surface, R_3 and R_3' are rational cubic scrolls, and Q_2 is a quadric (as usual the lower index denotes the degree). An immediate verification shows that if Q^0 is one of the surfaces in (4.3.1) then $\mathcal{N}_{C/V}$ has a positive summand; that is, $\mathcal{N}_{C/V} \simeq \mathcal{O}_C(-d) \oplus \mathcal{O}_C(d)$, with $d > 0$. But this contradicts the assumption that $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$. Hence $Q^0 = V$ and the morphism $q\colon T^0 \longrightarrow V$ is generically finite. Here we are using the fact that T^0 is proper over k.

(ii) The first two assertions are a consequence of deformation theory. As usual we have $\dim Q^0 \geqslant 2$. Suppose that $Q^0 = V$. Then the morphism $q\colon T^0 \longrightarrow V$ is generically finite, and hence its differential

$$\begin{array}{ccc} \mathcal{N}_{C/T^0} & \xrightarrow{dq} & \mathcal{N}_{C/V} \\ \wr\| & & \wr\| \\ \mathcal{O}_C \oplus \mathcal{O}_C & \longrightarrow & \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1) \end{array} \tag{4.3.2}$$

is nondegenerate at the general point of some curve C. However, the lower arrow of (4.3.2) shows that dq must have kernel of rank at least 1. This contradiction shows that $\dim Q^0 = 2$, and hence Q^0 is one of the surfaces of diagram (4.3.1). It cannot be the plane $\mathbf{P}^2$ or a quadric $Q_2 \subset \mathbf{P}^3$, since for either of these $\mathcal{N}_{C/V}$ would have a summand of positive degree greater than 1.

(iii) According to deformation theory two cases are possible here:

α) Δ^0 is nonsingular at a general point and $\dim \Delta^0 = 3$;

β) Δ^0 is nonreduced at the general point and $\dim \Delta^0 = 2$.

In either case the morphism $q\colon T^0_{\mathrm{red}} \longrightarrow V$ cannot be dominant, since the differential

$$dq\colon \mathcal{N}_{C/T^0_{\mathrm{red}}} \to \mathcal{N}_{C/V} \cong \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2)$$

cannot have rank 2 at the general point of C, because the trivial sheaf $\mathcal{N}_{C/T^0_{\mathrm{red}}}$ can only have a nonzero homomorphism into the second summand of $\mathcal{N}_{C/V}$.

Hence $Q^0 = q(T^0_{\mathrm{red}})$ is a surface from diagram (4.3.1). Clearly, in case α) $Q^0 = Q_2$ is a quadric of $\mathbf{P}^3$. Case β) cannot occur. Indeed, Q^0 cannot be the plane by considerations of the dimension of the family of conics on it. It is not difficult to check that for all the remaining surfaces in (4.3.1) $\mathcal{N}_{C/V}$ has positive summand $\mathcal{O}_C(1)$.

(iv) Here also $Q^0 = q(T^0_{\mathrm{red}})$ must be a surface from (4.3.1). In Lemma (4.2) we showed that $\mathcal{N}_{C/V}$ has type d) only if the plane P of C lies on V, or if $V = V_4$ is a quartic and V touches P along C. This final case is excluded since a smooth quartic 3-fold cannot contain any of the surfaces of (4.3.1) (see [14], and also (4.4, ii)). There remains the case $Q^0 = P$, a plane of V. The proposition is proved.

(4.4) THEOREM. *Suppose that* Pic $V \simeq \mathbf{Z}$, *and that V contains a line Z. Then the following assertions are true*:

(i) *V contains a smooth conic.*

(ii) *For every irreducible component T^0 of the family of conics T on V the morphism $q\colon T^0 \longrightarrow V$ is generically finite.*

(iii) *If $q > 8$ then only a finite number of conics pass through each point $v \in V$.*

PROOF. (i) According to (3.4) the lines of V sweep out a ruled surface $R \sim dH$, and the general line of R meets a further $d + 1$ lines. Hence there is a reducible conic C_0 on V consisting of 2 incident lines. Let I be the sheaf of ideals of the conic $C_0 \subset V$. Then

$$\mathcal{N}_{C_0/V} = \mathrm{Hom}\,(I/I_2,\ \mathcal{O}_{C_0})$$

is a locally free sheaf of rank 2, since C_0 is obviously a locally complete intersection, so that I/I^2 is a locally free sheaf.

According to Schlessinger's deformation theory we have

$$\dim \Delta^0 \geqslant h^0(\mathcal{N}_{C_0/V}) - h^1(\mathcal{N}_{C_0/V}),$$

where Δ^0 is an irreducible component of the scheme of conics Δ parametrizing deformations of C_0 in V. By the Riemann-Roch theorem we get

$$h^0(\mathcal{N}_{C_0/V}) - h^1(\mathcal{N}_{C_0/V}) = \deg \mathcal{N}_{C_0/V} + 2\,(1 - p_a(C)) = 2.$$

Hence dim $\Delta^0 \geqslant 2$. On the other hand, reducible conics on V form only a 1-dimensional family, since for $g \geqslant 4$ every line of V meets only a finite number of others (see (3.4, iii)).

In the case that $V = V_4 \subset \mathbf{P}^4$ is a quartic, the existence of smooth conics can easily be shown directly (see for example [14]). Thus (i) is proved.

(ii) According to Proposition (4.3), if $q: T^0 \longrightarrow V$ is not generically finite then $Q^0 = q(T^0_{\text{red}})$ is one of the surfaces of (4.3.1). Hence deg $Q^0 \leqslant 4$. By assumption Pic $V \simeq \mathbf{Z}$ and V contains a line, and hence Pic $V = \mathbf{Z} \cdot H$, with H the hyperplane class. Hence deg $G \geqslant 2g - 2 \geqslant 4$ for any surface $G \subset V$. Thus for the proof of (ii) we have to exclude the only possible case: $V = V_4 \subset \mathbf{P}^4$, and $Q^0 = S_4$, the Steiner surface. But it is easy to see (see [14], Lemma 4) that the Steiner surface S_4 cannot lie on a nonsingular quartic 3-fold V_4.

(iii) Let $v \in V$ be a closed point, and m_v its sheaf of ideals. For any integer $\nu \geqslant 0$ we have

$$h^0(\mathcal{O}_V(H) \otimes m_v^\nu) \geqslant h^0(\mathcal{O}_V(H)) - \binom{\nu+2}{3}.$$

In particular, for $\nu = 3$ and $g > 9$ we get

$$h^0(\mathcal{O}_V(H) \otimes m_v^3) \geqslant 2;$$

that is, dim $|H - 3v| \geqslant 1$. If there were a 1-dimensional family of conics passing through v, then the surface G_v swept out by these conics would have to be a fixed component of the linear system $|H - 3v|$, since the proper transform of any such conic has negative intersection with the proper transform of $|H - 3v|$ on blowing up v. But since Pic $V = \mathbf{Z} \cdot H$, every hyperplane section H is irreducible, so that $|H - 3v|$ cannot have any fixed components. This completes the proof of the theorem.

§5. Fano 3-folds of the first species: preliminary results

(5.1) Definition. A Fano 3-fold V will be said to be *of the first species* if Pic $V \simeq \mathbf{Z}$ (the term comes from Fano's "di 1ª specie" [4]).

Thus Fano 3-folds of the first species are characterized among projective varieties by the two conditions Pic $V \simeq \mathbf{Z}$ and $K_V^3 < 0$.

The study of the birational properties of such varieties is the principal content of Fano's classical papers. Basing himself on Fano's assertion that there exists a line on (the canonical model of) a Fano 3-fold of the first species with Pic $V = \mathbf{Z} \cdot K_V$ (see [20], (3.5, iv), and (3.6)), Roth classified in [11] all Fano 3-folds of the first species. However, as pointed out in our note [9], Roth's classification contains gaps. The most significant of these is the false assertion that such 3-folds (with Pic $V = \mathbf{Z} \cdot K_V$) exist only for $g \leqslant 10$. We will subsequently show (see (6.1, iii)) that there also exist Fano 3-folds with $g = 12$.

Let $\mathcal{O}_V(H)$ be the positive generator of Pic $V \simeq \mathbf{Z}$, and r the index of V (so that $rH \sim -K_V$). The classification of 3-folds of the first species and with index $r \geqslant 2$ is contained in our Part I [8], where a classification of all Fano 3-folds of index $r \geqslant 2$ is given, under the assumption (since proved; see [19]) of Hypothesis (1.14, Part I), that the linear system $|H|$ contains a smooth surface. One can show, although we will not do this here, that for $r \geqslant 2$ and Pic $V \simeq \mathbf{Z}$ we can get rid of this hypothesis.

In [8] we also gave a description of hyperelliptic 3-folds of the first species with index $r = 1$. A partial description of 3-folds of the first species and of the principal series has already been given in the present article: these are the complete intersections (see (1.3)) and the Grassmannian sections (see (1.4) and (1.5)).

In the remaining part of this article we describe Fano 3-folds of the first species with $g \geqslant 7$, under the assumption of the existence of lines (see (3.6) and [20]). The existence of lines and conics is used in (6.1) for the proof of the boundedness of the genus g ($g \leqslant 12$). To construct 3-folds with given g we use the birational technique (going back to Fano [5], [6]) of projection and double projection from lines on V.

From now on we will stick to the following notation.

(5.2) *Notation and conventions.*

$V = V_{2g-2} \subset \mathbf{P}^{g+1}$ is a Fano 3-fold of the first species.

$Z \subset V$ is a line on V.

$\sigma\colon V' \longrightarrow V$ is the blow-up of Z in V; $H^* = \sigma^* H$, where H is the hyperplane section, and $Z' = \sigma^{-1}(Z)$ is the exceptional ruled surface.

$\pi_Z\colon V \longrightarrow \mathbf{P}^{g-1}$ is the projection from the line Z.

$\varphi_Z = \pi_Z \circ \sigma\colon V' \longrightarrow \mathbf{P}^{g-1}$ is the morphism resolving the indeterminacy of π_Z.

$V'' = \pi_Z(V) = \varphi_Z(V')$ is the image of V under the projection π_Z.

$R_3 = \varphi_Z(Z')$ is the image of the ruled surface Z'.

$\pi_{2Z}\colon V \longrightarrow \mathbf{P}^{g-6}$ is the double projection from Z; that is, the rational map defined by the linear system $|H - 2Z|$.

$W = \pi_{2Z}(V)$ is the image of V under the double projection π_{2Z}.

$Q = Q_Z$ is the surface of V swept out by conics on V which meet Z.

Q' is the proper transform of Q in V'.

$Q'' = \varphi_Z(Q') \subset V''$ is the image of Q under π_Z.

Z_i, for $i = 1, \ldots, d+1$, are the lines of V meeting Z.

Z_i^0, for $i = 1, \ldots, d+1$, are the proper transforms of the Z_i on V'.

$\tau\colon \overline{V}' \longrightarrow V'$ is the blow-up of all the $d+1$ lines Z_i^0.

$\overline{Z}_i^0 = \tau^{-1}(Z_i^0)$ are the exceptional ruled surfaces above the blown-up lines Z_i^0, and $\overline{Z}' = \tau^{-1}(Z')$.

$z_i = Z_i^0 \cap Z'$ is the point of intersection of the line Z_i^0 with the ruled surface Z', for $i = 1, \ldots, d+1$.

$\overline{H}^* = \tau^* H^*$.

An isolated singularity $x \in X$ of a 3-fold X will be called an *ordinary double point* if after performing at x the blow-up $\sigma_x\colon X' \longrightarrow X$ the blow-up X' is nonsingular in a neighborhood of $Y' = \sigma_x^{-1}(x)$, and $Y' \simeq \mathbf{P}^1 \times \mathbf{P}^1$, with the normal sheaf given by

$$\mathcal{N}_{Y'/X'} \simeq p_1^* \mathcal{O}_{\mathbf{P}^1}(-1) \oplus p_2^* \mathcal{O}_{\mathbf{P}^1}(-1).$$

We begin with the following auxiliary result, which is also of independent interest.

(5.3) LEMMA. *In the notation of* (5.2) *suppose that Z has the properties* (i)–(iii) *of Lemma* (3.7). *Then $Z' \simeq \mathbf{F}_1$, where $\mathbf{F}_1$ is the standard ruled surface* (*obtained by blowing up one point in* $\mathbf{P}^2$), *and if $g \geqslant 5$ then the following assertions hold*:

(i) *The morphism $\varphi_Z\colon V' \longrightarrow V''$ is birational.*

(ii) *The linear system $|H^* - Z'|$ cuts out on Z' the complete linear system $|s + 2f|$, where s is the class of the exceptional section and f is the class of a fiber of the ruled* *surface Z'; and R_3 is a nonsingular scroll of degree 3 in $\mathbf{P}^4$, lying on V''.*

(iii) *V'' has $d+1$ isolated ordinary double points, the images of the $d+1$ lines $Z_1, \ldots, Z_{d+1}$ meeting Z (see (3.4, iv)), and has no other singularities.*

(iv) *Let $g_1 = g-2$; then $V'' \subset \mathbf{P}^{g_1+1}$ with $\deg V'' = 2g_1 - 2$, and furthermore the sheaf $\mathcal{O}_{V''}(-1)$ is the canonical dualizing sheaf $\mathcal{K}_{W''}$. Also* Pic $V'' = \mathbf{Z}\cdot H''$, *so that $V''_{2g_1-2} \subset \mathbf{P}^{g_1+1}$ is a Fano 3-fold of the first species, only having a finite number of ordinary double points.*

PROOF. Since $\check{\mathcal{N}}_{Z/V} \simeq \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$ and $Z' \simeq \mathbf{P}(\check{\mathcal{N}}_{Z/V})$, with $\check{\mathcal{N}}_{Z/V}$ the conormal sheaf, it follows that $Z' = \mathbf{F}_1$.

(i) In the projection $\pi_Z\colon \mathbf{P}^{g+1} \longrightarrow \mathbf{P}^{g-1}$ the inverse images of points $y \in \mathbf{P}^{g-1}$ are the planes $\mathbf{P}^2_y$ through Z. Hence if $y \in V''$ then its inverse image lies in $V \cap \mathbf{P}^2_y$. Since Pic $V \simeq \mathbf{Z}$ and $g \geqslant 5$, V is an intersection of quadrics of $\mathbf{P}^{g+1}$ (see (1.7)), and $\mathbf{P}^2_y \not\subset V$ (see (3.4, i)). Hence $V \cap \mathbf{P}^2_y$ is cut out by conics in $\mathbf{P}^2_y$. Hence, apart from the line Z, the scheme $V \cap \mathbf{P}^2_y$ can only contain either a further line Z_i (for $i = 1, \ldots, d+1$), or no more than one other point $x \in V$. Hence the restriction of π_Z to $V - (Z \bigcup_{i=1}^{d+1} Z_i)$ is a one-to-one mapping to $V'' - \varphi_Z(Z' \bigcup_{i=1}^{d+1} Z_i^0)$. Hence the morphism $\varphi_Z\colon V' \longrightarrow V''$ is birational. Note that it contracts each of the $d+1$ lines Z_i^0 into a point of V''.

(ii) Each nonsingular surface $H' \in |H^* - Z'|$ cuts out on Z' some section; that is, $H' \cap Z' \sim s + \alpha f$, with α some integer. We have

$$(s+\alpha f\cdot s+\alpha f)_{Z'} = (H^* - Z')^2\cdot Z' = 3 \tag{5.3.1}$$

(see 2.11, Part I). Since $Z' \simeq \mathbf{F}_1$, $s^2 = -1$. Hence $\alpha = 2$. Z' only contains a single curve having negative self-intersection, namely the negative section s. Since $(s\cdot Z')_V = (s+2f\cdot Z') - 2(f\cdot Z') = 3 - 2 = 1$, the morphism φ_Z cannot contract s to a point. Hence φ_Z cannot contract anything, apart from the $d+1$ lines Z_i^0.

Furthermore, if we prove that V'' is projectively normal, and hence a normal variety, then it will follow that $\varphi_{Z|Z'}\colon Z' \longrightarrow R_3$ is a birational morphism, and R_3 is a normal surface of degree 3 (see (5.3.1)). But this is only possible if $\varphi_{Z|Z'}$ is defined by the complete linear system $|s+2f|$; and then R_3 is a smooth scroll of $\mathbf{P}^4$.

Let us prove the projective normality of V''. Choose a smooth surface $H' \in |H^* - Z'|$ and set

$$\mathcal{L}' = \mathcal{O}_{V'}(H^* - Z'), \qquad \mathcal{L}'_{H'} = \mathcal{L}' \otimes \mathcal{O}_{H'}.$$

Since H' is a $K3$ surface, $\varphi_Z\colon H' \longrightarrow \varphi_Z(H')$ is a birational morphism and the complete linear system $|\mathcal{L}'_{H'}|$ is without fixed components and base points, it follows as shown in [13], Theorem 6.1, that the natural homomorphism

$$S^\bullet H^0(H', \mathcal{L}'_{H'}) \to \bigoplus_{n\geqslant 0} H^0(H', \mathcal{L}'^n_{H'}) \tag{5.3.2}$$

is surjective, where the left-hand side is the symmetric algebra on $H^0(H', \mathcal{L}'_{H'})$.

Since the sequence

$$H^0(V', \mathcal{L}') \to H^0(H', \mathcal{L}'_{H'}) \to H^1(V', \mathcal{O}_{V'}) = 0$$

is exact, the hypotheses of Lemma (2.9, Part I) are satisfied with $X = V$, $Y = H'$ and

$\mathscr{L} = \mathscr{L}'$. Becasuse of this lemma the natural homomorphism

$$S^*H^0(V', \mathscr{L}') \to \bigoplus_{n\geqslant 0} H^0(V', \mathscr{L}'^n) \tag{5.3.3}$$

is also surjective, so that

$$V'' = \varphi_Z(V') = \operatorname{Proj} \bigoplus_{n\geqslant 0} H^0(V', \mathscr{L}'^n)$$

is projectively normal in $\mathbf{P}^{g-1}$.

(iii) We have shown that outside $\bigcup_{i=1}^{d+1} Z_i^0$ the morphism φ_Z is a bijection with $V'' - \bigcup_{i=1}^{d+1} \varphi_Z(Z_i^0)$. Since V'' is normal, it follows from this by Zariski's theorem that

$$\varphi_Z \mid V' - \bigcup_{i=1}^{d+1} Z_i^0 : V' - \bigcup_{i=1}^{d+1} Z_i^0 \to V'' - \bigcup_{i=1}^{d+1} \varphi_Z(Z_i^0)$$

is an isomorphism. The inverse image Z_i^0 of each of the points $\varphi_Z(Z_i^0)$ is 1-dimensional, and hence each $\varphi_Z(Z_i^0)$ is a singular point of V''. Since $(Z_i^0 \cdot H^* - Z') = 0$, φ_Z extends to a morphism $\overline{\varphi}_Z\colon \overline{V}' \longrightarrow V''$, where $\overline{V}' \longrightarrow V'$ is the blow-up of all the $d + 1$ lines Z_i^0. Let $\overline{Z_i^0}$ be the exceptional ruled surface over Z_i^0. By hypothesis (see (3.7, iii))

$$\mathscr{N}_{Z_i^0/V'} \simeq \mathscr{O}_{Z_i^0}(-1) \oplus \mathscr{O}_{Z_i^0}(-1),$$

and hence $\overline{Z_i^0} \simeq \mathbf{P}^1 \times \mathbf{P}^1$; and, since $\overline{\varphi}_Z(\overline{Z_i^0}) = \varphi_Z(Z_i^0)$, $\varphi_Z(Z_i^0)$ is an ordinary double point of V''.

(iv) Using the multiplication table (2.11, Part I), we get

$$(H^* - Z')^3 = H^{*3} + 3H^* \cdot Z'^2 - Z'^3 = H^3 - 3H \cdot Z + \deg \det \mathscr{N}_{Z/V}$$

$$= 2g - 2 - 3 - 1 = 2g - 6,$$

and hence if we set $g_1 = g - 2$, then $\deg V'' = 2g_1 - 2$ and $V'' = V''_{2g_1-2} \subset \mathbf{P}^{g_1+1}$. Furthermore, it is well known that ordinary double points are Gorenstein; that is, the dualizing sheaf $\mathscr{K}_{V''}$ on V'' is invertible. Since

$$\mathscr{O}_{V'}(-K_{V'}) \simeq \mathscr{O}_{V'}(H^* - Z') \simeq \varphi_Z^* \mathscr{O}_{V''}(1)$$

and the morphism φ_Z is birational, we have

$$\mathscr{K}_{V''} = (\varphi_Z)_* \mathscr{O}_{V'}(K_{V'}) = \mathscr{O}_{V''}(-1).$$

It remains to show that $\operatorname{Pic} V'' \simeq \mathbf{Z}$, with $\mathscr{K}_{V''}^{-1} = \mathscr{O}_{V''}(1)$ a generator. We have that $\operatorname{Pic} V' \simeq \mathbf{Z} \oplus \mathbf{Z}$, with generators $\mathscr{O}_{V'}(H^*)$ and $\mathscr{O}_{V'}(Z')$. Note that the surface $R_3 = \varphi_Z(Z')$ is not a Cartier divisor on V'', since it passes through all of the $d + 1$ singular points $\varphi_Z(Z_i^0)$ of V'', but according to (iii) is itself nonsingular. One sees easily that no multiple rR_3 of R_3 can be a Cartier divisor either. On the other hand we know that the sheaf

$$(\varphi_Z)_* \mathscr{O}_{V'}(H^* - Z') = \mathscr{O}_{V''}(1)$$

is invertible, and that $\mathscr{O}_{V'}(H^* - Z')$ can be chosen as one of the generators of $\operatorname{Pic} V'$. Hence the sheaf

$$(\varphi_Z)_* \mathscr{O}_{V'}(\alpha H^* - \beta Z') = (\varphi_Z)_* \mathscr{O}_{V'}(H^* - Z')^{\alpha} \otimes \mathscr{O}_{V'}(Z')^{\alpha-\beta}$$

is invertible on V'' if and only if $\alpha = \beta$. It follows that Pic $V'' \simeq \mathbf{Z}$, with generator $\mathcal{O}_{V''}(1)$. The lemma is proved.

We will now prove some assertions concerned with the properties of double projections.

(5.4) LEMMA. *Under the conditions of Lemma* (5.3), *and using the notation* (5.2), *the following assertions* (i)–(viii) *are valid*:

(i) *If* $g \geqslant 5$ *then* $h^i(\mathcal{O}_{V'}(H^* - 2Z')) = 0$ *for* $i \geqslant 1$, $h^0(\mathcal{O}_{V'}(H^* - 2Z')) = g-5$, *and* $h^0(\mathcal{O}_{V'}(H^* - 3Z')) \leqslant 1$.

(ii) *If* $g \geqslant 7$ *then the linear system* $|H^* - 2Z'|$ *on* V' *is without fixed components, and its base locus consists just of the* $d + 1$ *lines* Z_i^0 (*see* (5.2) *for the notation*).

(iii) *Suppose that* $g \geqslant 7$, *and let* $L = |H^* - 2Z'| \cap Z'$ *be the trace of the linear system* $|H^* - 2Z'|$ *on* Z'. *Then* $L \subset |2s + 3f| = |-K_{Z'}|$, L *has no fixed components, and its base points are just the* $d + 1$ *points* $z_i = Z_i^0 \cap Z'$ (*and hence* $d \leqslant 7$).

(iv) *For* $g \geqslant 7$

$$g = 14 + h^0(\mathcal{O}_{V'}(H^* - 3Z')) - h^1(\mathcal{O}_{V'}(H^* - 3Z')) \tag{5.4.1}$$

and for $g \geqslant 7$ *and* $d \neq 7$

$$g \leqslant 13 + h^0(\mathcal{O}_{V'}(H^* - 3Z')) - d, \tag{5.4.2}$$

where d *is as in* (iii), *and* $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 0$ *or* 1 (*see* (i)).

(v) *For* $g \geqslant 7$ *suppose in addition that the* $d + 1$ *points* z_i *are simple base points for* L; *that is, that each* z_i *has multiplicity* 1 *in* L *and is resolved by a single blowing-up. Then the base lines* Z_i^0 *are also simple base lines for* $|H^* - 2Z'|$, *and if* $\tau\colon \overline{V}' \to V'$ *is the blow-up of all the* Z_i^0 *for* $i = 1, \dots, d + 1$, *and* $\varphi_{2Z}\colon \overline{V}' \to W \subset \mathbf{P}^{g-6}$ *is the map defined by the linear system* $|\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0|$ (*see* (5.2) *for the notation*), *then* φ_{2Z} *is a morphism.*

(vi) *Under the conditions of* (v), *if* $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1$ *and* $d \neq 7$ *then* $Q' \sim H^* - 3Z'$.

(vii) *If* $g \geqslant 9$ *and the conditions of* (v) *hold, then* φ_{2Z} *is a birational morphism* (*apart possibly for the one case* $g = 9$, $d = 5$ *and* $\deg \varphi_{2Z} = 2$) *which contracts the surface* $\overline{Q}'$ (*in the notation of* (5.2)) *on some irreducible curve* $Y \subset W$ *and which contracts each of the surfaces* $\overline{Z}_i^0 = \tau^{-1}(Z_i^0)$ *onto some line* $Y_i \subset W$.

(viii) *Under the conditions of* (vii) *we have the isomorphism* Pic $W \simeq \mathbf{Z}$; *furthermore,* W *is nonsingular if and only if* $\varphi_{2Z}|Z'\colon \overline{Z}' \to \varphi_{2Z}(\overline{Z}')$ *is an isomorphism.*

If W *is nonsingular, then it is a Fano* 3-*fold of the first species and of index* $r \geqslant 2$ *normally embedded in* $\mathbf{P}^{g-6}$.

PROOF. From the cohomology exact sequences associated to the short exact sequences of sheaves

$$0 \to \mathcal{O}_{V'}(-Z') \to \mathcal{O}_{V'} \to \mathcal{O}_{Z'} \to 0,$$

$$0 \to \mathcal{O}_{V'}(H^* - Z') \to \mathcal{O}_{V'}(H^*) \to \mathcal{O}_{Z'}(H^*) \to 0$$

and

$$0 \to \mathcal{O}_{V'}(H^* - 2Z') \to \mathcal{O}_{V'}(H^* - Z') \to \mathcal{O}_{Z'}(H^* - Z') \to 0,$$

we get $h^i(\mathcal{O}_{V'}(-Z')) = 0$ for $i \geqslant 1$, $h^i(\mathcal{O}_{V'}(H^* - Z')) = 0$ for $i \geqslant 1$, and, finally, for $g \geqslant 5$ (using (5.3, ii)), $h^i(\mathcal{O}_{V'}(H^* - 2Z')) = 0$ for $i \geqslant 1$. From this, using the Riemann-Roch formula, we get

$$h^0(\mathcal{O}_{V'}(H^* - 2Z')) = g - 5.$$

Let us prove the inequality $h^0(\mathcal{O}_{V'}(H^* - 3Z')) \leqslant 1$. Suppose that this is not the case; that is, that $|H^* - 3Z'|$ is a mobile linear system. Then the surface Q' of conics meeting Z (in the notation of (5.2)) is a fixed component of $|H^* - 3Z'|$, since $(H^* - 3Z' \cdot C') = -1$ for $C' \subset Q'$ the proper transform of a sufficiently general conic $C \subset Q$. Since $Q' \neq Z'$, Q must be a component of some hyperplane section $H \subset V$, which contradicts the assumption $\operatorname{Pic} V = \mathbf{Z} \cdot H$. The assertions in (i) are proved.

(ii) Since $\operatorname{Pic} V = \mathbf{Z} \cdot H$, the linear system $|H^* - 2Z'|$ cannot have fixed components (for $g \geqslant 7$) other than Z'. The surface Z' can also not be a fixed component, since otherwise $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = h^0(\mathcal{O}_{V'}(H^* - 2Z')) = g - 5 \geqslant 2$, contradicting (i). Let us prove that $|H^* - 2Z'|$ has no base points outside $\bigcup_{i=1}^{d+1} Z_i^0$. For this we use Lemma (5.3). For every $v' \in V' - \bigcup_{i=1}^{d+1} Z_i^0$ we can find, using Bertini's theorem and (5.3, iii), a smooth surface $H' \in |H^* - Z'|$ with $v' \in H'$. Furthermore, if $Y' \subset V'$ is some curve through v' then H' can be chosen so as not to contain any component of Y'. Since $h^1(\mathcal{O}_{V'}(-Z')) = 0$, $|H^* - 2Z'|$ cuts out on H' a complete linear system. By choice of H' this linear system is without fixed components. Since H' is a smooth $K3$ surface, every complete linear system without fixed components on it has no base points (see for example [13]). It follows that v' cannot be a base point of $|H^* - 2Z'|$, and since $v' \in V' - \bigcup_{i=1}^{d+1} Z_i^0$ was an arbitrary point, this proves that $|H^* - 2Z'|$ does not have any base points outside the $d + 1$ lines Z_i^0. On the other hand each of these lines is obviously a base locus for $|H^* - 2Z'|$, since $(Z_i^0 \cdot H^* - 2Z') = -1$.

(iii) By the adjunction formula we have

$$-K_{Z'} \sim (-K_{V'} - Z') \cap Z' = (H^* - 2Z') \cap Z'.$$

Hence $L \subset |-K_{Z'}| = |2s + 3f|$, where s is the class of the negative section and f is the fiber of the ruled surface $Z' \simeq \mathbf{F}_1$. Clearly, the $d + 1$ points z_i are base points for L, and L has no further base points, since $|H^* - 2Z'|$ has no base points on V' outside the $d + 1$ lines Z_i^0.

(iv) From the exact cohomology sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_{V'}(H^* - 3Z') \to \mathcal{O}_{V'}(H^* - 2Z') \to \mathcal{O}_{Z'}(H^* - 2Z') \to 0,$$

using (i) and (iii) we get $h^i(\mathcal{O}_{V'}(H^* - 3Z')) = 0$ for $i \geqslant 2$.

Computing the Euler characteristic we get (5.4.1). Since $h^1(\mathcal{O}_{V'}(H^* - 3Z')) = \dim |-K_{Z'}| - \dim L$, and L has no fixed components and only the $d + 1$ points z_i as base points, we have $h^1(\mathcal{O}_{V'}(H^* - 3Z')) \geqslant d + 1$ if $d < 7$. The inequality (5.4.2) follows immediately.

(v) The multiplicity of the linear system $|H^* - 2Z'|$ at the general point of Z_i^0 is at most equal to its multiplicity at the point z_i, which by hypothesis is 1. Hence $|H^* - 2Z'|$ has multiplicity 1 along each of the $d + 1$ lines Z_i^0.

Let $\tau: \overline{V}' \longrightarrow V'$ be the blow-up of the $d+1$ lines Z_i^0. The restriction of the linear system $|\overline{H}^* - 2\overline{Z}' - \Sigma_{i=1}^{d+1} \overline{Z}_i^0|$ to $\overline{Z}' = \tau^{-1}(Z')$ is without fixed components and base points by (iii) and by the assumption in (v). Let us prove that its restriction to each of the $d+1$ surfaces $\overline{Z}_i^0$ is also without fixed components and base points. According to (3.7), $\overline{Z}_i^0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$; let s_i and f_i be the classes of a section (with $s_i^2 = 0$) and a fiber of the ruled surface $\overline{Z}_i^0$. Then from the fact that $|H^* - 2Z'|$ has multiplicity 1 along Z_i^0 it follows that its proper transform on $\overline{V}'$ cuts out on $\overline{Z}_i^0$ some linear system of sections (that is, curves from the class $s_i + \alpha_i f_i$, with $\alpha_i \geqslant 0$ some integer). We have

$$2\alpha_i = (s_i + \alpha_i f_i \cdot s_i + \alpha_i f_i) = \left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^2 \cdot \overline{Z}_i^0$$
$$= 2\,(H^* \cdot Z_i^0)_{V'} - 4\,(Z' \cdot Z_i^0)_{V'} - \deg\det \mathcal{N}_{Z_i^0/V'} = 2 - 4 + 2 = 0;$$

and hence $\alpha_i = 0$ for $i = 1, \ldots, d+1$. Hence

$$\left|\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right| \cap \overline{Z}_i^0 \subseteq |s_i|$$

and since $\dim |s_i| = 1$ and the restricted linear system is mobile (this follows from the fact that it is mobile when restricted to the fiber over z_i, assumed in (v)), the complete linear system $|s_i|$ is cut out. Thus we have proved that $|\overline{H}^* - 2\overline{Z}' - \Sigma_{i=1}^{d+1} \overline{Z}_i^0|$ is without fixed components, and without base points on $\overline{Z}'$ or on any of the $d+1$ surfaces $\overline{Z}_i^0$; since we have already proved in (ii) that it has no base points outside these surfaces, it defines a morphism $\varphi_{2Z}: \overline{V}' \longrightarrow W$, where $W \subset \mathbf{P}^{g-6}$, since $h^0(\mathcal{O}_{V'}(H^* - 2Z')) = g - 5$ (see (i)).

(vi) Suppose that $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1$. Every conic $C \subset V$ which meets Z only meets it in 1 point, as follows from the fact that V is an intersection of quadrics. We have $(C' \cdot H^* - 3Z') = -1$, where $C' \subset V'$ is the proper transform of C. Hence if $|H^* - 3Z'| \neq \varnothing$, then the surface Q' of conics meeting Z is contained in $|H^* - 3Z'|$ as a component (that Q' is irreducible follows from $\operatorname{Pic} V = \mathbf{Z} \cdot H$). Let us show that in fact $Q' \sim H^* - 3Z'$. From the condition $\operatorname{Pic} V = \mathbf{Z} \cdot H$ a surface in $|H^* - 3Z'|$ can a priori only contain Q' together with some multiple of Z'. Let $Q' + \alpha Z' \sim H^* - 3Z'$, with $\alpha \geqslant 0$ an integer. Since $(C' \cdot H^* - 2Z') = 0$, the surface $\overline{Q}' \subset \overline{V}'$ has degree 0 relative to $\overline{H}^* - 2\overline{Z}' - \Sigma_{i=1}^{d+1} \overline{Z}_i^0$, so that

$$\overline{Q}' \cdot \left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^2 = 0. \tag{5.4.3}$$

In the proof of (v) we showed that $\overline{Z}_i^0$ also has degree 0. Hence $\tau^* Q'$ also has degree 0. Hence, using the multiplication table (2.11, Part I), we get

$$\alpha \overline{Z}' \cdot \left(H^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^2 = (\overline{H}^* - 3\overline{Z}') \cdot \left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^2$$
$$\alpha(8 - d - 1) \qquad = \qquad 2g - 28 + 2d$$

For $d \neq 7$ inequality (5.4.2) implies that $g \leqslant 14 - d$. Substituting this in the last equation we get $\alpha(8 - d - 1) \leqslant 0$, and hence $\alpha \leqslant 0$. This proves (vi).

(vii) Using the multiplication table (2.11, Part I) we get

$$\left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^3 = 2g - 22 + d + 1. \tag{5.4.4}$$

By Lemma (2.1, Part I) we have the inequality

$$\operatorname{codim} W + 1 = g - 8 \leqslant \deg W = \frac{2g - 22 + d + 1}{\deg \varphi_{2Z}}. \tag{5.4.5}$$

If $g \geqslant 9$, then it follows from (5.4.2) that $d \leqslant 5$. For $d \leqslant 4$ it follows immediately from (5.4.5) that $\deg \varphi_{2Z} = 1$; that is, φ_{2Z} is a birational morphism. For $d = 5$ one other case is possible: $g = 9$, $\deg \varphi_{2Z} = 2$. From (5.4.2) we get that in this case $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1$. In the following section during the proof of Theorem 6.1 we will show that this case does not in fact occur.

It is clear that φ_{2Z} contracts the surface $\overline{Q}'$ and each of the $\overline{Z}_i^0$. From the proof of (v) one sees that the $Y_i = \varphi_{2Z}(\overline{Z}_i^0)$ are lines of W. It remains to prove that $\overline{Q}'$ is not contracted to a point. This follows from the fact that $\overline{Q}' \cap \overline{Z}_i^0 \neq \varnothing$ for each i, or from the fact that the curve $\overline{Q}' \cap \overline{Z}'$ cannot be contracted under the map $\varphi_{2Z}|\overline{Z}'\colon \overline{Z}' \longrightarrow \varphi_{2Z}(\overline{Z}')$. The detailed checking will be carried out during the proof of Theorem (6.1).

(viii) The group Pic $\overline{V}'$ has rank $d + 3$ and is generated by the classes of $\overline{H}^*$, $\overline{Z}'$ and the $\overline{Z}_i^0$, $i = 1, \dots, d + 1$. The morphism φ_{2Z} contracts all the $\overline{Z}_i^0$ and the surface $\overline{Q}'$. Standard arguments then deduce that Pic $W \simeq \mathbf{Z}$. In exactly the same way we can prove that the group of Weil divisor classes Cl W is also isomorphic to $\mathbf{Z}$. Hence Pic $W \subset$ Cl $W \simeq \mathbf{Z}$. Let us show that in fact we have equality Pic $W =$ Cl W. For this it is enough to check that on V' the group Pic V' is generated by $(H^* - 2Z')$ and the class of the contracted surface Q'; that is, that the class of Z' can be expressed in terms of them. Let a and b be the integers such that $Q' \sim aH^* - bZ'$. If $C \subset Q$ is a sufficiently general conic meeting Z then $\mathcal{N}_{C/V} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$ (see (4.4, ii)). Let C' be the proper transform of C on V'; then

$$\mathcal{N}_{C'/V'} \simeq \mathcal{O}_{C'}(-1) \oplus \mathcal{O}_{C'}.$$

It follows that $(C' \cdot Q')_{V'} = -1$, since $(C' \cdot C')_{Q'} = 0$. We have

$$(C' \cdot Q') = (C' \cdot aH^* - bZ') = 2a - b = -1.$$

Hence $Q' \sim a(H^* - 2Z') - Z'$, and

$$Z' \sim a(H^* - 2Z') - Q'. \tag{5.4.6}$$

From this we conclude that the image E on W of the class of $(\overline{H}^* - 2\overline{Z}' - \Sigma_{i=1}^{d+1} \overline{Z}_i^0)$ is a generator both of Pic W and of Cl W, and furthermore, the image $F = \varphi_{2Z}(\overline{Z}')$ of the surface $\overline{Z}'$ is equivalent to aE.

From the fact that $Q' \cdot (\overline{H}^* - 2\overline{Z}' - \Sigma_{i=1}^{d+1} \overline{Z}_i^0)^2 = 0$, we get

$$a(2g - 21 + d) + (d - 7) = 0. \tag{5.4.7}$$

Hence and from (5.4.1) and (5.4.2) we get a list of all possible values of g, d and a (of course, with $g \geqslant 9$):

g	d	a
13	1	1
12	2	1
11	1	3
11	3	1
10	3	2
10	4	1
9	4	3
9	5	1

(5.4.8)

Now let us prove the criterion for the nonsingularity of W. First suppose that $F = \varphi_{2Z}(\overline{Z}') \simeq \overline{Z}'$ is a smooth surface. Then W cannot have singularities on F, since $F \sim aE$ is a Cartier divisor. Furthermore, from the fact that W has a smooth hyperplane section it follows that W cannot have more than isolated singularities. It is easily seen that W is projectively normal in $\mathbf{P}^{g-6}$. Hence every isolated singularity is the image of the contraction of some subvariety $\overline{X}' \subset \overline{V}'$, and $\overline{X}' \cap \overline{Z}' = \varnothing$, since otherwise the singular point would lie on F. But a curve of V' not meeting Z' cannot be contracted by φ_{2Z}, since it has a nonzero intersection number (equal to its degree) with $H^* - 2Z'$. This proves that F smooth implies W smooth.

For the proof of the converse implication note that F can only have singularities if the birational morphism $\varphi_{2Z}|\overline{Z}'$: $\overline{Z}' \longrightarrow F$ contracts some curves. Let $\overline{X}' \subset \overline{Z}'$ be such a curve. Since W is nonsingular, $\overline{Q}'$ is an exceptional divisor of the first kind (more precisely, it becomes an exceptional divisor of the first kind after contracting all of the $\overline{Z}_i^0$) on $\overline{V}'$, so that its image on W should be a nonsingular curve $Y \subset W$. It follows from this that the intersection curve $\overline{Y}' = \overline{Q}' \cap \overline{Z}'$ is also nonsingular, since it is isomorphic to Y. Hence the curve $\overline{X}'$ cannot lie on the contractible surface $\overline{Q}'$. One checks similarly that $\overline{X}'$ also cannot lie on any of the $d + 1$ surfaces $\overline{Z}_i^0$. There are no other surfaces contracted by φ_{2Z}. Hence φ_{2Z} must contract an isolated curve. But then W must have a singular point. This contradiction shows that W smooth implies that $F = \varphi_{2Z}(\overline{Z}')$ is a smooth surface.

The final assertion in (viii) follows at once from previous arguments. The lemma is proved.

(5.5) REMARKS. The term "double projection" for φ_{2Z} comes from the fact that the map π_{2Z} defined by the linear system $|H - 2Z|$ on V can be represented as the composite of two projections:

a) the projection from the line Z, π_Z: $V \longrightarrow V''$;

b) the projection π_{R_3}: $V'' \longrightarrow W$ from the ruled surface R_3, the "image" of the line Z under the projection π_Z.

The second projection is induced by the projection $\mathbf{P}^{g-1} \longrightarrow \mathbf{P}^{g-6}$ from the linear subspace $\mathbf{P}^4$—the linear span of the scroll R_3. To resolve the indeterminacy of the projection π_{R_3} one has to blow up $R_3 \subset V''$ into a Cartier divisor. Let δ: $V''' \longrightarrow V''$ be this blow-up. Then V''' is a smooth 3-fold: δ is the most economic resolution of the singularities of V'', in the sense that the inverse image of each singular point is a smooth rational curve—the blow-up of the corresponding point on R_3. Let $\overline{R}_3 = \delta^{-1}(R_3)$. Then $\delta|\overline{R}_3$: $\overline{R}_3 \longrightarrow R_3$ is the blow-up of R_3 in the $d + 1$ points of R_3 which are singular on V''. Under the conditions of general position of (5.4) it is not difficult to show that V''' is the image of the variety $\overline{V}'$ on contracting the $d + 1$ surfaces $\overline{Z}_i^0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$ on the other component from the contraction τ.

§6. Fano 3-folds of the first species: the main theorem

(6.1) THEOREM. *Let $V = V_{2g-2} \subset \mathbf{P}^{g+1}$, $g \geq 7$, be a Fano 3-fold of the first species. Suppose that V contains a line, and let π_{2Z}: $V \longrightarrow W \subset \mathbf{P}^{g-6}$ be the double projection (see 5.2) from a sufficiently general (in the sense of Lemma (3.7)) line $Z \subset V$. Let E denote the hyperplane section of W. Then the following assertions hold*:

(i) $g \leq 12$.

(ii) *If $g = 12$, then $W = W_5 \subset \mathbf{P}^6$ is a Fano 3-fold of the first species and of index 2 and degree 5 (with possibly one singular point); the map ρ_Y: $W \longrightarrow V$ inverse to π_{2Z} is given by the linear system $|3E - 2Y|$, with $Y \subset W$ a normal rational curve of degree 5 in $\mathbf{P}^5$.*

(iii) *There do not exist any Fano 3-folds of the first species with $g = 11$.*

(iv) *If $g = 10$, then $W = W_2 \subset \mathbf{P}^4$ is a quadric and ρ_Y: $W \longrightarrow V$ is given by the linear system $|5E - 2Y|$, where Y is a smooth curve of genus 2 and degree 7 in $\mathbf{P}^4$.*

(v) *If $g = 9$, then $W = \mathbf{P}^3$ and ρ_Y: $\mathbf{P}^3 \longrightarrow V$ is given by the linear system $|7E - 2Y|$, where Y is a smooth curve of genus 3 and degree 7.*

(vi) *If $g = 8$, then π_{2Z}: $V \longrightarrow \mathbf{P}^2$ is a rational map with fibers (after resolving the determinacy) curves of genus 2, and such that the inverse images of lines of $\mathbf{P}^2$ are rational surfaces.*

(vii) *If $g = 7$, then π_{2Z}: $V \longrightarrow \mathbf{P}^1$ is a rational map whose general fiber (after resolving the indeterminacy) is a del Pezzo surface of degree 5 with 8 points blown up; V is a rational 3-fold, and the projection from a line maps it into a complete intersection of 3 quadrics of $\mathbf{P}^6$ containing a smooth rational ruled surface $R_3 \subset \mathbf{P}^4$.*

(6.2) COROLLARY (FANO). *Fano 3-folds of the first species with $g = 7$ or $g \geq 9$ are rational (of course, assuming that there exist lines on them [20]).*

(6.3) REMARK. In [6] it is asserted that Fano 3-folds of the first species with $g = 5$, 6 or 8 are irrational, and that a 3-fold with $g = 8$ is birational to a smooth cubic of $\mathbf{P}^4$. The irrationality of V_8 with $g = 5$ is proved in [16] and in [1]. The author has succeeded in reestablishing the construction of the birationality of V_{14} ($g = 8$) with a cubic 3-fold. The proof will be published.

(6.4) PROOF OF THE THEOREM. (i) From (5.4.2) one gets at once that $g \leq 13$, since $d \geq 1$ (see (3.4, iv)). Let us show that there do not exist any 3-folds with $g = 13$. In this case (5.4.2) becomes an equality with $d = 1$ and $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1$. Note that the points of intersection z_1 and z_2 of the lines Z_1^0 and Z_2^0 with Z' must be distinct, since otherwise the 3 lines Z, Z_1 and Z_2 on V would have to lie in one plane, which is impossible since V is an intersection of quadrics. Furthermore, according to Lemma (3.7), the line Z can be chosen so that the points z_1 and z_2 do not lie on the negative section of the ruled surface Z'.

Consider first the case that z_1 and z_2 do not both lie on the same fiber of the ruled surface Z'. In this case all the conditions of Lemma (5.4) are fulfilled, and according to this lemma the image of the double projection π_{2Z}: $V \longrightarrow W$ is a smooth Fano 3-fold of the first species and of degree 6, $W_6 \subset \mathbf{P}^7$. W is smooth, as one checks using the criterion (5.4, viii).

However, it is known (4.2, Part I) that such 3-folds do not exist. Hence in this case a 3-fold V with $g = 13$ cannot exist either.

Now suppose that z_1 and z_2 lie on the same fiber of Z'. Following through the proof of Lemma (5.4) in this situation, one can check that the double projection $\pi_{2Z}\colon V \longrightarrow W$ can be resolved to a birational morphism $\varphi_{2Z}\colon \overline{V}' \longrightarrow W$ which contracts the fiber of Z' on which z_1 and z_2 lie into an isolated singularity of W. Apart from this singular point, as in the first case W satisfies $\deg W = 6$ and $\operatorname{Pic} W = \mathbf{Z} \cdot E$. An analysis of the proof of Theorem (4.2) in this case shows that such 3-folds W do not exist. Hence $g \neq 13$ and (i) is proved.

REMARK. It is known (see 4.2, Part I) that there does exist in $\mathbf{P}^7$ a Fano 3-fold $W_6 \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ of index 2. A map ρ_Y, the inverse of a double projection π_{2Z}, allows us to construct a Fano 3-fold $V \subset \mathbf{P}^{14}$ of index 1 with $g = 13$. However, V is not a 3-fold of the first species: $\operatorname{Pic} V \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. The map ρ_Y is given by the linear system

$$|3E-2Y-Y_1-Y_2|,$$

where E is the hyperplane section, Y is a smooth curve of genus 1 and degree 7 in $\mathbf{P}^6$, and Y_1 and Y_2 are two lines, chords of Y.

(ii) Now let us consider the case $g = 12$. It follows from (5.4.2) that $d = 1$ or 2. The case $d = 1$ does not occur (see the table (5.4.8)). If $d = 2$ then $h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1$, and (5.4.2) turns into an equality; that is, in (5.4.1)

$$h^1(\mathcal{O}_{V'}(H^*-3Z')) = d+1.$$

Note that all 3 of the points z_1, z_2 and z_3 cannot lie on one fiber of the ruled surface Z', since otherwise this fiber would be a base curve of the linear system $|H^* - 2Z'|$, contradicting (5.4, ii). Furthermore, by Lemma (3.7) the line Z can be chosen such that none of the 3 points z_1, z_2 and z_3 lie on the negative section of the ruled surface Z'. We consider separately two cases:

a) The 3 points $z_1, z_2, z_3 \in Z'$ are in general position; that is, no two of them lie on a fiber of Z', and all 3 do not lie on a section of Z' in the class of $s + f$.

b) The 3 points $z_1, z_2, z_3 \in Z'$ are not in general position.

Case a). Here all the conditions of Lemma (5.4) are fulfilled, and according to this lemma the double projection π_{2Z} maps V to a smooth Fano 3-fold of the first species $W_5 \subset \mathbf{P}^6$. The degree of W is computed from (5.4.4). There exists just one such 3-fold up to projective equivalence, namely a linear section of the Grassmannian $G(1, 4)$ of lines in $\mathbf{P}^4$ (see 4.2, iii, Part I). Now to convince ourselves of the existence of V we carry out the construction of the inverse map $\rho_Y\colon W \longrightarrow V$ to π_{2Z}. For this let us find first the curve $Y \subset W$ onto which the surface of conics $Q = Q_Z$ is contracted (for the notation, see (5.2)). Let $Y' = Q' \cap Z'$ be the curve of intersection of the contractible surface $Q' \sim H^* - 3Z'$ with Z' (see (5.4, vi)). Clearly $Y' \sim 3s + \alpha f$ for some integer α, which we find from the equation

$$(3s+\alpha f \cdot 3s+\alpha f)_{Z'} = (H^*-3Z')^2 \cdot Z' = 15. \qquad (6.4.1)$$

Hence $\alpha = 4$. The curve Y' passes through the points z_1, z_2 and z_3 with certain multiplicities. To discover these, we first compute the multiplicity of the surface Q' at the general points of the lines Z_1^0, Z_2^0 and Z_3^0.

Let $H' \in |H^* - 2Z'|$ be a sufficiently general smooth surface. This exists by Lemma (5.4) and by Bertini's theorem. Then $Q' \cap H' = r_1 Z_1^0 + r_2 Z_2^0 + r_3 Z_3^0 + X$, where r_i, $i = 1, 2, 3$, is the multiplicity of Q' at the lines Z_i^0. Computing the intersection number $Q' \cdot Z_i^0$ in two different ways, we get

$$-2 = (H^* - 3Z' \cdot Z_i^0) = (Q' \cdot Z_i^0) = -r_i + (X \cdot Z_i^0)_{H'}.$$

Since $(X \cdot Z_i^0)_{H'} \geqslant 0$, we have $r_i \geqslant 2$ for $i = 1, 2, 3$. Hence a fortiori the points z_i have multiplicity at least 2 on Y'.

Set $\overline{Y}' = \overline{Q}' \cap \overline{Z}'$ on $\overline{V}'$. As was mentioned at the end of the proof of Lemma (5.4), because W is nonsingular, $\overline{Q}'$ becomes an exceptional divisor of the first kind after contracting the 3 surfaces $\overline{Z}_i^0$ (the irreducibility of $\overline{Q}'$ follows from the assumption Pic $V = \mathbf{Z} \cdot H$). Any irreducible conic $C \subset Q$ meets Z in one point. It follows from this that the curve $\overline{Y}'$ is a section of the ruled surface $\overline{Q}'$. $\overline{Y}'$ is irreducible, since otherwise either $\overline{Q}'$ would be reducible, which is not the case, or $\overline{Y}'$ would contain some fiber of the ruled surface $\overline{Q}'$ as a component. But in this case such a fiber, belonging to the intersection $\overline{Q}' \cap \overline{Z}'$, would be contracted under the morphism $\varphi_{2Z}\colon \overline{Z}' \longrightarrow \varphi_{2Z}(\overline{Z}')$, contradicting the fact that z_1, z_2 and z_3 are in general position on Z'.

Thus $\overline{Y}'$ is a nonsingular section of the ruled surface $\overline{Q}'$, and is isomorphic to the curve $Y \subset W$ on which $\overline{Q}'$ is contracted. From this it also follows that Y and $\overline{Y}'$ are smooth curves.

The morphism $\tau\colon \overline{V}' \longrightarrow V'$ takes $\overline{Y}'$ to Y'. Hence Y' is irreducible. From the formula for the genus of a curve on a surface we get

$$g(\overline{Y}') = \frac{(3s + 4f \cdot 3s + 4f + K_{Z'})}{2} + 1 - \sum_{i=1}^{3} \frac{t_i(t_i - 1)}{2}$$
$$= 3 - \sum_{i=1}^{3} \frac{t_i(t_i-1)}{2} \geqslant 0, \qquad (6.4.2)$$

where $t_i \geqslant 2$ are the multiplicities of Y' at the points z_i for $i = 1, 2, 3$. Hence we get $t_1 = t_2 = t_3 = 2$ and $g(\overline{Y}') = 0$. Hence $g(Y) = 0$.

Let us compute the degree of Y. This is equal to

$$\left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{3} \overline{Z}_i^0\right) \cdot \overline{Y}' = (2s + 3f \cdot 3s + 4f) - 6 = 5.$$

Note that each of the 3 lines Y_i into which the surfaces $\overline{Z}_i^0$ are contracted meets Y in 2 points; that is, it is a 2-chord of Y.

Now let us find the image of the hyperplane section H under the double projection $\pi_{2Z}\colon V \longrightarrow W$, or, more precisely, the image of the linear system $|\overline{H}^*|$ on $\overline{V}'$ under the morphism $\varphi_{2Z}\colon \overline{V}' \longrightarrow W$. From general considerations it is clear that

$$\varphi_{2Z}(|\overline{H}^*|) = \left| nE - mY - \sum_{i=1}^{3} m_i Y_i \right|,$$

where n, m, and the m_i are certain integers. Let us compute them. We have

$$5n = \overline{H}^* \cdot \left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{3} \overline{Z}_i^0\right)^2 = 15,$$

whence $n = 3$. The integer m is equal to the intersection number of the surface $\overline{H}^*$ with the fiber of the surface $\overline{Q}'$ which contracts to Y; that is, $m = (C \cdot H) = 2$, where $C \subset Q$ is an arbitrary conic. Similarly $m_i = (Z_i \cdot H) = 1$ for $i = 1, 2, 3$.

Thus

$$\varphi_{2Z}(|\overline{H}^*|) = \left|3E - 2Y - \sum_{i=1}^{3} Y_i\right| = |3E - 2Y|.$$

The final equality holds because the chords Y_i are automatically base curves of the linear system $|3E - 2Y|$. The linear system $\varphi_{2Z}(|\overline{H}^*|)$ has no further base points, since, outside $\overline{Q}' \cup \bigcup_1^3 \overline{Z}_i^0$, φ_{2Z} is an isomorphism and the linear system $|\overline{H}^*|$ is free from base points. Hence the map ρ_Y inverse to π_{2Z} should be defined by the linear system

$$\left|3E - 2Y - \sum_{i=1}^{3} Y_i\right|,$$

where $Y \subset W$ is a smooth rational curve of degree 5 in $\mathbf{P}^5$, and the Y_i for $i = 1, 2, 3$ are 2-chords of Y. It remains to see that such a linear system exists.

We show first that W contains a curve Y with the required properties. Let $f\colon E \longrightarrow \mathbf{P}^2$ be the representation of a del Pezzo surface of degree 5 (a smooth hyperplane section of W) as a blow-up of 4 points $x_1, \ldots, x_4 \in \mathbf{P}^2$ in general position. Then the proper transform on E of a smooth conic of $\mathbf{P}^2$ passing through just one of the points x_i satisfies all the conditions for Y: the three 2-chords Y_i are the proper transform on E of the 3 lines of $\mathbf{P}^2$ passing through a pair of the remaining points. For computational purposes we will require information about the normal sheaf $\mathcal{N}_{Y/W}$. Since Y lies on a smooth hyperplane section E and $(Y \cdot Y)_E = 3$, $\mathcal{N}_{Y/W}$ can be represented as an extension

$$0 \to \mathcal{O}_Y(3) \to \mathcal{N}_{Y/W} \to \mathcal{O}_Y(5) \to 0,$$

where as usual in this article $\mathcal{O}_Y(d)$ denotes the invertible sheaf of degree d on Y. Hence $h^0(\mathcal{N}_{Y/W}) = 10$ and $h^1(\mathcal{N}_{Y/W}) = 0$, and hence it follows according to local deformation theory that the family of curves in a neighborhood of Y is smooth and 10-dimensional.

There are only two possibilities:

$$\mathcal{N}_{Y/W} \simeq \mathcal{O}_Y(4) \oplus \mathcal{O}_Y(4), \quad \mathcal{N}_{Y/W} \simeq \mathcal{O}_Y(3) \oplus \mathcal{O}_Y(5).$$

In both cases the computations lead to varieties V with the same numerical characteristics. The different normal sheaves only affect the geometrical properties of the surface Q. It is likely that for a sufficiently general curve Y 1) holds; for our purposes this is not essential.

Choose one such curve Y and let Y_1, Y_2 and Y_3 be its 2-chords. The linear system $|3E - 2Y - Y_1 - Y_2 - Y_3|$ cuts out on E, in addition to the base curves $2Y + Y_1 + Y_2 + Y_3$, also a pencil of conics $|C|$ with $(C \cdot C)_E = 0$ and $\dim |C| = 1$. It follows from this that it has no further base points outside the curves Y and the three Y_i. Each curve C meets Y in 3 points and is contracted to a point by the map ρ_Y defined by the linear system

$$|3E - 2Y - Y_1 - Y_2 - Y_3|,$$

Hence the surface E on which the curve Y and its chords Y_i lie is contracted to some curve.

Let $W' \longrightarrow W$ be the blow-up of Y and the three Y_i, let E^* be the total transform of E on W'', let Y' and Y_i' be the inverse images of the corresponding curves, and let E' be the proper transform of the surface E containing Y, so that $E' \sim E^* - Y' - Y_1' - Y_2' - Y_3'$. From the exact sequence

$$0 \to H^0(\mathcal{O}_{W'}(2E^* - Y')) \to H^0\left(\mathcal{O}_{W'}\left(3E^* - 2Y' - \sum_{i=1}^{3} Y_i'\right)\right)$$

$$\to H^0\left(\mathcal{O}_{E'}\left(3E^* - 2Y' - \sum_{i=1}^{3} Y_i'\right)\right)$$

we find that

$$\dim\left|3E - 2Y - \sum_{i=1}^{3} Y_i\right| = 13.$$

Hence $\rho_Y(W) = V \subset \mathbf{P}^{13}$. As in Lemma (5.4) we check that ρ_Y defines a morphism ρ_Y': $W' \longrightarrow V$. One of the methods used to check that ρ_Y' is a birational morphism is the following. By Lemma (2.1, Part I), deg $\rho_Y' = 1$ or 2. Furthermore, if deg $\rho_Y' = 2$, then V is a rational 3-fold scroll of $\mathbf{P}^{13}$ (see 2.7, Part I). The morphism ρ_Y' contracts down the 3 surfaces Y_i' and the surface E'. It follows from this that Pic $V =$ Cl $V \simeq \mathbf{Z}$. This is sufficient to get a contradiction to the assumption that deg $\rho_Y' = 2$. An alternative method is a direct algebro-geometric analysis.

The degree of V is computed in the usual way:

$$\left(3E^* - 2Y' - \sum_{i=1}^{3} Y_i'\right)^3 = 22.$$

We leave out the detail of standard computations so as not overload what has already become a rather weighty exposition. The reader can easily reestablish them if desired.

Case b). This case only differs from the consideration of case a) in that the image W gets one singular point.

(iii) Let $g = 11$; then from (5.4.2) we have $d = 1$, 2 or 3. We restrict ourselves to the consideration of the general case; that is, when the points $z_1, \ldots, z_{d+1} \in Z'$ are in general position. The case when these points are not in general position leads, as in (ii, b), to a variety W having the same numerical invariants, but having isolated singularities. From the table (5.4.8) one sees that the case $d = 2$ is excluded. $d = 1$ is also excluded, since it contradicts (5.4.5). There remains the case $d = 3$. By Lemma (5.4), in this case W is a Fano 3-fold of the first species in $\mathbf{P}^5$ and deg $W = 4$; that is, $W \subset \mathbf{P}^5$ is the intersection of two quadrics (see 4.2, Part I).

As in the proof of (ii), Case a), let $Y' = Q' \cap Z'$; then, since $Q' \sim H^* - 3Z'$ (see (5.4, vi)), we have that $Y' \sim 3s + 4f$ on Z'. The curve Y' passes through $z_1, \ldots, z_4$ and has multiplicity at least 2 at each of them. From the formula for the genus (6.4.2) we deduce that Y' cannot be irreducible. From this as in (ii) we deduce that the surface Q' is irreducible, which contradicts the assumption Pic $V = \mathbf{Z} \cdot H$. This shows that there do not exist Fano 3-folds V containing a line and with $g = 11$ and Pic $V = \mathbf{Z} \cdot H$. An analysis of the cases where we do not have general position confirms this conclusion.

REMARK. If we leave out the condition Pic $V = \mathbf{Z} \cdot H$ and carry through the construction analogous to that in (ii) we arrive at the following result.

There exists a Fano 3-fold V with $g = 11$ and Pic $V \simeq \mathbf{Z} \oplus \mathbf{Z}$, *which is obtained as the image of a rational map $\rho_Y\colon W \longrightarrow V$ given by the linear system $|3E - 2Y - \Sigma_1^4 Y_i|$, where $W = W_4 \subset \mathbf{P}^5$ is a complete intersection of two quadrics of $\mathbf{P}^5$, $Y = C \cup Z$, where C is a smooth conic and Z is a line disjoint from C, and the Y_i are lines meeting both C and Z. Furthermore, the lines Y and Y_i, $i = 1, \ldots, 4$, all belong to a smooth hyperplane section E of W.*

(iv) Let $g = 10$; then, according to (5.4.2), $1 \leqslant d \leqslant 4$. Let us again restrict ourselves to considering the case that the points $z_1, \ldots, z_{d+1}$ are in general position on Z'. Table (5.4.8) excludes the cases $d = 1$ or 2. The case $d = 4$ is also excluded; indeed, by Lemma (5.4) we would have deg $W = 3$; that is, $W \subset \mathbf{P}^4$ is a cubic. As in (iii) the curve $Y' = Q' \cap Z'$ would again turn out to be reducible, which contradicts the condition Pic $V = \mathbf{Z} \cdot H$.

There remains the case $d = 3$. Here $W \subset \mathbf{P}^4$ is a quadric. Set $F = \varphi_{2Z}(\overline{Z}')$. From (5.4.8) and (5.4.6) we get $F \sim 2E$ and $Q' \sim 2H^* - 5Z'$. Let $Y' = Q' \cap Z'$, so that $Y' \sim 5s + 7f$ on Z'. Let us determine the multiplicity of the curve Y' at each of the 4 points z_i. Let $H' \in |H^* - 2Z'|$ be a sufficiently general surface. Then $Q' \cap H' = r_1 Z_1^0 + \cdots + r_4 Z_4^0 + X$, where r_i is the multiplicity of Q' at the general point of the line Z_i^0. Computing the intersection number $Q' \cdot Z_i^0$ in two different ways, we get

$$-3 = (2H^* - 5Z' \cdot Z_i^0) = (Q' \cdot Z_i^0) = -r_i + (X \cdot Z_i^0)_{H'}.$$

Since $(X \cdot Z_i^0)_{H'} \geqslant 0$, we get $r_i \geqslant 3$ for $i = 1, \ldots, 4$. Hence Y' has multiplicity at least 3 at each of the z_i. From the formula for the genus, as in (6.4.2) we find that $r_1 = \cdots = r_4 = 3$ and $g(\overline{Y}') = 2$, where $\overline{Y}'$ is the proper transform of Y' on $\overline{V}'$. Here we are using the fact that Q', and hence the curve Y', are irreducible, for otherwise $h^0(\mathcal{O}_{V'}(H^* - 3Z')) \neq 0$, so that $Q' \sim H^* - 3Z'$, which is not the case.

As in the proof of (ii) it can be shown that the curve $\overline{Y}' \simeq Y$ is nonsingular. Hence $Y \subset W$ is a smooth curve of genus 2, and

$$\deg Y = (5s + 7f \cdot 2s + 3f)_{Z'} - 3 \cdot 4 = 7.$$

The 4 lines $Y_i = \varphi_{2Z}(\overline{Z}_i^0)$ are obviously 3-chords of Y. As in the proof of (ii) we get that

$$\varphi_{2Z}(|\overline{H}^*|) = \left|5E - 2Y - \sum_{i=1}^{4} Y_i\right|.$$

This information is already sufficient for the construction of the inverse map $\rho_Y\colon W \longrightarrow V$. We will not give here the details of the construction.

(v) Let $g = 9$; then $1 \leqslant d \leqslant 5$. Again let us restrict ourselves to the general case. The values 1, 2 and 3 for d are excluded by (5.4.5). If $d = 5$, then, according to (5.4.2),

$$h^0(\mathcal{O}_{V'}(H^* - 3Z')) = 1.$$

Then by Lemma (5.4, vi) we have $Q' \sim H^* - 3Z'$, and as in (iii) the curve $Y' = Q' \cap Z'$ turns out to be reducible, contradicting the condition Pic $V = \mathbf{Z} \cdot H$.

There remains the case $d = 4$. By Lemma (5.4) we have deg $W = 1$; that is, $W \simeq \mathbf{P}^3$. The subsequent computations are analogous to those of the previous paragraphs. Here

$F \sim 3E$, $Q' \sim 3H^* - 7Z'$, the curve Y' contains 5 points $z_1, \dots, z_5$ with multiplicity 4 each, $\overline{Y}' \simeq Y$ is a smooth curve of genus 3 and degree 7, with 5 lines Y_i as 4-chords of Y, and, finally,

$$\varphi_{2Z}(|H^*|) = \left| 7E - 2Y - \sum_{i=1}^{5} Y_i \right|.$$

(vi) If $g = 8$, then $h^0(\mathcal{O}_{V'}(H^* - 2Z')) = g - 5 = 3$, and hence the double projection π_{2Z} maps V onto $\mathbf{P}^2$. The fibers of the morphism $\varphi_{2Z}\colon \overline{V}' \to \mathbf{P}^2$ are curves; hence

$$0 = \left(\overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{d+1} \overline{Z}_i^0\right)^3 = 2g - 22 + d + 1. \qquad (6.4.3)$$

and so $d = 5$. We have

$$K_{V'} \sim -H^* + Z', \qquad K_{\overline{V}'} \sim -\overline{H}^* + \overline{Z}' + \sum_{i=1}^{6} \overline{Z}_i^0.$$

Let $H' \in |H^* - 2Z'|$ be a sufficiently general surface, and let

$$\overline{H}' \in \left| \overline{H}^* - 2\overline{Z}' - \sum_{i=1}^{6} \overline{Z}_i^0 \right|$$

be its proper transform on $\overline{V}'$. Then obviously $\tau|\overline{H}'\colon \overline{H}' \to H'$ is an isomorphism. From the adjunction formula we get

$$-K_{H'} \sim (H^* - 2Z') \cdot Z' \sim H' \cdot Z', \qquad -K_{\overline{H}'} \sim \overline{H}' \cdot \overline{Z}'. \qquad (6.4.4)$$

Hence the anticanonical system $|-K_{H'}|$ is nonempty: it contains an irreducible elliptic curve $D = H' \cap Z' \sim 2s + 3f$ on Z'. Let us show that $h^1(\mathcal{O}_{H'}) = 0$. Consider the exact sequence

$$0 = H^1(\mathcal{O}_{V'}) \to H^1(\mathcal{O}_{H'}) \to H^2(\mathcal{O}_{V'}(-H')).$$

By duality $h^2(\mathcal{O}_{V'}(-H')) = h^1(\mathcal{O}_{V'}(-Z'))$. As we saw at the beginning of the proof of Lemma (5.4), $h^1(\mathcal{O}_{V'}(-Z')) = 0$. Hence by Castelnuovo's rationality criterion H' and also $\overline{H}'$ are rational surfaces. Let X be a fiber of the morphism $\varphi_{2Z}\colon \overline{V}' \to \mathbf{P}^2$. Then $X \subset \overline{H}'$ and $(X \cdot X)_{H'} = 0$. By the formula for the genus we obtain

$$g(X) = \frac{(X \cdot X)_{\overline{H}'} + (X \cdot K_{\overline{H}'})_{\overline{H}'}}{2} + 1 = \frac{\overline{H}' \cdot \overline{H}' \cdot \overline{Z}'}{2} + 1 = 2.$$

(vii) Let $g = 7$. Then π_{2Z} maps V onto $\mathbf{P}^1$. From (6.4.3) we get $d = 7$. In the notation of (vi),

$$(K_{\overline{H}'} \cdot K_{\overline{H}'})_{\overline{H}'} = (D \cdot D)_{\overline{H}'} = (H^* - 2Z')^2 \cdot Z' = -3. \qquad (6.4.5)$$

Furthermore, on the general fiber $\overline{H}' \in |\overline{H}^* - 2\overline{Z}' - \Sigma_1^8 \overline{Z}_i^0|$ there are 8 pairwise disjoint exceptional curves of the first kind $\overline{H}' \cap \overline{Z}_i^0$, $i = 1, \dots, 8$. As a result of contracting these the self-intersection number $(K_{\overline{H}'} \cdot K_{\overline{H}'})_{\overline{H}'}$ increases from -3 to 5. As in (vi) one shows that $\overline{H}'$ is a rational surface. Let F be the image of $\overline{H}'$ on contracting the curves $\overline{H}' \cap \overline{Z}_i^0$. To prove that F is a del Pezzo surface of degree 5 it remains to prove that the anticanonical sheaf $\mathcal{O}_F(-K_F)$ is ample. We will use the numerical criterion of ampleness, so that it is enough to show that any curve $X' \subset H'$ has nonempty intersection with $D + \Sigma_1^8 Z_i^0$. But this is obvious, since otherwise $0 = X' \cdot (H^* - 2Z') \Rightarrow X' \cdot H^* = 0$; that is, $X' \subset Z'$. But then $(X' \cdot D)_{Z'} = (X' \cdot 2s + 3f)_{Z'} \neq 0$.

Another method of proving that F is a del Pezzo surface of degree 5 is got from the following considerations. Let $\pi_Z\colon V \longrightarrow V''$ be a projection from a sufficiently general (in the sense of (3.7)) line Z. Then, according to (5.3), V'' is a variety of degree 8 of $\mathbf{P}^6$ containing a scroll R_3, and having $d + 1 = 8$ ordinary double points $v_1, \dots, v_8$ lying on R_3. The pencil of hyperplanes of $\mathbf{P}^6$ through R_3 (or equivalently, through the linear span $\mathbf{P}^4$ of R_3; see (5.5)) cuts out on V'', residually to R_3, a pencil of surfaces F of degree 5 of $\mathbf{P}^5$. By Bertini's theorem the general member of this pencil can only be singular at $v_1, \dots, v_8$. However, it is easily seen that it is in fact nonsingular. Indeed, $|F|$ cuts out on R_3 a pencil of irreducible elliptic curves $|2s + 3f - \Sigma_1^8 v_i|$. Hence the general member of $|F|$ does not have singularities at the 8 points v_i. Hence F is a del Pezzo surface. It is clear that F is the image of H' under the morphism $\varphi_Z\colon V' \longrightarrow V''$.

It is well known (see for example [3]; a modern proof has been given by Swinnerton-Dyer) that a del Pezzo surface of degree 5 defined over an arbitrary field is rational over the same field. Hence V is rational, since the general fiber of the double projection $\pi_{2Z}\colon V \longrightarrow \mathbf{P}^1$ is rational.

It is not difficult to show that V'' is a complete intersection of 3 quadrics of $\mathbf{P}^6$; that is, according to (1.7), that it is not trigonal.

The representation of V'' as a complete intersection of 3 quadrics of $\mathbf{P}^6$ passing through R_3 is the base for a proof of the existence of V. Let $(x_0, \dots, x_6)$ be homogeneous coordinates of $\mathbf{P}^6$. These can be chosen so that the scroll R_3 is given by the system of equations

$$x_0x_3 - x_1x_2 = 0,$$

$$x_0x_4 - x_2^2 = 0,$$

$$x_1x_4 - x_2x_3 = 0$$

in $\mathbf{P}^4$: $x_5 = x_6 = 0$. Then the general form of V'' will be given by equations of the form

$$\begin{aligned} x_0x_3 - x_1x_2 + x_5L_{11} + x_6L_{12} &= 0, \\ x_0x_4 - x_2^2 + x_5L_{21} + x_6L_{22} &= 0, \\ x_1x_4 - x_2x_3 + x_5L_{31} + x_6L_{32} &= 0, \end{aligned} \tag{6.4.6}$$

with the L_{ij} arbitrary linear forms in $(x_0, \dots, x_6)$ for $i = 1, 2, 3$ and $j = 1, 2$. If the L_{ij} are sufficiently general, then direct computations show that the V'' given by (6.4.6) does in fact have 8 double points lying on R_3. V is constructed by means of an inverse map $V'' \longrightarrow V$ to the projection π_Z. The proof of the theorem is complete.

(6.5) Table of Fano 3-folds of the first species

Case	r	H^3	g	3-fold V	Reference
1	4	1	33	$\mathbf{P}^3$	(4.2, Part I)
2	3	2	28	$Q_2 \subset \mathbf{P}^4$ a quadric	(4.2, Part I)
3	2	1	5	$V_1 \to W_4$ a double cover of the cone over the Veronese	(4.2, Part I)
4	2	2	9	$V_2 \to \mathbf{P}^3$ a double space with quartic ramification	(4.2, Part I)
5	2	3	13	$V_3 \subset \mathbf{P}^4$ a cubic	(4.2, Part I)
6	2	4	17	$V_{2\cdot 2} \subset \mathbf{P}^5$ an intersection of 2 quadrics	(4.2, Part I)
7	2	5	21	$V_5 \subset \mathbf{P}^6$ a section of the Grassmannian $G(1, 4) \subset \mathbf{P}^9$ by a $\mathbf{P}^6$	(4.2, Part I)
8	1	2	2	$V_2 \to \mathbf{P}^3$ a double space with sextic ramification	(7.2, Part I)
9	1	4	3	$V_4 \subset \mathbf{P}^4$ a quartic	(1.3)
10	1	4	3	$V_4' \to Q_2$ a double cover with ramification in a surface of degree 8	(7.2, Part I)
11	1	6	4	$V_{2\cdot 3} \subset \mathbf{P}^5$ an intersection of a quadric and a cubic	(1.3)
12	1	8	5	$V_{2\cdot 2\cdot 2} \subset \mathbf{P}^6$ a complete intersection of 3 quadrics	(1.3)
13	1	10	6	$V_{10} \subset \mathbf{P}^7$ the intersection of the Grassmannian $G(1, 4) \subset \mathbf{P}^9$ by a $\mathbf{P}^7$ and a quadric	(1.4)
14	1	12	7	$V_{12} \subset \mathbf{P}^8$	(6.1)
15	1	14	8	$V_{14} \subset \mathbf{P}^9$ the intersection of the Grassmannian $G(1, 5) \subset \mathbf{P}^{14}$ by a $\mathbf{P}^9$	(1.4)
16	1	16	9	$V_{16} \subset \mathbf{P}^{10}$	(6.1)
17	1	18	10	$V_{18} \subset \mathbf{P}^{11}$	(6.1)
18	1	22	12	$V_{22} \subset \mathbf{P}^{13}$	(6.1)

Here r is the index of V, g its genus, and H is a positive generator of Pic $V \simeq \mathbf{Z}$.

The form of the varieties in Cases 3–7 depends on Hypothesis (1.14, Part I) on the existence of a smooth divisor in $|H|$; this is proved in [19]. Cases 8–18 depend also on Conjecture (3.6) on the existence of lines, proved in [20].

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*Added in translation.