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FANO 3-FOLDS. I

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Abstract. This article contains a classification of special Fano varieties; we give a description of the projective models of Fano 3-folds of index $r \geqslant 2$ and of hyperelliptic Fano 3-folds.

Bibliography: 31 titles.

Introduction

In this article we begin the study of smooth projective 3-folds V, defined over an algebraically closed field k of characteristic 0, and having an ample anticanonical class K_V^{-1}. These will be referred to as Fano 3-folds; Gino Fano studied these varieties (and even not necessarily smooth ones) in a series of papers [4]–[10]. An exposition of his results can be found in Leonard Roth's book [24]. Interest in Fano varieties has recently been stimulaed in connection with progress achieved in the birational geometry of 3-folds (see [3], [12], [19] and [27]–[29]).

The simplest, and as far as birational geometry is concerned the essential examples of Fano 3-folds are projective space $\mathbf{P}^3$, the smooth hypersurfaces V_d of degree $d \leqslant 4$ in $\mathbf{P}^4$ and the following smooth complete intersections:

$V_{2\cdot 2}$—the intersection of two quadrics in $\mathbf{P}^5$;

$V_{2\cdot 3}$—the intersection of a quadric and a cubic in $\mathbf{P}^5$;

$V_{2\cdot 2\cdot 2}$—the intersection of 3 quadrics in $\mathbf{P}^6$.

This exhausts the list of Fano 3-folds which can be represented as complete intersections in projective space.

Fano's theory has two aspects:

(A) the biregular classification—the description of the projective models of Fano varieties and their numerical invariants;

(B) the birational theory--studying the problem of rationality or unirationality, and determining the group of birational automorphisms of Fano varieties.

The present article is concerned with (A). As is generally admitted, Fano's theory is in an extremely unsatisfactory state. For this reason we have been obliged to carry out

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practically all the investigations anew. Apart from this, Fano's treatment completely ignored the classification of hyperelliptic varieties (§7, Theorem 7.2); in the main he studied varieties embedded in projective space by means of their anticanonical sheaf. The definition we have chosen requires us also to study Fano varieties V for which the linear system $|K_V^{-1}|$ has base points (see §3). In §§4–6 we give a treatment of Fano 3-folds of index $r \geqslant 2$ which is more complete than Fano's [9]; and in §§5 and 6 we give a complete description of the family of lines on the varieties of index $r = 2$ and degree $\geqslant 5$. The family of lines in the cases of degrees 3 and 4 are already described in the literature (see [30] and [22]). In §1 we treat a number of basic properties of Fano 3-folds which follow immediately from the definitions and from the Riemann-Roch theorem. Already here the condition char $k = 0$ is essential, since we use the Kodaira vanishing theorem. In §2 we collect together certain technical material which we subsequently need.

We remark that the definition we have chosen excludes the consideration of anticanonical varieties with singularities. Such varieties would arise if instead of insisting on the ampleness of K_V^{-1} we merely required that for some integer $m \geqslant 1$ the sheaf K_V^{-m} defines a birational morphism $\varphi_{K_V^{-m}} : V \longrightarrow \varphi_{K_V^{-m}}$. The absence of any geometric theory of singularities of 3-folds prevents us for the moment from working with this wider definition.

In this article we will make the hypothesis (see (1.14)) that $|H_V|$ contains a smooth surface, where H_V is a primitive submultiple of K_V^{-1} in Pic V, which we will use to define a map from V to projective space; this is a weakening of the standard requirement that $|H_V|$ is without fixed components and base points. I am not aware of any example for which (1.14) fails.

We note here the essential methods used:

(1) the reduction of properties of V to properties of a smooth surface $H \in |H_V|$; that is, to the thoroughly investigated properties of Del Pezzo surfaces [20] and K3 surfaces [25];

(2) a modernization of Fano's method of projecting a variety from points or from curves of low degree, which is used to prove the existence or nonexistence of varieties V with given invariants;

(3) the geometric properties of the scrolls $X = \mathbf{P}(E)$, with E a locally free sheaf of rank $m \geqslant 2$ over the line $\mathbf{P}^1$, and of linear systems on X (see (2.5) and Reid's lemma (7.4)).

The main results of this article are Theorems (3.2), (4.2) and (7.2).

While working on this article I have received great benefit from discussions with many mathemticians; I would like here to express my deep gratitude to all of them. In particular I would like to thank Ju. I. Manin, A. N. Tjurin and M. Reid; their help and constant interest in my work has to a large extent enabled me to complete this article. Manin kindly provided me with his copies of Fano's articles, and Reid lent me his notes on the rational scrolls $X = \mathbf{P}(E)$, which I have used in this paper.

§1. Elementary properties of Fano 3-folds

(1.1) DEFINITION. A smooth complete irreducible algebraic variety V of dimension 3 over a field k will be called a *Fano 3-fold* if the anticanonical invertible sheaf K_V^{-1} on V is ample.

For the duration of this article the field k will be assumed to be an algebraically closed subfield of the complex number field $\mathbf{C}$.

(1.2) NOTATION. For any Cartier divisor D on a variety X, $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf, and in particular $K_V^{-1} = \mathcal{O}_V(-K_V)$, where K_V is a canonical divisor of V. For any invertible sheaf $L = \mathcal{O}_X(D)$ the symbol $|L|$ or $|D|$ denotes the complete linear system of effective divisors formed by the divisors of zeros of sections in $H^0(X, L)$. By definition $\dim |L| = \dim H^0(X, L) - 1$. The symbol $\sim$ denotes linear equivalence of divisors. The invertible sheaf L defines a rational map φ_L (or $\varphi_{|D|}$ if $L = \mathcal{O}_X(D)$) from X to $\mathbf{P}(H^0(X, L)) \simeq \mathbf{P}^{\dim|L|}$ (the isomorphism depending on the choice of basis in $H^0(X, L)$). If $M \subset |L|$ is a linear subsystem, then M defines a rational map which we will denote φ_M.

If X is a smooth complete variety, and $Y_1, \ldots, Y_r$ are reduced irreducible subvarieties of X of codimension codim $Y_i \geqslant 2$, and $\nu_1, \ldots, \nu_r$ are nonnegative integers, then the symbol $|L - \nu_1 Y_1 - \cdots - \nu_r Y_r|$ (or $|D - \nu_1 Y_1 - \cdots - \nu_r Y_r|$ if $L = \mathcal{O}_X(D)$) will denote the linear subsystem of $|L|$ consisting of all divisors $D \in |L|$ such that, for each $i = 1, \ldots, r$, $\nu_{y_i}(D) \geqslant \nu_i$, where $\nu_{y_i}(D)$ is the multiplicity of D at the generic point $y_i \in Y_i$.

For F an arbitrary coherent sheaf on a variety X we will often write $h^j(X, F)$, or simply $h^j(F)$, instead of $\dim H^j(X, F)$.

Let $A(X)$ denote the Chow ring of numerical equivalence classes of cycles of X. For Y and Z any two cycles on X, $(Y \cdot Z)$ or $Y \cdot Z$ will denote the product of the corresponding classes in $A(X)$. We will occasionally write Z^2 instead of $(Z \cdot Z)$, Z^3 instead of $(Z \cdot Z \cdot Z)$, etc.

We recall the Riemann-Roch theorem for a smooth complete 3-fold X and invertible sheaf $\mathcal{O}_X(D)$:

$$\sum_{i=0}^{3} (-1)^i h^i(X, \mathcal{O}_X(D)) = \frac{1}{6} D^3 - \frac{1}{4} D^2 \cdot K_X + \frac{1}{12} D \cdot (K_X^2 + c_2(X)) - \frac{1}{24} K_X \cdot c_2(X);$$
$$K_X \cdot c_2(X) = -24(1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)),$$

where $c_2(X) \in A(X)$ is the second Chern class of the tangent sheaf T_X.

From the Riemann-Roch theorem, Serre duality and the Kodaira vanishing theorem we immediately obtain

(1.3) PROPOSITION. *Let V be a Fano variety. Then*:

(i) $h^i(\mathcal{O}_V(-mK_V)) = 0$ *if* $i > 0$ *and* $m \geqslant 0$, *or if* $i < 3$ *and* $m < 0$; *in particular,* $h^i(\mathcal{O}_V) = 0$ *for* $i > 0$.

(ii) $h^0(\mathcal{O}_V(-mK_V)) = (m(m+1)(2m+1)/12)(-K_V)^3 + 2m + 1$, *and, in particular,*

$$h^0(\mathcal{O}_V(-K_V)) = \dim|-K_V| + 1 = \frac{-K_V^3}{2} + 3 \geqslant 4.$$

Let $F \in |-K_V|$ be an effective divisor in the anticanonical linear system of a Fano 3-fold V. Then from the cohomology exact sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{O}_V(-F) \to \mathcal{O}_V \to \mathcal{O}_F \to 0,$$

and from Proposition (1.3), together with the adjunction formula for any effective divisor D on V:

$$\mathcal{O}_D(K_D) \simeq \mathcal{O}_D \otimes_{\mathcal{O}_V} \mathcal{O}_V(D+K_V)$$

we get

(1.4) Proposition. (i) *$h^0(\mathcal{O}_F) = h^2(\mathcal{O}_F) = 1$, and $h^1(\mathcal{O}_F) = 0$.*
(ii) *$\mathcal{O}_F(K_F) \simeq \mathcal{O}_F$; that is, $K_F \sim 0$.*

(1.5) Corollary. *If $F \in |-K_V|$ is a smooth surface, then F is a* K3 *surface (see for example* [25]).

Suppose that there exists a smooth surface $F \in |-K_V|$, and let $\mathcal{O}_F(-K_V) = \mathcal{O}_F \otimes \mathcal{O}_V(-K_V)$ be the invertible sheaf on F obtained by restricting the anticanonical sheaf $\mathcal{O}_V(-K_V)$. Then $\mathcal{O}_F(-K_V)$ is ample, and from the Riemann-Roch theorem

$$h^0(\mathcal{O}_F(-K_V)) = \frac{-K_V^3}{2} + 2 \geqslant 3.$$

Let $X \in |\mathcal{O}_F(-K_V)|$ be any curve. Since $\mathcal{O}_F(-K_V)$ is ample on F it follows that $h^0(\mathcal{O}_X) = 1$ (see [21]). By the adjunction formula we have $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X \otimes \mathcal{O}_F(X)$, and hence $\deg K_X = (X \cdot X)_F = -K_V^3$. From this follows immediately

(1.6) Proposition. *If $F \in |-K_V|$ is a smooth surface, $X \in |\mathcal{O}_F(-K_V)|$ is a curve, and X has genus $g = g(X) = h^1(\mathcal{O}_X)$, then the following assertions are true*:
(i) $-K_V^3 = 2g - 2$.
(ii) *If $\mathcal{O}_V(-K_V)$ is very ample, then $\varphi_{|-K_V|}(V) = V_{2g-2}$ is a smooth variety of degree $-K_V^3 = 2g-2$ in $\mathbf{P}^{g+1}$, the hyperplane sections of which are* K3 *surfaces, and the curve sections of which are canonical curves $X_{2g-2} \subset \mathbf{P}^{g-1}$ of genus g.*

(1.7) Definition. The integer invariant $g = g(V) = -K_V^3/2 + 1$ will be called the *genus* of the Fano 3-fold V.

We introduce one more integer invariant. For this, note that the Picard group Pic V coincides with the Néron-Severi group NS(V), since $h^1(\mathcal{O}_V) = 0$ ((1.3, i)), and hence is a finitely generated Abelian group. Hence there exist at most a finite number of classes of invertible sheaves $L \in \operatorname{Pic} V$ and integers $s \geqslant 1$ such that $L^s \simeq K_V^{-1}$.

(1.8) Definition. The maximal integer $r \geqslant 1$ such that $H^r \simeq K_V^{-1}$ for some invertible sheaf $H \in \operatorname{Pic} V$ is called the *index* of the Fano 3-fold V.

Obviously the invertible sheaf H in (1.8) is ample, and if H is some divisor for which $H = \mathcal{O}_V(H)$ then from the Riemann-Roch theroem, Serre duality, the Kodaira vanishing theorem and the adjunction formula one easily obtains

(1.9) PROPOSITION. (i) *$h^i(H^j) = 0$ if $i \geqslant 1$ and $j \geqslant 1 - r$, where r is the index of V, or if $i < 3$ and $j < 0$.*

(ii) *For $j \geqslant 1 - r$ we have*

$$h^0(\mathcal{O}_V(jH)) = \frac{j(r+j)(r+2j)}{12} H^3 + \frac{2j}{r} + 1,$$

and in particular

$$h^0(\mathcal{O}_V(H)) = \frac{(r+1)(r+2)}{12} H^3 + \frac{2}{r} + 1 \geqslant 3.$$

(iii) *If $r \geqslant 2$, then $h^0(\mathcal{O}_H) = 1$, $h^1(\mathcal{O}_H) = h^2(\mathcal{O}_H) = 0$, and the canonical invertible sheaf of H is given by $K_H \simeq \mathcal{O}_H \otimes \mathcal{O}_V(-(r-1)H)$.*

(1.10). DEFINITION. A smooth projective surface with ample anticanonical sheaf is called a *Del Pezzo surface.*

From (iii) of Proposition (1.9) we at once get

(1.11) COROLLARY. *If V has index $r(V) \geqslant 2$, $H^r \simeq K_V^{-1}$ and $H \in |H|$ is a smooth surface, then H is a Del Pezzo surface.*

Using elementary properties of Del Pezzo surfaces (see for example [20]), we get

(1.12) PROPOSITION. *Let V have index $r \geqslant 2$, let $H^r \simeq K_V^{-1}$, and suppose that the linear system $|H|$ contains a smooth surface H. Then*

(i) $1 \leqslant r \leqslant 4$;

(ii) *if $r = 2$ then $1 \leqslant H^3 \leqslant 9$;*

(iii) *if $r = 3$ then $H^3 = 2$;*

(iv) *if $r = 4$ then $H^3 = 1$.*

PROOF. For a Del Pezzo surface H we have $1 \leqslant (K_H \cdot K_H) \leqslant 9$ (the left-hand inequality follows from the ampleness of $\mathcal{O}_H(-K_H)$, and the right-hand one from Noether's formula applied to rational surfaces). Substituting the expression for K_H in (1.9, iii), we get

$$1 \leqslant (r-1)^2 H^3 \leqslant 9. \tag{1.12.1}$$

(i), (ii) and (iv) follow from this, and for $r = 3$ we get two possibilities, $H^3 = 1$ and $H^3 = 2$; but if $r = 3$ then $H^3 = 1$ is impossible, since $-K_V^3 = r^3 H^3 = 2g - 2 \equiv 0 \pmod 2$. This proves (iii), and completes the proof.

(1.13) DEFINITION . Set $d = d(V) = H^3$. If H is very ample, then $d(V)$ is the degree of $\varphi_H(V)$ in $\mathbf{P}^{\dim|H|}$

In what follows we will study the map $\varphi_H : V \longrightarrow \mathbf{P}^{\dim|H|}$ defined by the invertible sheaf H for which $H^r \simeq K_V^{-1}$, r being the index of V. Instead of the usual hypothesis that the linear system $|H|$ is without fixed components and base points we will impose the following slightly weaker requirement:

(1.14) Hypothesis. *There exists an invertible sheaf* $\mathcal{H} \in \operatorname{Pic} V$ *such that* $\mathcal{H}^r \simeq K_V^{-1}$ (*where r is the index of V*), *and such that the linear system* $|\mathcal{H}|$ *contains a smooth surface H.*

It follows immediately from (1.14) that $|\mathcal{H}|$ is without fixed components.

The restriction $\mathcal{H} \otimes \mathcal{O}_H$ of $\mathcal{H}$ to H will be denoted $\mathcal{H}_H$. The invertible sheaf $\mathcal{H}$ in (1.14) is determined up to an r-torsion element of Pic V. However, the ampleness of $\mathcal{H}$ together with the hypothesis (1.14) excludes the existence of any torsion in Pic V, as is shown by the following assertion:

(1.15) Proposition. (i) $\operatorname{Pic} V \simeq H^2(V, \mathbf{Z})$.

(ii) *Under hypothesis* (1.14), Pic V *is torsion-free.*

Proof. We may assume that $k = \mathbf{C}$. Then the isomorphism (i) follows immediately from the cohomology exact sequence associated to the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_V \xrightarrow{\exp} \mathcal{O}_V^* \to 1,$$

and the fact that $h^i(\mathcal{O}_V) = 0$ for $i > 0$.

Let us prove (ii). By the Lefschetz theorem we have an embedding $H^2(V, \mathbf{Z}) \hookrightarrow H^2(H, \mathbf{Z})$. Since H is a K3 surface or a Del Pezzo surface (see (1.15) and (1.11)), $H^2(H, \mathbf{Z})$ is torsion-free. Hence $H^2(V, \mathbf{Z})$, and by (i) also Pic V, is torsion-free. The proposition is proved.

§2. Some preliminary results

We begin with the following elementary fact.

(2.1) Lemma. *Let* $X \subset \mathbf{P}^N$ *be a variety* (*that is, a reduced irreducible k-scheme*) *not lying in any hyperplane* $\mathbf{P}^{N-1} \subset \mathbf{P}^N$; *then*

$$\deg X \geqslant \operatorname{codim} X + 1. \tag{2.1.1}$$

Proof. Both sides of (2.1.1) remain unaltered on passing to a hyperplane section of X, and if the hyperplane section is sufficiently general the conditions of the lemma will continue to hold. Hence we can assume that $\dim X = 1$. Intersecting the curve X by a sufficiently general hyperplane E, we obtain a finite number deg X of points which span E. (2.1.1) obviously holds for these. The lemma is proved.

(2.2) Corollary. *Let V be a Fano 3-fold, let r be the index of V, and let* $\mathcal{H} = \mathcal{O}_V(H)$ *be an invertible sheaf for which* $\mathcal{H}^r \simeq K_V^{-1}$. *Suppose that* $|\mathcal{H}|$ *is without fixed components and base points. Let* $\deg \varphi_{\mathcal{H}}$ *be the degree of the morphism* $\varphi_{\mathcal{H}} : V \to \varphi_{\mathcal{H}}(V)$ ($\deg \varphi_{\mathcal{H}} < \infty$, *since* $\mathcal{H}$ *is ample*), *and set* $d = H^3$. *Then the following assertions are true*:

(i) $$\frac{d}{\deg \varphi_{\mathcal{H}}} \geqslant \frac{(r+1)(r+2)}{12} d + \frac{2}{r} - 2; \tag{2.2.1}$$

(ii) $\deg \varphi_H = 1$ *or* 2.

PROOF. Substitute $X = \varphi_H(V)$ in (2.1.1). The degree of $\varphi_H(V)$ is $d/\deg \varphi_H$, and the codimension is computed from the formula in (1.9, ii). Hence we get the inequality (2.2.1), proving (i).

One checks easily that (2.2.1) has positive integral solutions for d, r and $\deg \varphi_H$ only if $\deg \varphi_H = 1$ or 2. This gives (ii) and proves the proposition.

(2.3) REMARK. The conditions in (2.2) imply the conditions in Proposition (1.12) by Bertini's theorem. However, the inequality $r \leqslant 4$ can also be obtained directly from (2.2.1), with $r = 4$ implying that $d = 1$ and $\deg \varphi_H = 1$. If $r = 3$, then $d = 2$ and $\deg \varphi_H = 1$ ($d = 1$ is excluded since the right-hand side of (2.2.1) must be an integer). Let $\deg \varphi_H = 2$; then for $r = 2$ (2.2.1) again has the unique solution $d = 2$ (since the right-hand side must be an integer). If $\deg \varphi_H = 2$ and $r = 1$ then (2.2.1) becomes an equality; that is, the corresponding variety $W = \varphi_H(V)$ satisfies

$$\deg W = \operatorname{codim} W + 1. \tag{2.3.1}$$

The varieties W satisfying (2.3.1) are described in the classical literature (see also [25]). We reproduce this description here, and will make use of it in the classification of hyperelliptic Fano varieties in §7, and of trigonal varieties in the sequel.

(2.4). Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$ be nonnegative integers, and let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(d_m)$ be a locally free sheaf of rank m on $\mathbf{P}^1$. Set $X = \mathbf{P}_{\mathbf{P}^1}(\mathcal{E})$, and let $f: X \longrightarrow \mathbf{P}^1$ be the natural projection. Let $\mathcal{L} = f^*\mathcal{O}_{\mathbf{P}^1}(1)$ and let $\mathcal{M} = \mathcal{O}_{X/\mathbf{P}^1}(1)$ be the tautological invertible sheaf on X (such that $f_*\mathcal{M} = \mathcal{E}$). Let L and M denote divisors on X such that $\mathcal{L} = \mathcal{O}_X(L)$ and $\mathcal{M} = \mathcal{O}_X(M)$. The following assertions are proved in [25].

(2.5) LEMMA. *In the notation of* (2.4) *the following assertions are true*:

(i) $\mathcal{M}$ *is generated by its global section, and* $\mathcal{M}$ *is very ample if and only if* $d_i > 0$ *for every* $i = 1, \ldots, m$.

(ii) $h^i(X, \mathcal{M}^j) = 0$ *for* $i \geqslant 1$, $j \geqslant 0$, *and*

$$h^0(X, \mathcal{M}) = h^0(\mathbf{P}^1, f_*\mathcal{M}) = \sum_{i=1}^{m} (d_i + 1);$$

(iii) *The natural homomorphism of graded algebras*

$$\alpha : S^*H^0(X, \mathcal{M}) \to \bigoplus_{n \geqslant 0} H^0(X, \mathcal{M}^n)$$

is an epimorphism, where S^*E denotes the symmetric algebra generated by the vector space E.

(iv) *The kernel I of α is generated by elements of degree* 2 (*or* $I = 0$); *that is, the variety* $\varphi_{\mathcal{M}}(X) \subset \mathbf{P}^{\Sigma d_i + m - 1}$ *is an intersection of the quadrics which pass through it* (*or* $\varphi_{\mathcal{M}}(X) = \mathbf{P}^{\Sigma d_i + m - 1}$) (*compare* [25]).

(v) $\deg \varphi_{\mathcal{M}}(X) = \operatorname{codim} \varphi_{\mathcal{M}}(X) + 1$.

(2.6) REMARK. (iii)–(v) are only proved in [25] for a very ample sheaf $\mathcal{M}$. However, the inductive argument used there also goes through in our more general case (see also (2.9) and (2.10) below). Note also that if M is not very ample then there exists some index i_0 with $1 \leqslant i_0 < m$ such that $d_1 \geqslant \cdots \geqslant d_{i_0} > 0$ and $d_{i_0+1} = \cdots = d_m = 0$. The projection $E \to \bigoplus_{i>i_0} \mathcal{O}_{\mathbf{P}^1}$ defines an embedding

$$\mathbf{P}(\bigoplus_{i>i_0} \mathcal{O}_{\mathbf{P}^1}) \simeq \mathbf{P}^{m-i_0-1} \times \mathbf{P}^1 \to \mathbf{P}(\mathcal{E}) = X.$$

The variety $\varphi_M(X)$ is in this case a cone with vertex

$$\varphi_{\mathcal{M}}(\mathbf{P}(\bigoplus_{i>i_0} \mathcal{O}_{\mathbf{P}^1})) \simeq \mathbf{P}^{m-i_0-1}$$

over a base which is isomorphic to $P(\bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbf{P}^1}(d_i))$. The restriction $\varphi_M \mid \mathbf{P}(\bigoplus_{i>i_0} \mathcal{O}_{\mathbf{P}^1})$ coincides with the projection $\mathrm{pr}_1 : \mathbf{P}^{m-i_0-1} \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{m-i_0-1}$.

(2.7) Definition. If M is very ample, the variety $\varphi_M(X)$ will be called a *rational scroll.*

We now state the classical result in which we are interested.

(2.8) Lemma (see [25] and [23]). *Let $W \subset \mathbf{P}^N$ be a variety not lying in any hyperplane; suppose that* $\dim W \geqslant 2$ *and that* $\deg W = \operatorname{codim} W + 1$. *Then W is one of the following:*

(i) *a quadric hypersurface in $\mathbf{P}^N$;*

(ii) *a rational scroll;*

(iii) *a cone over a rational scroll; that is, a variety $\varphi_M(X)$, with X and M as in (2.4), but with M not ample;*

(iv) *the Veronese surface in $\mathbf{P}^5$;*

(v) *the cone over the Veronese surface;*

(iv) *the cone over a rational normal curve of degree n in $\mathbf{P}^n$.*

We will also use the following facts.

(2.9) Lemma. *Let X be a complete irreducible variety with* $\dim X \geqslant 2$, *and let L be an invertible sheaf on X; let $Y \in |L|$ be an effective divisor, and set $L_Y = \mathcal{O}_Y \otimes L$. Suppose that the sequence $H^0(X, L) \to H^0(Y, L_Y) \to 0$ is exact. Then, if the graded ring $\bigoplus_{n\geqslant 1} H^0(Y, L_Y^n)$ is generated by $H^0(Y, L_Y)$, it follows that the graded ring*

$$\bigoplus_{n\geqslant 1} H^0(X, \mathcal{L}^n)$$

is generated by $H^0(X, L)$.

Proof (compare (6.6) of [25]). Let $s_0 \in H^0(X, L)$ be the section defining the divisor Y. Consider the exact sequence of sheaves

$$0 \to \mathcal{L}^n \xrightarrow{s_0} \mathcal{L}^{n+1} \to \mathcal{L}_Y^{n+1} \to 0.$$

Let $s_0, s_1, \ldots, s_N$ be a basis of $H^0(X, L)$; then by the hypotheses of the lemma the restriction $\bar{s}_1, \ldots, \bar{s}_N$ to Y is a basis of $H^0(Y, L_Y)$. Let $h_{n+1} \in H^0(X, L^{n+1})$ be any element, and let $\bar{h}_{n+1} \in H^0(Y, L_Y^{n+1})$ be its restriction to Y. Then again by hypothesis $\bar{h}_{n+1} = p(\bar{s}_1, \ldots, \bar{s}_N)$, with p some homogeneous polynomial of degree $n+1$. The section $h_{n+1} - p(s_1, \ldots, s_N)$ vanishes on Y and is therefore of the form $s_0 \cdot h_n$ for some $h_n \in H^0(X, L^n)$. By induction $h_n = q(s_0, \ldots, s_N)$, with q a homogeneous polynomial of degree n, and hence $h_{n+1} = p(s_1, \ldots, s_N) + s_0 q(s_0, \ldots, s_N)$; that is, $s_0, \ldots, s_N$ generate the ring $\bigoplus_{n>0} H^0(X, L^n)$. The lemma is proved.

(2.10) LEMMA (compare (7.9) of [25]). *Let X be a complete irreducible variety with* dim $X \geqslant 2$, *let L be a very ample sheaf on X and let $\varphi_L : X \longrightarrow \mathbf{P}^N$ be the corresponding embedding. Let $\mathbf{P}^{N-1} \subset \mathbf{P}^N$ be a hyperplane, intersecting X in the subvariety $Y = X \cap \mathbf{P}^{N-1}$. Then, for any integer $m \geqslant 2$, if the homomorphism*

$$\beta : S^{m-1}H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^{m-1})$$

is surjective (where S^{m-1} denotes the $(m-1)$th symmetric product), it follows that for every hypersurface $F_M(Y)$ of degree m in $\mathbf{P}^{N-1}$ containing Y there exists a hypersurface $F_m(X)$ of degree m in $\mathbf{P}^N$ containing X and such that the restriction of $F_m(X)$ to $\mathbf{P}^{N-1} \subset \mathbf{P}^N$ coincides with $F_m(Y)$. For $m = 2$, $F_2(X)$ is uniquely determined by $F_2(Y)$. Furthermore, if $Y \subset \mathbf{P}^{n-1}$ is the intersection of the quadrics containing it, then so is $X \subset P^N$.

PROOF. We have the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{F_{m-1}(X)\} \arrow[r] \arrow[d] & \{F_m(X)\} \arrow[r, "\alpha_m"] \arrow[d] & \{F_m(Y)\} \arrow[d] & \\
0 \arrow[r] & H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m-1)) \arrow[r] \arrow[d, "\beta"'] & H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) \arrow[r] \arrow[d] & H^0(\mathbf{P}^{N-1}, \mathcal{O}_{\mathbf{P}^{N-1}}(m)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathcal{L}^{m-1}) \arrow[r] \arrow[d] & H^0(X, \mathcal{L}^m) \arrow[r] & H^0(Y, \mathcal{L}_Y^m) & \\
 & 0 & & &
\end{tikzcd}$$

β is surjective by hypothesis, so that by the Snake Lemma α_m is also surjective. If $m = 2$ then $\{F_1(X)\} = \varnothing$, since X, under the embedding φ_L, is not contained in any hyperplane, and hence α_2 is an isomorphism.

Let us prove the final assertion of the lemma. Let $F_2^1(Y), \ldots, F_2^s(Y)$ be a basis for the space of quadrics containing Y, and let $F_2^1(X), \ldots, F_2^s(X)$ be the corresponding basis of the space of quadrics containing X. If Y is an intersection of the quadrics containing it, then any hypersurface containing Y is given by a form $F_m(Y)$, with

$$F_m(Y) = \sum_{i=1}^{s} G_{m-2}^i F_2^i(Y),$$

with the G_{m-2}^i forms of degree $m-2$ on $\mathbf{P}^{N-1}$. Let H_{m-2}^i be forms of degree $m-2$ on $\mathbf{P}^N$ whose restriction to $\mathbf{P}^{N-1}$ is G_{m-2}^i (for $i = 1, \ldots, s$). Then for $F_m(X)$ any hypersurface of $\mathbf{P}^N$ containing X we have

$$F_m(X) - \sum_{i=1}^{s} H_{m-2}^i F_2^i(X) = F_{m-1}(X)L,$$

with L the linear form defining $\mathbf{P}^{N-1} \subset \mathbf{P}^N$, and $F_{m-1}(X)$ some hypersurface of degree $m-1$ containing X. By induction $F_{m-1}(X)$ belongs to the ideal generated by the quadrics $F_2^i(X)$. Hence $F_m(X)$ also belongs to this ideal. The lemma is proved.

In the rest of this paper we will frequently use the multiplication table in the Chow ring of a blown-up 3-fold.

(2.11) LEMMA (see [13] and [14]). *Let X be a nonsingular 3-fold, and let $\sigma: X' \longrightarrow X$ be the blowing up of X in the smooth center $Y \subset X$. Let $Y' = \sigma^{-1}(Y)$, and let f be either the class of a line in Y' if Y' is a plane, or the class of a fiber of Y' if Y' is a ruled surface in the Chow ring $A(X')$. Then the following assertions are true*:

(i) $A(X') \simeq \sigma^* A(X) \oplus \mathbf{Z}Y' \oplus \mathbf{Z}f$ *as additive groups, with* $\sigma_*(Y') = \sigma_*(f) = 0$, *and* $\sigma_*\sigma^* A(X) = A(X)$.

(ii) *The multiplicative structure of $A(X')$ is given by the following multiplication table*:

(a) *If Y is a point, then*

$$(Y')^2 = -f, \quad (Y')^3 = -(Y' \cdot f) = 1,$$

and (2.11.1)

$$(Y' \cdot \sigma^* Z) = (f \cdot \sigma^* Z) = 0 \quad \textit{for all } Z \in A(X).$$

(b) *If Y is a smooth curve, let $c_1(X)$ be the first Chern class of X, $\mathcal{N}_Y$ the normal sheaf to Y, and $c_1(\mathcal{N}_Y)$ its first Chern class; then*

$$\begin{gathered}
(Y')^2 = -\sigma^*(Y) + c_1(\mathcal{N}_Y) f, \\
(Y')^3 = -c_1(\mathcal{N}_Y), \\
(Y' \cdot f) = -1, \\
(Y' \cdot \sigma^*(D)) = (Y \cdot D) f, \quad (f \cdot \sigma^*(D)) = 0 \quad \textit{for all } D \in A^1(X), \\
(Y' \cdot \sigma^*(C)) = (f \cdot \sigma^*(C)) \quad \textit{for all } C \in A^2(X),
\end{gathered} \tag{2.11.2}$$

$A^i(X)$ denoting the ith component of the Chow ring $A(X)$, graded by codimension.

Furthermore, the first Chern classes on Y satisfy the usual relation

$$c_1(\mathcal{N}_Y) + 2 - 2g(Y) = (c_1(X) \cdot Y), \tag{2.11.3}$$

where $g(Y)$ is the genus of Y.

§3. Base points in the linear system $|H|$

(3.1) PROPOSITION. *Under the hypothesis* (1.14) *the linear system $|H|$ on a Fano 3-fold V does not have base points, except in the following two cases*:

(a) $r = 2$, $H^3 = 1$; *$|H|$ has a unique base point*;

(b) $r = 1$; *H contains smooth irreducible curves Z and Y, with Z a curve of genus* 0,

Y a fiber of an elliptic pencil $|Y|$ on H with $(Z \cdot Y)_H = 1$, and $H_H = \mathcal{O}_H(Z + mY)$ with $m \geqslant 3$ an integer; $|H|$ has the unique base curve Z and has no other base points.

PROOF. Since $h^1(\mathcal{O}_V) = 0$, we have an exact sequence

$$H^0(V, \mathcal{H}) \to H^0(H, \mathcal{H}_H) \to 0. \tag{3.1.1}$$

Hence any base curve of the linear system $|H|$ is a fixed component of the linear system $|H_H|$ on H, and conversely, every fixed component of $|H_H|$ is a base curve of $|H|$. If $|H|$ has no base curves, then the base points of the linear system $|H|$ are precisely the base points of the restriction $|H_H|$. First suppose that $r \geqslant 2$. Then by (1.11) H is a Del Pezzo surface, and $|-K_H| = |H_H^{r-1}|$. From the theory of Del Pezzo surfaces it is well known that $|H_H|$ is without fixed components, and only in the unique case $r = 2$, $(K_H \cdot K_H) = 1$ can it have a single base point. Thus this point is the only possible base point of $|H|$, and this gives case (a).

Now supposet that $r = 1$; H is then a K3 surface, by (1.6), and the sheaf H_H is ample on H. In [25] it is shown that for any ample sheaf L on a K3 surface H, provided that the linear system $|L|$ is without fixed components, it has no base points; and $|L|$ can only have fixed components if $L = \mathcal{O}_H(Z + mY)$, with Z and Y as in (b), and $m \geqslant 3$. Hence if $H_H = \mathcal{O}_H(Z + mY)$ then $|H|$ has a unique base curve Z and has no other base points—since there are none in the linear system $|mY|$ on H. The proposition is proved.

(3.2) REMARK. In (4.2, iv) we will show that 3-folds having the properties (3.1, a) exist, and we will even give a description of the equations defining them. For each such V, H defines a rational map $\varphi_H : V \longrightarrow \mathbf{P}^2$ whose fibers are elliptic curves. Every fiber X is irreducible, since its "degree" $(X \cdot H) = 1$. The question as to the existence of Fano 3-folds of the type (3.1, b) is answered by the following result:

(3.3) THEOREM. *Let V be a Fano 3-fold as in Proposition (3.1, b). Then $m = 3$, and there exists a Fano 3-fold $V_{(a)}$ of type (3.1, a), and a birational morphism $\tau : V \longrightarrow V_{(a)}$, which is the blow up with center a smooth elliptic curve $X \subset V_{(a)}$, with $X \in |H_{H_{(a)}}|$ for some surface $H_{(a)} \in |H_{(a)}|$.*

Conversely if $V_{(a)}$ is a Fano 3-fold of the type (3.1, a), and $X \in |H_{H_{(a)}}|$ is a smooth elliptic curve on $V_{(a)}$, then let $V_{(b)} \longrightarrow V_{(a)}$ be the blow up of $V_{(a)}$ with center X; then $V_{(b)}$ is a Fano 3-fold of type (3.1, b).

PROOF. Let $H \in |H|$ be a smooth surface, and let $H_H = \mathcal{O}_H(Z + mY)$ be as in (3.1, b). Then the linear system $|H|$ has precisely the single curve Z as its base locus. Since $(Z \cdot Z) = -2$ on the K3 surface H, it follows that $H^3 = 2m - 2$, and according to (1.3, ii) we have dim $|H| = m + 1$. The map $\varphi_H : V \longrightarrow \mathbf{P}^{m+1}$ is undefined only on $Z \subset V$, and the image of V in $\mathbf{P}^{m+1}$ is a certain surface W of degree deg $W = m$. Indeed, the restriction of φ_H to H is a morphism, defined by the invertible sheaf $\mathcal{O}_H(mY)$, according to the exact sequence (3.1.1). The linear system $|mY|$ on H is composed with the elliptic pencil $|Y|$, and, as is easily verified, $\varphi_H \mid H$ maps H to a normal rational curve of degree m in $\mathbf{P}^m$. But this curve is none other than the hyperplane section of $\varphi_H(V)$ by the hyperplane of $\mathbf{P}^{m+1}$ corresponding to H.

Let $\sigma : V' \longrightarrow V$ be the blow up of the curve Z, and let $Z' = \sigma^{-1}(Z)$ be the ruled surface. Let S' denote the unique section of the rational ruled surface $Z' \longrightarrow Z$, which is a curve with negative self-intersection, and let F' denote a fiber. Let H' and Y' be the proper transform of H and Y, and let $\varphi' = \varphi_H \circ \sigma$.

The following assertions hold.

(1) *$\varphi' : V' \longrightarrow W$ is a morphism.*

Indeed, φ' is given by the invertible sheaf $\mathcal{O}_{V'}(H')$, and the linear system $|H'|$ on V' is without base points, since the curve $Z \subset V$ can easily be seen to be the scheme-theoretic intersection of the divisors of $|H|$.

(2) $-K_{V'} \sim H'$.

This follows from general formulas for the behavior of the canonical class of varieties under blowing up.

(3) *The general fiber of the morphism φ' is a geometrically irreducible smooth elliptic curve.*

Indeed, among the fibers of φ' one finds the proper transforms of the curves of the elliptic pencil $|Y|$ of H.

(4) *The restriction of φ' to Z' is a birational morphism $\varphi' \mid Z' : Z' \longrightarrow W$, taking the fibers of the ruled surface $Z' \longrightarrow Z$ into lines of W; and hence W is either a nonsingular rational scroll, or a cone over a normal rational curve.* (*compare* (2.8))

This follows from the fact that $(Z' \cdot Y') = 1$ and $(H' \cdot F') = 1$ (see (2.11.1)), and from the fact that $\deg W = \operatorname{codim} W + 1$.

(5) *Let $\mathbf{F}_n$ denote the standard rational scroll having an exceptional section with self-intersection number $-n$. Then Z', together with its projection onto Z, is isomorphic to either $\mathbf{F}_{m-2}$ or $\mathbf{F}_m$.*

For the proof consider the normal sheaf $\mathcal{N}_Z$ to Z in V. Since Z lies on a nonsingular K3 surface H and its normal sheaf in H is isomorphic to $\mathcal{O}_Z(-2)$, since $(Z \cdot Z) = -2$ on H, we get the exact sequence

$$0 \to \mathcal{O}_Z(-2) \to \mathcal{N}_Z \to \mathcal{O}_Z(m-2) \to 0. \tag{3.3.1}$$

From the corresponding exact cohomology sequence we get the following two possibilities for the dimension of the cohomology:

(a) $h^0(\mathcal{N}_Z) = m-2,\ h^1(\mathcal{N}_Z) = 0$;

(b) $h^0(\mathcal{N}_Z) = m-1,\ h^1(\mathcal{N}_Z) = 1$.

On the other hand, since $\mathcal{N}_Z$ is a locally free sheaf of rank 2 on a smooth rational curve, it is of the form $\mathcal{O}_Z(d_1) \oplus \mathcal{O}_Z(d_2)$ for some integers d_1 and d_2. Comparing the values for the cohomology, we get:

(a′) $d_1 = -1,\ d_2 = m-3$;

(b′) $d_1 = -2,\ d_2 = m-2$.

(5) now follows from the fact that $Z' \simeq \mathbf{P}(\check{\mathcal{N}}_Z)$.

(6) *If* $Z' \simeq \mathbf{F}_{m-2}$ *then* $\varphi' | Z' : Z' \longrightarrow W$ *is the isomorphism* $\varphi_{S'+(m-1)F'}$ *defined by the invertible sheaf* $\mathcal{O}_{Z'}(S' + (m-1)F')$. *And if* $Z' \simeq \mathbf{F}_m$ *then* $\varphi' | Z' : Z' \longrightarrow W$ *is the morphism* $\varphi_{S'+mF'}$ *defined by* $\mathcal{O}_{Z'}(S' + mF')$, *and contracts* S' *to a singular point* $w_0 \in W$; *that is, in this case* W *is a cone with vertex* w_0.

Indeed, according to (4) $\varphi' | Z' : Z' \longrightarrow W$ is the morphism defined by some linear subsystem of the complete linear system $|S' + aF'|$, where the integer a is determined by the condition $(S' + aF' \cdot S' + aF') = m$. Computing the dimension of $|S' + aF'|$ by Riemann-Roch, we get that for $a = m - 1$ if $Z' \simeq \mathbf{F}_{m-2}$, or for $a = m$ if $Z' \simeq \mathbf{F}_m$, we have $\dim |S' + aF'| = m + 1$. Hence the morphism is given by the complete linear system $|S' + aF'|$ for the appropriate value of a. From this all the assertions of (6) follow easily.

(7) *Let* F *be a fiber of the scroll* W, *and let* S *be the negative section if* $W \simeq \mathbf{F}_{m-2}$, *or the vertex* w_0 *if* W *is a cone. Let* $E' = \varphi'^{-1}(S)$. *Then* E' *is an irreducible surface in* V' *containing a pencil of irreducible (possibly singular) elliptic curves, having as section the curve* S'*–the negative section of the surface* Z'. *Furthermore,* $E' \cap Z' = S'$, *and* E' *is nonsingular in a neighborhood of* S'.

All these assertions follow easily from (6). Indeed, the linear pencil of elliptic curves on E' is cut out by the pencil of surfaces $|\varphi'^{-1}(F)|$, or $|\varphi'^{-1}(F) - E'|$ if W is a cone. All the curves of this pencil are irreducible; otherwise there would exist a component C for which $(C \cdot H') = (C \cdot Z') = 0$, and hence $(C \cdot H' + Z') = (C \cdot \sigma^*H) = (\sigma_*C \cdot H) = 0$. Since C cannot be a fiber of the surface Z', this contradicts the ampleness of $\mathcal{O}_V(H)$. The remaining assertions are just as simple, and we omit the proofs.

We now proceed with the direct proof of the main assertions of Theorem (3.3), which we restate as two lemmas:

(3.4) Lemma. *Under the hypotheses of Theorem* (3.3), *the integer* m *cannot be greater than* 3, *and the surface* Z' *defined at the beginning of the proof of* (3.3) *is isomorphic to* $\mathbf{F}_1$.

Proof. By the adjunction formula we compute the canonical (dualizing) sheaf on the surface $E' \subset V'$ defined in (7):

$$\mathcal{K}_{E'} \simeq \begin{cases} \mathcal{O}_{E'}(-(m-1)X'), & \text{if } Z' \simeq \mathbf{F}_{m-2}, \\ \mathcal{O}_{E'}(-mX'), & \text{if } Z' \simeq \mathbf{F}_m, \end{cases}$$

where X' is a fiber of the elliptic pencil on E'. Since $m \geqslant 3$, we have $h^0(E', K_{E'}^{-1}) > 0$, and hence $h^0(E', K_{E'}) = 0$. By duality $h^2(E', \mathcal{O}_{E'}) = 0$. From the fact that E' has a linear pencil of curves of genus 1 it follows that $h^1(E', \mathcal{O}_{E'}) \leqslant 1$. On the other hand, the restriction exact sequence

$$H^1(X', \mathcal{O}_{X'}) \longrightarrow H^1(E', \mathcal{O}_{E'}) \longrightarrow H^2(X', \mathcal{O}_{X'}(-E')),$$

together with Serre duality:

$$h^2(X', \mathcal{O}_{X'}(-E')) = h^0(X', \mathcal{O}_{X'}(K_{X'} + E')) = 0$$

shows that $h^1(X', \mathcal{O}_{X'}) \geqslant 1$. This proves that $h^1(\mathcal{O}_{X'}) = 1$.

Let us compute $(S' \cdot S')$ on E'. This is permissible, since, as we have seen in (7), E' is nonsingular in a neighborhood of S'. Since $S' = E' \cap Z'$, we have

$$(S' \cdot S')_{E'} = (E' \cdot Z' \cdot Z')_{V'} = (S' \cdot Z')_{V'} = \begin{cases} m-3, & \text{if } Z' \simeq \mathbf{F}_{m-2}, \\ m-2, & \text{if } Z' \simeq \mathbf{F}_m. \end{cases}$$

The Riemann-Roch inequality gives

$$h^0(\mathcal{O}_{E'}(S')) \geqslant \begin{cases} m-2, & \text{if } Z' \simeq \mathbf{F}_{m-2}, \\ m, & \text{if } Z' \simeq \mathbf{F}_m. \end{cases}$$

Since S' is a section of the elliptic pencil on E', the left-hand side of this inequality cannot be greater than 1, for otherwise E' would have a linear pencil of curves, whose sections are curves of genus 1, which is impossible. Hence there remains the single possibility $m = 3$ and $Z' \simeq \mathbf{F}_1$. The lemma is proved.

(3.5) LEMMA. *In the previous notation, suppose that $m = 3$, $Z' \simeq \mathbf{F}_1$ and $E = \sigma(E')$. Then E is an exceptional smooth ruled surface on V over the elliptic curve X as base. The contraction morphism $\tau : V \longrightarrow V_{(a)}$ of E maps V onto a Fano 3-fold $V_{(a)}$ of type* (3.1, a), *and $\tau(E) = X$ is an elliptic curve in the linear system $|H_{H_{(a)}}|$ for some nonsingular surface $H_{(a)} \in |H_{H_{(a)}}|$. Conversely, the blow up of any such smooth curve X on any Fano 3-fold of type* (3.1, a) *leads to a Fano variety of type* (3.1, b).

PROOF. Let us show that the condition for the contractibility of a ruled surface onto a nonsingular curve is also satisfied by the surface E'. Let $N_{S'}$ be the normal sheaf to S' in V'; then $N_{S'}$ can be represented as an extension

$$0 \rightarrow \mathcal{O}_{S'} \rightarrow \mathcal{N}_{S'} \rightarrow \mathcal{O}_{S'}(-1) \rightarrow 0,$$

which splits as a direct sum $N_{S'} = \mathcal{O}_{S'} \oplus \mathcal{O}_{S'}(-1)$, since S' is a smooth rational curve, and $\operatorname{Ext}^1(\mathcal{O}_{S'}(-1), \mathcal{O}_{S'}) = 0$. We have $h^0(N_{S'}) = 1$ and $h^1(N_{S'}) = 0$.

According to general deformation theory, V' contains a 1-parameter family of deformations of the curve S', over a base which is nonsingular at the point corresponding to S' (see [11]). Furthermore, since the summand $\mathcal{O}_{S'}$ in $N_{S'}$ is the normal sheaf to S' in E', every deformation of S' belongs to E', and these do not intersect one another in view of the fact that the normal sheaf is trivial. It follows that E' is a ruled surface with base X'; that is, there exists a morphism $\beta\colon E' \longrightarrow X'$. Furthermore, the elliptic pencil on E' defines a morphism $\alpha : E' \longrightarrow \mathbf{P}^1$. Hence we can define the product morphism $\alpha \times \beta :$ $E' \longrightarrow X' \times \mathbf{P}^1$. Since the fiber of α and a fiber of β meet in a single point, and α has irreducible fibers, one sees easily that $\alpha \times \beta$ is an isomorphism.

Let us prove that X' is a nonsingular curve. If $x_0 \in X'$ is a singular point (there can only be one such point, and it must be a node or a cusp since $h^1(\mathcal{O}_{X'}) = 1$), and if S_0' is the smooth curve of V' consisting of singular points of E', then from the fact that $E' \simeq X' \times \mathbf{P}^1$ it follows that $N_{S'_0} \simeq \mathcal{O}_{S'_0} \oplus \mathcal{O}_{S'_0}$ and hence $c_1(N_{S'_0}) = 0$. On the other hand, by (2.11.3) we have

$$c_1(\mathcal{N}_{S'_0}) = -2 + (H' \cdot S'_0) = 1.$$

This contradiction shows that X' is nonsingular, and hence that $E' \simeq X' \times \mathbf{P}^1$ is also nonsingular.

The ruled surface $E' \longrightarrow X'$ satisfies the contractibility criterion:

$$(E' \cdot S')_{V'} = (E' \cdot Z' \cdot Z') = (S' \cdot S')_{Z'} = -1;$$

actually this is immediately visible since the second summand in the normal bundle $N_{S'}$ is $\mathcal{O}_{S'}(-1)$.

Now note that since $E' \cap Z' = S'$ it follows that the morphism $\sigma : V' \longrightarrow V$ contracting Z' preserves the structure of ruled surface of $E = \sigma(E')$, and that the numerical criterion for the contractibility of E is still satisfied; and $\sigma(S') = Z$, and $(Z \cdot E) = -1$, $(Z \cdot H) = m - 2 = 1$.

Let $\tau : V \longrightarrow V_{(a)}$ be the morphism contracting E in some nonsingular variety $V_{(a)}$; then $\tau(E) = X$ is a smooth elliptic curve on $V_{(a)}$. Let us show that the anticanonical invertible sheaf $K^{-1}_{V_{(a)}}$ is ample on $V_{(a)}$. We have $\tau_* c_1(V) = c_1(V_{(a)})$, since τ is a birational morphism, and hence $-K_{V_{(a)}} \sim \tau_*(H)$; also by the well-known formulas for the behaviour of canonical class under blowing up, we have $\tau^*(-K_{V_{(a)}}) \sim H + E$.

By the numerical criterion for ampleness we have to show that $(-K_{V_{(a)}} \cdot C) > 0$ for every curve $C \subset V_{(a)}$. For this it is enough to show that

$$(\tau^*(-K_{V_{(a)}}) \cdot \tau^* C) = (H + E \cdot \tau^* C) > 0.$$

If $C \neq X$, then the cycle $\tau^* C$ is numerically equivalent to a curve $C^0 + \nu Z$, with $\nu \geqslant 0$ and C^0 the proper transform of C. We have

$$(H + E \cdot \tau^* C) = (H + E \cdot C^0 + \nu Z) = (H \cdot C^0) + (E \cdot C^0) + \nu(H + E \cdot Z) > 0,$$

since $(H + E \cdot Z) = (H \cdot Z) + (E \cdot Z) = 1 - 1 = 0$, $(H \cdot C^0) > 0$ by the ampleness of $\mathcal{O}_V(H)$, and $(E \cdot C^0) \geqslant 0$ since $C^0 \not\subset E$. If $C = X$, then

$$(H + E \cdot \tau^* C) = (H + E \cdot Y + \mu Z) = (H + E \cdot Y) = (H \cdot Y) + (E \cdot Y) = 2.$$

Thus $K^{-1}_{V_{(a)}}$ has been proved to be ample, so that $V_{(a)}$ is a Fano 3-fold.

Now let us show that $V_{(a)}$ is of index 2. For this note that $-K_{V_{(a)}} \sim \tau_*(H) = \tau_*(H - E)$, since $\tau_*(E) = 0$, and $H - E \sim 2G$, where G is the inverse image of the fiber of the ruled surface $W \simeq \mathbf{F}_1$ under $\varphi_H : V \longrightarrow W$. Thus $K^{-1}_{V_{(a)}}$ is divisible by 2 in Pic $V_{(a)}$. On the other hand,

$$\begin{aligned} -K^3_{V_{(a)}} &= \tau^*(-K_{V_{(a)}})^3 = (H + E)^3 = H^3 + 3H^2E + 3HE^2 + E^3 \\ &= 4 + 6 + 0 - 2 = 8. \end{aligned}$$

Hence if $H_{(a)}$ is a divisor of $V_{(a)}$ such that $-K_{V_{(a)}} \sim 2H_{(a)}$, then $H^3_{(a)} = 1$; if $\mathcal{H}_{(a)} = \mathcal{O}_{V_{(a)}}(H_{(a)})$, then $H^2_{(a)} \simeq K^{-1}_{V_{(a)}}$.

Thus the Fano variety $V_{(a)}$ has index $r = 2$ and degree $d = 1$, and thus is of type (3.1, a). It is clear that the curve X lies on some surface $\tau(G) \in |\mathcal{H}_{(a)}|$ and passes through the base point of the linear system $|\mathcal{H}_{(a)}|$. We can assume that $H_{(a)} = \tau(G)$, and since $(-K_{V_{(a)}} \cdot X) = 2$ we have $(H_{(a)} \cdot X) = 1$ and $X \in |\mathcal{H}_{H_{(a)}}|$. Thus the direct part of Lemma (3.5) is proved.

(3.6) PROOF OF THE "CONVERSE" PART OF LEMMA 3.5. Let $\tau : V \longrightarrow V_{(a)}$ be the blow up of $V_{(a)}$ with center in a smooth curve $X \in |\mathcal{H}_{H_{(a)}}|$, and let $E = \tau^{-1}(X)$. We have

$$-K_V \sim \tau^*(-K_{V_{(a)}}) - E \sim 2\tau^*(H_{(a)}) - E.$$

We can assume that $H_{(a)}$ is a smooth surface, and $(X \cdot X)_{H_{(a)}} = 1$. If $\mathcal{N}_X$ is the normal sheaf to X in $V_{(a)}$, then it can be represented as an extension

$$0 \to \mathcal{O}_X(1) \to \mathcal{N}_X \to \mathcal{O}_X(1) \to 0,$$

which as is easily seen splits as a direct sum

$$\mathcal{N}_X \simeq \mathcal{O}_X(1) \oplus \mathcal{O}_X(1).$$

Hence $E \simeq \mathbf{P}(\check{\mathcal{N}}_X) \simeq \mathbf{P}^1 \times X$.

The linear system $|\tau^*\mathcal{H}_{(a)}|$ has a unique base curve $Z = \tau^{-1}(v)$, where $v \in V_{(a)}$ is the base point of $|\mathcal{H}_{(a)}|$. In the exact sequence

$$\begin{aligned} 0 \to H^0(V, \mathcal{O}_V(\tau^*H_{(a)} - E)) &\to H^0(V, \mathcal{O}_V(\tau^*H_{(a)})) \\ &\to H^0(E, \mathcal{O}_E \otimes \mathcal{O}_V(\tau^*H_{(a)})) \end{aligned}$$

the final arrow is an epimorphism, and the final vector space is 1-dimensional, since $|\tau^*\mathcal{H}_{(a)}|$ cuts out on E only the curve Z, which is a fiber of the projection $\mathbf{P}^1 \times X \longrightarrow X$, and does not move in a linear system, since X is an elliptic curve. It follows that the surface $\tau^{-1}(H_{(a)}) - E$ cuts out on E some fiber Y of the projection $E \longrightarrow \mathbf{P}^1$. Indeed, $E \cap (\tau^{-1}(H_{(a)}) - E)$ is numerically equivalent to a cycle of the form $Y + aZ$. But

$$(Y + aZ \cdot Y + aZ)_E = (\tau^*H_{(a)} - E)^2 \cdot E = 0$$

according to (2.11.1). Hence $a = 0$.

Now let us show that the linear system $|-K_V| = |2\tau^*H_{(a)} - E|$ has Z as its unique base curve. For this note that this system cuts out on E curves numerically equivalent to a cycle of the form $Y + bZ$. The coefficient b is determined by the condition

$$(Y + bZ \cdot Y + bZ)_E = (2\tau^*H_{(a)} - E)^2 E = 2,$$

and hence $b = 1$. Obviously any linear system on E of the form $|Y_0 + Z_0|$, where Y_0 and Z_0 are fibers of the first and second projections respectively of $E \simeq \mathbf{P}^1 \times X$, contains Z_0 as fixed component. Hence the base locus of the linear system $|-K_V|$ is some fiber Z_0 of E and only this. But for any fiber Z_0 distinct from Z one easily gives an example of a reducible divisor in $|\tau^*H_{(a)}| + |\tau^*H_{(a)} - E|$ not containing Z_0. Hence the base curve of $|-K_V|$ is Z.

It is not difficult to show, using Bertini's theorem, that $|-K_V|$ contains a smooth surface.

To prove that $\mathcal{O}_V(-K_V)$ is very ample it is sufficient to check that $(-K_V \cdot Z) > 0$. By direct computation, one gets

$$(-K_V \cdot Z) = (2\tau^* H_{(a)} - E \cdot Z) = 1.$$

Finally, according to Proposition (3.1), V can only be of type (3.1, b). The proof of Lemma (3.5), and with it that of Theorem (3.3), is thus complete.

§4. Fano 3-folds of index r $\geqslant 2$. The statement and beginning of proof of Theorem (4.2)

(4.1) Let V be a Fano 3-fold of index r satisfying the hypothesis (1.13); let H be an invertible sheaf, and $H \in |H|$ a surface as in (1.14). Then according to (1.12) we have $1 \leqslant r \leqslant 4$. Recall that $d = H^3$, and that $1 \leqslant (r-1)^2 d \leqslant 9$ for $r \geqslant 2$ (see (1.12.1)). For any variety X and a very ample invertible sheaf $\mathcal{O}_X(1)$ we write X_m to indicate that X has degree m with respect to $\mathcal{O}_X(1)$. In this section we begin the proof (for its completion, see §6) of the following theorem.

(4.2) THEOREM. *Let V be a Fano 3-fold of index $r \geqslant 2$ satisfying the hypothesis* (1.14). *Then the following assertions hold*:

(i) *If $r \geqslant 3$, then $\varphi_H : V \xrightarrow{\sim} \mathbf{P}^3$ is an isomorphism for $r = 4$, and $\varphi_H : V \xrightarrow{\sim} V_2 \subset \mathbf{P}^4$ is an isomorphism of V with a smooth quadric of $\mathbf{P}^4$ for $r = 3$.*

(ii) *If $r = 2$, then a variety V only exists for $1 \leqslant d \leqslant 7$; for $d \geqslant 3$, $\varphi_H : V \xrightarrow{\sim} V_d \subset \mathbf{P}^{d+1}$ is an embedding of V as a subvariety V_d of degree d in $\mathbf{P}^{d+1}$, with V_d projectively normal; and if $d \geqslant 4$, then V_d is the intersection of the quadrics containing it. Conversely, for any $d \geqslant 3$, every smooth projectively normal 3-fold $V_d \subset \mathbf{P}^{d+1}$ not lying in any hyperplane is a Fano 3-fold, and has index 2, apart from the case $r = 4$, $d = 8$, when V_8 is the image of $\mathbf{P}^3$ in $\mathbf{P}^9$ under the Veronese embedding.*

(iii) *If $r = 2$ and $3 \leqslant d \leqslant 7$, then; for $d = 7$, V_7 is the projection of the Veronese 3-fold $V_8 \subset \mathbf{P}^9$ from some point of V_8;*

for $d = 6$, $V_6 \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ in its Segre embedding;

for $d = 5$, $V_5 \subset \mathbf{P}^6$ is unique up to projective equivalence, and can be realized in either of the two following ways:

(a) *as the birational image of a quadric $W \subset \mathbf{P}^4$ under the map defined by the linear system $|\mathcal{O}_W(2) - Y|$ of quadrics passing through a twisted cubic Y;*

(b) *as the section of the Grassmannian* Gr(2, 5) *of lines in $\mathbf{P}^4$ by 3 hyperplanes in general position*;

for $d = 4$, V_4 is any smooth intersection of two quadrics in $\mathbf{P}^5$; for $d = 3$, V_3 is any smooth cubic hypersurface of $\mathbf{P}^4$.

(iv) *If $r = 2$ and $d = 1$ or 2, then*:

for $d = 2$, $\varphi_H : V \longrightarrow \mathbf{P}^3$ is a double covering with smooth ramification surface $D_4 \subset \mathbf{P}^3$ of degree 4, and any such variety is a Fano 3-fold with $r = 2$ and $d = 2$; and every

Fano 3-fold with $r = 2$ and $d = 2$ can be realized as a smooth hypersurface of degree 4 in the weighted projective space $\mathbf{P}(x_0, \ldots, x_4)$, where x_4 has degree 2, and the remaining x_i have degree 1;

for $d = 1$, $\varphi_H : V \longrightarrow \mathbf{P}^2$ is a rational map with a single point of indeterminacy, and with irreducible elliptic fibers; and V can be realized in either of the two following ways:

(a) *$\varphi_{K_V^{-1}} : V \longrightarrow W_4$ is any double cover of the cone W_4 over the Veronese surface $F_4 \subset \mathbf{P}^5$, having smooth ramification divisor $D \subset W_4$ by a cubic hypersurface not passing through the vertex of the cone;*

(b) *any smooth hypersurface of degree 6 in the weighted projective space $\mathbf{P}(x_0, \ldots, x_4)$, where x_0, x_1 and x_2 have degree 1, x_3 has degree 2, and x_4 degree 3.*

(4.3) PROOF OF (4.2, i). Both assertions follow at once from (1.9, ii), (1.12, iii, iv) and (3.1). In the case $r = 3$ one has only to show that the quadric $W = \varphi_H(V)$ is nonsingular. This follows from the following general assertion.

(4.4) PROPOSITION. *Let V be a Fano 3-fold of index $r \geqslant 1$ satisfying the hypothesis (1.13). If $|H|$ is without base points (see (3.1)) and $\deg \varphi_H = 1$ (that is, the morphism $\varphi_H : V \longrightarrow \varphi_H(V)$ is birational), then the following assertions are true:*

(i) *The natural homomorphism of graded algebras*

$$S^* H^0(V, \mathcal{H}) \to \bigoplus_{n \geqslant 0} H^0(V, \mathcal{H}^n) \tag{4.4.1}$$

is surjective.

(ii) *H is very ample, and the image $\varphi_H(V)$ is projectively normal.*

(iii) *The ideal I_V (the kernel of (4.4.1)) is zero if $r = 4$, is generated by elements of degree 2 if $r = 3$ or if $r = 2$ and $d \geqslant 4$, and is generated by elements of degree 2 and 3 if $r = 1$ and $d > 4$.*

(4.5) PROOF. By Lemma (2.9) we only have to prove (i) for a smooth surface $H \in |H|$ and for the invertible sheaf H_H. Since $|H|$ is without fixed components and base points, by (3.1.1) the linear system $|H_H|$ on H is also without fixed components and base points. Since $\dim |H_H| \geqslant 2$ by (1.9, ii), by Bertini's theorem there exists a smooth curve $X \in |H_H|$, which is irreducible since H_H is ample (see [21]). By the same Lemma (2.9) we then only have to prove (i) for the curve X and the invertible sheaf $H_X = \mathcal{O}_X \otimes H$.

Now if $r \geqslant 2$, then, as one checks easily by the adjunction formula, X has genus $g(X) \leqslant 1$, and under the conditions of the proposition $\deg H_X \geqslant 2g + 1$. Hence our assertion follows from a result of [18].

If $r = 1$ then $H_X \simeq K_X$ is the canonical invertible sheaf, $\deg H_X = 2g(X) - 2$, and by hypothesis the morphism $\varphi_{H_X} : X \longrightarrow \varphi_{H_X}(X)$ is birational; that is, X is a nonhyperelliptic curve. In this case our assertion follows from a classical theorem of Max Noether (see, for example, [31]). This proves (i).

The first assertion of (ii) follows from (i), as was shown in [18], and the second is merely a restatement of (i).

For the proof of (iii), use Lemma (2.10). By this lemma it is sufficient to prove the assertion for a curve $X \in |H_H|$ and the sheaf H_X. If $r = 4$, then (iii) follows from (4. 2, i), which has already been proved.

According to the main result of [26], the ideal I_X is generated by elements of degree 2 if $\deg H_X \geqslant 2g(X) + 2$. This proves (iii) for $r = 3$ and $r = 2$, $d \geqslant 4$. The case $r = 1$ follows from the classical Noether-Enriques-Petri theorem on canonical curves (see, for example, [31]). The proposition is proved.

(4.6) PROOF OF (4.2, ii). If $r = 2$ and $d \geqslant 3$, then from (2.2.1) it follows that $\deg \varphi_H = 1$. According to (3.1) the linear system $|H|$ is without base points. Hence all the hypotheses of Proposition 4.4 are fulfilled. Hence, since $\dim |H| = d + 1$ (1.9, ii), we get that for $d \geqslant 3$ the morphism $\varphi_H : V \longrightarrow \mathbf{P}^{d+1}$ is an embedding, and if $d \geqslant 4$, the image $\varphi_H(V)$ is projectively normal, and is an intersection of quadrics.

Conversely, if $V = V_d$ is a smooth 3-fold of degree $d \geqslant 3$ in $\mathbf{P}^{d+1}$, not lying in any hyperplane, then its smooth hyperplane section H is a rational surface (see [20]). Hence, using the exact sequence

$$0 \to H^0(V, \mathcal{O}_V(m-1)) \to H^0(V, \mathcal{O}_V(m)) \xrightarrow{\alpha_m} H^0(H, \mathcal{O}_H(m)) \to \dots$$

and the Kodaira vanishing theorem, one shows easily by induction that α_m is an epimorphism for every $m \geqslant 0$. From this it follows that if V is projectively normal, then so is the section $H \subset \mathbf{P}^d$. Hence H is a Del Pezzo surface; that is, $K_H^{-1} \simeq \mathcal{O}_H(1)$ (see [20]). By the Lefschetz theorem one has a natural embedding $\operatorname{Pic} V \hookrightarrow \operatorname{Pic} H$. Hence, using the adjunction formula, we get that $K_V^{-1} \simeq \mathcal{O}_V(2)$; that is, V is a Fano 3-fold of index $r \geqslant 2$. From (4.2, i) it follows that the index can only be distinct from 2 if $r = 4$, $d = 8$, and V_8 is the image of $\mathbf{P}^3$ in $\mathbf{P}^9$ under the Veronese map.

For the proof of (4.2, ii) it remains to show that there do not exist Fano 3-folds $V_d \subset \mathbf{P}^{d+1}$ of index 2 for $d = 8$ and 9, since according to (1.12.1) $d \leqslant 9$.

First consider the case $d = 9$. Let H be the smooth hyperplane section of V_9. Then H is a Del Pezzo surface of degree 9 in P^9; that is, it is the image of $\mathbf{P}^2$ under the Veronese map by means of the invertible sheaf $\mathcal{O}_{\mathbf{P}^2}(3)$. We can assume that $k = \mathbf{C}$. By the Lefschetz theorem there exists an epimorphism

$$H_2(H, \mathbf{Z}) \xrightarrow{i_*} H_2(V, \mathbf{Z}) \to 0, \tag{4.6.1}$$

where $i : H \longrightarrow V_9$ is the natural embedding. Since $H_2(H, \mathbf{Z}) \simeq \mathbf{Z}$ and $\operatorname{rk} H_2(V_9, \mathbf{Z}) \neq 0$, i_* is an isomorphism. By duality $H^2(V_9, \mathbf{Z}) \simeq H^2(H, \mathbf{Z})$; hence, by (1.15, i), $\operatorname{Pic} V_9 \simeq \operatorname{Pic} H$, with the isomorphism given by the natural restriction. But this is impossible, since $\operatorname{Pic} V_9$ is generated by H, whereas the restriction of H to H is divisible by 3. This contradiction shows that such a 3-fold V_9 does not exists.

Now let $d = 8$. For the hyperplane section H there are now two possibilities:

(a) H is the image of $\mathbf{P}^1 \times \mathbf{P}^1$ under the anticanonical embedding;

(b) H is the image of the standard ruled surface F_1 under the anticanonical embedding.

Since $\operatorname{Pic} V \simeq H^2(V, \mathbf{Z})$ and H is a primitive element of $\operatorname{Pic} V$, because $r(V) = 2$, by Poincaré duality there exists an element $Z \in H_2(V, \mathbf{Z})$ such that $(H \cdot Z) = 1$. According to (4.6.1) the cycle Z can be chosen to lie on the hyperplane section H. Since the surface H is rational, any cycle of $H_2(H, \mathbf{Z})$ is homologous to some divisor. We can therefore assume that Z is a divisor of H.

In case (a) we therefore get an immediate contradiction, since any divisor in $H \subset \mathbf{P}^8$ has even degree.

In case (b), $H \subset \mathbf{P}^8$ contains a unique line—the image of the exceptional curve of F_1. Let us show that in this case V_8 must contain a surface swept out by lines. Let $X = H_1 \cap H_2$ be the smooth intersection of two hyperplanes H_1 and H_2; then X is an irreducible curve of genus 1. The hyperplanes H_1 and H_2 span a pencil of hyperplane sections of V having base locus X, defining a rational map $\psi : V_8 \longrightarrow \mathbf{P}^1$ which is indeterminate only at X. Every nonsingular fiber of this pencil contains a single line; and these lines sweep out some ruled surface on V_8, whose closure we denote by $P \subset V_8$. X does not lie on P; indeed, if $Z \subset H$ is a line then $(Z \cdot X)_H = (Z \cdot H) = 1$. Hence if $X \subset P$ then X is either a section of the ruled surface P, or a component of a fiber. Obviously, neither of these is possible, since X is an irrational curve. Hence the intersection $P \cap H$ consists of the line Z only. From this one computes easily the degree of the surface P:

$$(P \cdot H^2) = (Z \cdot H) = 1;$$

that is, P is a plane.

The plane $P \subset V_8$ satisfies the numerical criterion for contractibility to a nonsingular point:

$$(P \cdot Z) = (P \cdot P \cdot H) = (Z \cdot Z)_H = -1.$$

The contraction is achieved by the invertible sheaf $\mathcal{O}_V(H + P)$. Indeed, since

$$\mathcal{O}_P \otimes \mathcal{O}_V(H+P) \simeq \mathcal{O}_P(1) \otimes \mathcal{O}_P(-1) \simeq \mathcal{O}_P$$

and $h^1(\mathcal{O}_V(H)) = 0$ by (1.9, i), we have the exact sequence

$$0 \to H^0(\mathcal{O}_V(H)) \to H^0(\mathcal{O}_V(H+P)) \to H^0(P, \mathcal{O}_P) \to 0. \tag{4.6.2}$$

From this it follows that $\dim |H + P| = \dim |H| + 1$; that is, P is not a fixed component of the linear system $|H + P|$, and by a standard method one checks that because $\mathcal{O}_V(H)$ is very ample, $\mathcal{O}_V(H + P)$ defines a birational morphism contracting P to a nonsingular point. For $V = V_8$ we have $h^0(\mathcal{O}_V(H + P)) = 11$ and $(H + P)^3 = 9$, so that $\varphi_{|H+P|}(V_8) = V_9$ is a smooth 3-fold of degree 9 in $\mathbf{P}^{10}$. Since $\varphi_{|H+P|}$ is a birational morphism, we have $\varphi_{|H+P|*}(K_{V_8}) = K_{V_9}$, and hence

$$-K_{V_9} \sim 2\varphi_{|H+P|*}(H) = 2\varphi_{|H+P|*}(H+P);$$

that is, $K_{V_9}^{-1} \simeq \mathcal{O}_{V_9}(2)$. Hence V_9 is a Fano 3-fold of index $r \geqslant 2$. Since $-K_{V_9}^3 = 2^3(H + P)^3 = 2^3 \cdot 9$, we have $r = 2$. But we have already shown that such Fano 3-folds do not exist. Hence Fano 3-folds $V_8 \subset \mathbf{P}^3$ of index 2 do not exist either. This completes the proof of (4.2, ii).

(4.7) REMARKS. (a) One can see that every projectively normal 3-fold of degree 9 in $\mathbf{P}^{10}$ having no more than isolated singularities is the cone over the Del Pezzo surface of degree 9 in $\mathbf{P}^9$. Every projectively normal 3-fold of degree 8 in $\mathbf{P}^9$ having only isolated singularities is either the cone over the Del Pezzo surface $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^8$, or the cone over $\mathbf{F}_1$ (that is, the projection of the cone $V_9 \subset \mathbf{P}^{10}$ from some nonsingular point), or the smooth Fano variety of index 4, the Veronese image of $\mathbf{P}^3$.

(b) It is also not difficult to prove that any smooth 3-fold $V_d \subset \mathbf{P}^{d+1}$ with $3 \leqslant d \leqslant 5$ not lying in any hyperplane is projectively normal, and is hence a Fano 3-fold of index 2.

§5. The family of lines on the Fano 3-folds $V_d \subset \mathbf{P}^{d+1}$

(5.1) PROPOSITION. *Let $V = V_d \subset \mathbf{P}^{d+1}$ with $3 \leqslant d \leqslant 7$ be a Fano 3-fold of index 2, let* $\mathrm{Gr}(2, d + 2)$ *be the Grassmannian of lines in $\mathbf{P}^{d+1}$, and let $T \subset \mathrm{Gr}(2, d+2)$ be the subscheme parametrizing the family of lines lying on V. Then the following assertions hold:*

(i) *T is closed, and each irreducible component of T is a smooth complete surface.*

(ii) *Let $T_0 \subset T$ be an irreducible component, and let S_0 be the family of lines parametrized by T_0; that is, the restriction to T_0 of the universal family $\mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is the tautological locally free sheaf of rank 2 on* $\mathrm{Gr}(2, d+2)$. *Consider the diagram of natural maps:*

$$\begin{array}{ccc} S_0 & \xrightarrow{\psi_0} & V \\ {\scriptstyle\pi}\downarrow & & \\ T_0 & & \end{array}$$

Then either ψ_0 is surjective, and through the general point of V there pass a finite number of lines, or it has image $\psi_0(S_0) = P_0$, a plane lying on V.

PROOF. (i) T is the Hilbert scheme of closed subschemes of V with Hilbert polynomial $p(n) = n + 1$. Its existence follows from general theorems [11], and the fact that it is closed is proved in just the same way as the analogous assertion in [17], Lecture 15. Also, T is nonempty. Indeed, any smooth hyperplane section H of V is a Del Pezzo surface of degree d with $3 \leqslant d \leqslant 7$, and it is well known (see for example [15]) that there exist a finite number of lines on H, which implies the existence of lines on V.

For the proof of the nonsingularity of T let us turn to the infinitesimal deformation theory (see [11]). Let $Z \subset V$ be any line, and let $\mathcal{N}_Z$ be its normal sheaf in V. There exists a smooth hyperplane section H containing Z; indeed, the linear system of hyperplane sections passing through V obviously has base locus precisely Z. Hence it is without fixed components, and by Bertini's theorem its general element can only have singular points along Z. The claim then follows from a dimension count, since hyperplanes containing Z form a linear system of dimension $d - 1$, whereas the hyperplanes containing the tangent plane to some point of Z form a variety of dimension $\leqslant d - 2$.

Since $(Z \cdot Z)_H = -1$, we have the exact sequence

$$0 \to \mathcal{O}_Z(-1) \to \mathcal{N}_Z \to \mathcal{O}_Z(1) \to 0. \tag{5.1.1}$$

From the corresponding cohomology exact sequence we get

$$h^0(\mathcal{N}_Z) = 2, \qquad h^1(\mathcal{N}_Z) = 0. \tag{5.1.2}$$

According to [11] it follows from this that T is smooth and of dimension 2 at the point corresponding to Z. This proves (i).

In assertion (ii), if ψ_0 is surjective, then, since $\dim S_0 = \dim V = 3$, ψ_0 is generically finite-to-one, and through almost every point of V there pass a finite number of lines. If ψ_0 is not surjective, then, since S_0 is complete and irreducible, its image $\psi_0(S_0) = P_0$ is a closed irreducible subvariety of V. Clearly one cannot have $\dim P_0 \leqslant 1$. There remains the possibility $\dim P_0 = 2$; that is, P_0 is a surface containing a 2-dimensional family of lines. From the classification of surfaces it is well known that such a surface can only be the plane. The proposition is proved.

(5.2) PROPOSITION. *In the notation of* (5.1) *suppose that* $\psi_0 : S_0 \longrightarrow V$ *is surjective, and let* $D_0 \subset S_0$ *be the subvariety of points of* S_0 *where* ψ_0 *is not étale. Then the following assertions are true*:

(i) *If* $\deg \psi_0 > 1$, *there exists a closed curve* $C_0 \subset T_0$ *such that* $\pi_0^{-1}(C_0) = D_0$.

(ii) *The dimension of any fiber of* ψ_0 *is no greater than* 1, *and either* ψ_0 *is a finite morphism, or there exist a finite number of* 1*-dimensional fibers.*

(iii) *For any point* $t \in T_0$ *let* $\mathcal{N}_t$ *be the normal sheaf to the corresponding line* $Z_t \subset V$. *Then there are two possibilities*:

$$\begin{aligned} &\text{(a) } \mathcal{N}_t \simeq \mathcal{O}_{Z_t} \oplus \mathcal{O}_{Z_t}, \\ &\text{(b) } \mathcal{N}_t \simeq \mathcal{O}_{Z_t}(-1) \oplus \mathcal{O}_{Z_t}(1), \end{aligned} \tag{5.2.1}$$

and if $t \in T_0 - C_0$ *then* $\mathcal{N}_t$ *is of type* (a).

PROOF. (i) Since $\operatorname{char} k = 0$, ψ_0 is a separable morphism, and hence D_0 is a closed subset of S_0 distinct from S_0.

On the other hand, D_0 is nonempty. Indeed, the inverse image of $\psi_0^{-1}(H)$ of the general hyperplane section H is an irreducible surface by Bertini's theorem, and, since H is rational and $\deg \psi_0 > 1$, the morphism $\psi_0^{-1}(H) \longrightarrow H$ is ramified. Hence ψ_0 itself is also ramified; that is, D_0 is nonempty. By Zariski's theorem on the purity of the branch locus, D_0 is a divisor of S_0.

We compute the canonical class of S_0 by two methods. On the one hand,

$$K_{S_0} \sim \psi_0^*(K_V) + D_0' = \psi_0^*(2H) + D_0'. \tag{5.2.2}$$

where D_0' is an effective divisor supported precisely on D_0. On the other hand, since $S_0 = \mathbf{P}(E_0)$, where E_0 is a locally free sheaf of rank 2, the restriction to T_0 of the tautological sheaf E on $\mathrm{Gr}(2, d+2)$, letting $M_0 = \psi_0^* \mathcal{O}_V(1)$ be the Grothendieck tautological sheaf on $\mathbf{P}(E_0)$ one gets the following formula for the canonical class K_{S_0} (see for example [14]):

$$\mathcal{K}_{S_0} \simeq \mathcal{M}_0^{-2} \otimes \pi_0^* \mathcal{K}_{T_0} \otimes \pi_0^* \det \mathcal{E}_0. \tag{5.2.3}$$

Let C_0' denote a divisor on T_0 such that $\mathcal{O}_{T_0}(C_0') = K_{T_0} \otimes \det E_0$; comparing (5.2.2) and (5.2.3) we get $D_0' \sim \pi_0^*(C_0')$. Since D_0' is an effective divisor, and $\pi_0 : S_0 \longrightarrow T_0$ is a locally trivial fiber bundle with fiber P^1, the divisor C_0' can be chosen to be effective; for example, $C_0' = \pi_0(D_0')$. Setting $C_0 = \pi_0(D_0)$, we obtain a proof of (i).

For the proof of (ii), note that if $\dim \psi_0^{-1}(v) = 2$ for some point $v \in V$, then every 2-dimensional irreducible component of $\psi_0^{-1}(v)$ would be contained in D_0. But according to (i) any such component contains a 1-dimensional family of lines from S_0, and hence ψ_0 must map it onto some ruled surface in V, and not into a point v. This contradiction proves that a fiber of ψ_0 cannot have dimension 2.

Let $W = \{v \in V \mid \dim \psi_0^{-1}(v) > 0\}$. Then W is a closed subset of V. If $\dim W > 0$, then for any curve $Y \subset W$ every 2-dimensional component $\psi_0^{-1}(Y)$ is contained in D_0, and by what we have seen above cannot be mapped to Y. Hence $\dim W \leqslant 0$, which proves (ii).

The fact that in (iii) only the two possibilities (a) and (b) can occur follows at once from a consideration of the exact sequence (5.11) and the values of the cohomology (5.1.2). Let $t \in T_0 - C_0$. Then according to (i) ψ_0 is étale along the fiber $\pi_0^{-1}(t) = S_{0,t}$. Hence the differential $d\psi_0$ of ψ_0 defines a monomorphism of the tangent bundles

$$d\psi_0 | S_{0,t} : T_{S_0 | S_{0,t}} \to T_{V | Z_t}$$

and since $\psi_0 | S_{0,t} : S_{0,t} \longrightarrow Z_t$ is an isomorphism, $d\psi_0$ induces a monomorphism of normal bundles $N_{S_{0,t}} \longrightarrow N_{Z_t}$. Since $N_{S_{0,t}}$ is a trivial rank 1 bundle, it follows from this that N_t is an extension

$$0 \to \mathcal{O}_{Z_t} \to \mathcal{N}_t \to \mathcal{O}_{Z_t} \to 0,$$

which must be split, since Z_t is a line. This proves (iii), and with it the proposition.

(5.3) Proposition. (i) *A Fano 3-fold $V = V_d \subset \mathbf{P}^{d+1}$ does not contain any planes for $3 \leqslant d \leqslant 6$, and contains a unique plane if $d = 7$.*

(ii) *Through every point $v \in V_d$ for $3 \leqslant d \leqslant 7$ there passes at least one line.*

Proof. (ii) for $3 \leqslant d \leqslant 6$ follows from (i) and from (5.1, ii). In the case $d = 7$ both (i) and (ii) follow from the explicit description of $V_7 \subset \mathbf{P}^8$ given in (6.1) and (6.2), independently of the proposition in hand.

It remains to prove (i) for $3 \leqslant d \leqslant 6$. Let $P \subset V$ be a plane; then P satisfies the condition for contractibility to a nonsingular point. Indeed, for any line $Z \subset P$, the normal sheaf N_Z of Z in V fits into an exact sequence

$$0 \to \mathcal{O}_Z(1) \to \mathcal{N}_Z \to \mathcal{O}_Z(-1) \to 0,$$

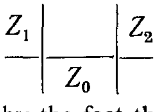
since $\det N_Z \simeq \mathcal{O}_Z$ according to (5.1.1). Hence $(Z \cdot P) = -1$. Just as in (4.6, b) the contraction of P is given by the invertible sheaf $\mathcal{O}_V(H + P)$, where H is the hyperplane section, and we have an exact sequence (4.6.2). As in (4.6, b) we get $\varphi_{|H+P|} : V_d \longrightarrow V_{d+1}$. The inverse map is the projection from the point $v = \varphi_{|H+P|}(P)$, $\chi_V : V_{d+1} \longrightarrow V_d$. Since the restriction of $\varphi_{|H+P|}$ is an isomorphism $V_d - P \longrightarrow V_{d+1} - v$, the restriction of the projection χ_v is the inverse isomorphism $\chi_v \mid V_{d+1} - v : V_{d+1} - v \longrightarrow V_d - P$. But if at least one line Z of V_{d+1} passes through v, then the projection χ_v from v must contract Z to a point, and the restriction $\chi_v \mid V_{d+1} - v$ cannot be an isomorphism. Thus if (ii) holds for V_{d+1} then (i) holds for V_d; for $d \leqslant 6$, (i) for V_d implies (ii) for V_d by (5.1, ii). The proof of the proposition thus follows from the fact that (ii) holds for V_7 (Corollary (6.2)).

(5.4) REMARK. In the case $d = 3$ the scheme T of lines on V_3 has been thorougly investigated (see [1] and [30]); T is the irreducible smooth Fano surface of lines on the cubic 3-fold V_3. In the case $d = 4$, T has also been thoroughly investigated (see for example [22] and [29]); it turns out that it is irreducible, and has the structure of an Abelian surface, isomorphic to the intermediate Jacobian $J(V_4)$ of the 3-fold V_4. Furthermore, it is isomorphic to the Jacobian $J(C)$ of a certain curve of genus 2 which is closely associated with V_4.

In §6 we compute T in the remaining cases $d = 5$, 6 and 7 (see (6.6), (6.4) and (6.2)). In these cases T has rather a simple structure: each of its irreducible component is isomorphic to either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$.

§6. Fano 3-folds of index $r \geqslant 2$; completion of the proof of Theorem (4.2)

(6.1) PROOF OF (4.2, iii): *the case* $d = 7$. Let $V = V_7 \subset \mathbf{P}^8$. Then its smooth hyperplane section H is a Del Pezzo surface of degree 7. It is well known [15] that any such surface is the image of the plane by means of the linear system of cubics passing through 2 distinct points. H contains 3 lines Z_0, Z_1, Z_2 which form the following configuration:



Among the 3 lines Z_0 is distinguished by the fact that it meets both of the remaining lines, whilst Z_1 and Z_2 meet only Z_0. As in the proof of (4.6, b) one shows that as H varies in a sufficiently general pencil of hyperplane sections the lines Z_0 sweep out a plane $P_0 \subset V_7$. In the process of proving (5.3) we showed that P_0 satisfies the criterion for contractibility, and that the morphism performing the contraction $\varphi_{|H+P_0|} : V_7 \longrightarrow V_8$ maps V_7 onto a Fano variety $V_8 \subset \mathbf{P}^9$, while the inverse map of $\varphi_{|H+P_0|}$ is the projection $\chi_{v_0} : V_8 \longrightarrow V_7$ from the point $v_0 = \varphi_{|H+P_0|}(P_0)$. From the proof of (4.6) one sees that V_8 can only be the Veronese image of $\mathbf{P}^3$. Hence $V_7 \subset \mathbf{P}^8$ is the projection of the Veronese image V_8 from some point $v \in V_8$. It will be enough to show that the image of such a projection is smooth and projectively normal. Then from (4.2, ii) it will follow that it is a Fano variety of index 2, and obviously of degree 7. Let us prove this.

Let $\sigma: V' \longrightarrow V_8$ be the blow up with center $v \in V_8$, and let $P = \sigma^{-1}(v)$. Then the projection χ_v from v defines a morphism

$$\varphi_{|\sigma^*H-P|}: V' \to \mathbf{P}^8,$$

since $\dim |\sigma^*H - P| = \dim |H| - 1 = 8$. Since $(\sigma^*H - P)^3 = 8 - 1 = 7$ by (2.11.1), it follows that $\varphi_{|\sigma^*H-P|}: V' \longrightarrow \varphi_{|\sigma^*H-P|}(V')$ is a birational morphism onto a 3-fold of degree 7. Since, on blowing up a point, $K_{V'} \sim \sigma^*(K_V) + 2P$, it follows that $K_{V'}^{-1/2} \simeq \mathcal{O}_{V'}(\sigma^*H - P)$. It is enough to show, according to (4.4), that the sheaf $\mathcal{O}_{V'}(\sigma^*H - P)$ is ample. Let $Z \subset V'$ be any curve. We us the numerical criterion of ampleness.

(a) If $Z \not\subset P$ then $(\sigma^*H - P \cdot Z) = \deg \sigma_*Z - (P \cdot Z)$; that is, $(\sigma^*H - P \cdot Z)$ is the difference between the degree of the curve σ_*Z and its multiplicity at v. Hence $(\sigma^*H - P \cdot Z) \geqslant 0$, and for an irreducible curve Z equality is only possible in the single case $\deg \sigma_*Z = 1$; that is, when $\sigma(Z)$ is a line passing through v. But since V_8 is the image of the Veronese map of $\mathbf{P}^3$, it does not contain any lines. Hence, $(\sigma^*H - P \cdot Z) > 0$.

(b) If $Z \subset P$, suppose that Z is a curve of degree m on the plane $P \simeq \mathbf{P}^2$; then $(\sigma^*H - P \cdot Z) = -(P \cdot Z) = m > 0$.

Hence $K_{V'}^{-1}$ is ample, so that V' is a Fano variety, and $\varphi_{|\sigma^*H-P|}: V' \longrightarrow V_7$ is an isomorphism. This proves the case $d = 7$ of (4.2, iii).

From the proof one sees that V_7 is obtained from $\mathbf{P}^3$ by blowing up one point. From this one at once obtains

(6.2) COROLLARY. *The surface T of lines on $V_7 \subset \mathbf{P}^8$ consists of two components T_0 and T_1, each of which is isomorphic to $\mathbf{P}^2$. If S_0 and S_1 are the corresponding families of lines, and $\psi_i: S_i \longrightarrow V_7$ are the natural maps* ($i = 0, 1$), *then $\psi_0(S_0) = P_0 \subset V_7$ is the plane $\varphi_{|\sigma^*H-P|}(P)$, and $\psi_1: S_1 \longrightarrow V_7$ is an isomorphism. In particular, through every point $v \in V_7$ there passes a line.*

(6.3) PROOF OF (4.2, iii): *the case* $d = 6$. In the usual notation, let $Z \subset V$ be a line of $V = V_6$ such that $N_Z \simeq \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)$. Such a line exists by (5.2). As in the proof of (5.2, i), one shows that there exists a smooth hyperplane section $H \in |\mathcal{O}_V(1)|$ containing the line Z.

Let $\sigma: V' \longrightarrow V$ be the blow up of the line Z. Set $Z' = \sigma^{-1}(Z)$ and $H' = \sigma^*\mathcal{O}_V(1) \otimes \mathcal{O}_{V'}(-Z')$. Then the following assertions hold.

(a) $Z' \simeq P(\check{N}_Z) \simeq \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$.

(b) *The linear system $|H'|$ is without fixed components and base points.*

(c) *The morphism $\varphi_{H'}: V' \longrightarrow \varphi_{H'}(V)$ is birational, and* $\deg \varphi_{H'}(V) = 3$; *if π_Z: $V \longrightarrow \mathbf{P}^5$ is the projection from the line Z, then $\varphi_{H'} = \pi_Z \cdot \sigma$.*

Let us prove (c). Let $H' \in |H'|$, and set $W = \varphi_{H'}(V)$. Then, by (1.11.2), $(H')^3 = (\sigma^*H - Z')^3 = H^3 - 3(H \cdot Z) - (Z')^3 = 3$. Hence $\varphi_{H'}$ is birational, and $\deg W = 3$. From the exact sequence

$$0 \to H^0(\mathcal{O}_{V'}(\sigma^*H - Z')) \to H^0(\mathcal{O}_{V'}(\sigma^*H))$$
$$\to H^0(\mathcal{O}_{Z'}(\sigma^*H \cdot Z)) \simeq H^0(Z, \mathcal{O}_Z(1)) \to 0$$

we find that $\dim |H'| = 5$. Hence $W \subset \mathbf{P}^5$, and we obviously have the condition $\deg W = \operatorname{codim} W + 1$. It is also clear that $\varphi_{H'} = \pi_Z \circ \sigma$.

(d) *The morphism* $\varphi_{H'} | Z' : Z' \longrightarrow \varphi_{H'}(Z') \subset \mathbf{P}^3$ *is an isomorphism of* Z' *onto a smooth quadric* $Q = \varphi_{H'}(Z') \subset \mathbf{P}^3$.

Indeed, by (1.11.2), $(H' \cdot H' \cdot Z') = (\sigma^*H - Z')^2 \cdot Z' = 2$. Hence Z' maps onto some surface Q. Since $\varphi_{H'}$ is birational and W is a normal variety (see (2.5, iii)), the morphism $\varphi_{H'}|Z'$ is also birational. Hence $\deg Q = (H')^3 = 2$; that is, Q is a 2-dimensional quadric. This cannot be a cone, since $Z' \simeq \mathbf{P}^1 \times \mathbf{P}^1$ does not contain any curves with negative self-intersection, so that there does not exist any birational morphism of Z' to a cone.

(e) *Let* $E \subset V$ *be the ruled surface swept out by lines which meet the line* Z, *and let* E' *be the proper transform of* E *on* V'. *Then* E' *consists of* 2 *disjoint irreducible components* E_1', E_2', *the morphism* $\varphi_{H'} | V' - E' : V' - E' \longrightarrow W - \varphi_{H'}(E')$ *is an isomorphism, and* $\varphi_{H'}$ *contracts* E' *onto two disjoint lines* Y_1 *and* Y_2 *in* W.

For the proof, note that through the general point $v \in V$ there pass 3 lines. Indeed, a sufficiently general tangent hyperplane to v intersects V in a Del Pezzo surface H_v of degree 6 with a double point v, and H_v contains all lines of V through v. Considering H_v as the image of the blow up of $\mathbf{P}^2$ in 3 points x_1, x_2, x_3 under the anticanonical map, one sees easily that in this case the 3 points lie on a certain line, which is contracted into the singular point v; the 3 blown up lines above the x_i are the only lines of H_v passing through v.

Now, since we can choose the line Z to be outside the ramification divisor of $S \longrightarrow V$ (see (5.2)), then through the general point $z \in Z$ there pass 2 further lines lying on V. On the other hand, on a nonsingular surface H containing Z, 2 of the 6 lines of H meet Z. Hence if $Y' = E' \cap Z'$, $Y' \sim \alpha s + \beta f$, where f is a fiber of the ruled surface Z', and s is a section, then $\alpha = 2$; and $(Y' \cdot X')_{Z'} = 2$, where $X' = H' \cap Z' \sim s + f$. From this we conclude that $\beta = 0$.

Obviously, the projection π_2 from Z contracts the surface E; and the 2 lines passing through some point $v \in Z$ project into distinct points, provided that they do not lie together with Z in some plane $\mathbf{P}^2 \subset \mathbf{P}^7$. But 3 lines of V_6 cannot lie in one plane, since V_6 is an intersection of quadrics, and does not contain a plane, by (5.3). Hence $Y' \subset Z'$ cannot be a section with multiplicity 2, for otherwise $\varphi_{H'}$ would not be defined on Y'. There remains therefore only one possibility: Y' consists of 2 disjoint sections Y_1' and Y_2', and hence E' consists of two irreducible disjoint components E_1' and E_2'. Clearly the images on W of E_1' and E_2' coincide with the images of Y_1' and Y_2', and are therefore lines Y_1 and Y_2 in W.

Let $C \subset V$ *be an irreducible curve which contracts under the projection* $\pi_Z : V \longrightarrow W$. A necessary and sufficient condition for this to happen is that $(H \cdot C) = (Z' \cdot C')$, where C' is the proper transform of C on V'. The right-hand side of this equality can be interpreted as the number of points (counted with multiplicities) of intersection of C with the line

$Z \subset V$; the right-hand side is the degree of C in $\mathbf{P}^7$. From the condition $(H \cdot C) = (Z' \cdot C')$ one sees that C is a plane curve; let P be the plane containing it. Then $C \cup Z \subset P \cap V$. But since V_6 is an intersection of quadrics, every such quadric either cuts out on P a curve of degree 2, or contains P. According to (5.3) P is not contained in V, so that $C \cup Z$ can only be a curve of degree 2; that is, C is a line meeting Z , so that $C \subset E$.

Thus the birational morphism $\varphi_{H'} \mid V' - E' : V' - E' \longrightarrow W - (Y_1 \cup Y_2)$ does not contract anything, and is thus a finite morphism of degree 1; and since W is a normal variety, $\varphi_{H'} \mid V' - E'$ is an isomorphism.

(f) *$W \subset \mathbf{P}^5$ is nonsingular, and is the image of $\mathbf{P}^1 \times \mathbf{P}^2$ under the Segre embedding.*

Since codim $W + 1 = \deg W = 3$, it follows from Lemma (2.8) that W is either as in assertion (f), or is a cone over a smooth surface $F_3 \subset \mathbf{P}^4$, or a cone over a rational cubic $C_3 \subset \mathbf{P}^3$. We have to exclude the latter two possibilities. Suppose that W is a cone; then from (e) it follows that its vertex has to lie on the quadric $Q \subset W$. But it is easily seen that a cone W cannot contain a smooth quadric which contains the vertex (0- or 1-dimensional). This contradiction proves (f).

(g) *There is an isomorphism $V \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.*

Identify W with $\mathbf{P}^1 \times \mathbf{P}^2$; then Q is identified with $\mathbf{P}^1 \times \mathbf{P}^1$ under some linear embedding of the second factor $\mathbf{P}^1$ in $\mathbf{P}^2$, and Y_1 and Y_2 are identified with the inverse images of certain points $y_1, y_2 \in \mathbf{P}^1$ under the projection $\mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ onto the second factor. The birational transformation inverse to the projection $\pi_Z : V \longrightarrow W$ is constructed in the obvious fashion: the two lines Y_1 and Y_2 are blown up, and the proper transform of $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is blown down. This transformation is compatible with the projection onto the first factor $\mathbf{P}^1 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^1$, and transforms the fiber $\mathbf{P}^2$ into $\mathbf{P}^1 \times \mathbf{P}^1$. This proves (g).

Letting $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ denote the projection onto the ith factor, we have

$$\mathcal{K}^{-1}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \simeq p_1^*\mathcal{O}_{\mathbf{P}^1}(2) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(2) \otimes p_3^*\mathcal{O}_{\mathbf{P}^1}(2)$$

and hence under the identification $V \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ we have

$$\mathcal{H}_V \simeq \bigotimes_{i=1}^{3} p_i^*\mathcal{O}_{\mathbf{P}^1}(1);$$

that is, $V_6 \subset \mathbf{P}^7$ is the image of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ under its Segre embedding. Thus we have completed the treatment of the case $d = 6$ of Theorem (4.2, iii).

(6.4) COROLLARY. *The surface T of lines on $V_6 \subset \mathbf{P}^7$ is the disjoint union of* 3 *surfaces $T_i \simeq \mathbf{P}^1 \times \mathbf{P}^1$ $(i = 1, 2, 3)$. Each of the families S_i is isomorphic to $V_6 \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, with ψ_i coinciding with the ith projection of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ onto $\mathbf{P}^1 \times \mathbf{P}^1$.*

(6.5) PROOF OF (4.2, iii): *the case $d = 5$.* This case is in many ways analogous to the consideration in (6.3) of the case $d = 6$. Let Z be a line of $V = V_5$ such that $N_Z \simeq \mathcal{O}_Z \oplus \mathcal{O}_Z$, and let $\sigma: V' \longrightarrow V$ be the blow up of Z. Then in the same notation as in (6.3) we can state and prove as in (6.3) assertions similar to (a), (b) and (d); the assertion analogous to (6.3, c) takes the following form.

(c) *The morphism* $\varphi_{H'} : V' \longrightarrow W \subset \mathbf{P}^4$ *is birational, where W is a quadric of* $\mathbf{P}^4$. *If* $\pi_Z : V \longrightarrow \mathbf{P}^4$ *is the projection from the line Z, then* $\varphi_{H'} = \pi_Z \circ \sigma$.

The proof is similar to the corresponding proof in (6.3, c).

The remaining assertions show certain substantial differences from the corresponding ones in (6.3), and we give complete statements and proofs for them.

(e) *In the notation of* (6.3, e), E' *is an irreducible surface, and the morphism*

$$\varphi_{\mathcal{H}'} \,|\, V' - E' : V' - E' \to W - \varphi_{\mathcal{H}'}(E')$$

is an isomorphism; and $\varphi_{H'}$ *contracts* E' *onto a smooth rational curve* $Y \subset Q$ *of degree* 3, *where* $Q = \varphi_{H'}(Z')$ *is a 2-dimensional quadric.*

Indeed, exactly as in (6.3, e) one shows that through almost every point $z \in Z$ there pass 2 other lines of V, not lying together with Z in a plane $\mathbf{P}^2 \subset \mathbf{P}^6$. On a Del Pezzo surface $H_5 \subset \mathbf{P}^5$ every line meets 3 other lines. From this we deduce that $Y' = E' \cap Z' \sim 2s + f$ on Z'. Clearly, $\varphi_{H'}(E') = \varphi_{H'}(Y')$, and setting $Y = \varphi_{H'}(Y')$ we have $Y' \simeq Y$, and $Y \subset Q$ is a curve of genus 0 and degree 3. Such a curve will be singular if and only if it is reducible; let us show that Y is nonsingular, and hence irreducible. For this note that the morphism $\varphi_{H'}$ factors through the blow up $\sigma_Y : V'' \longrightarrow V$:

$$\begin{array}{ccc} V' & \to & V'' \\ {\scriptstyle \varphi_{\mathcal{H}'}}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ W & = & W \end{array}$$

Suppose that Y is a singular curve; then V'' is a singular normal variety (this is well known, and is easily checked directly). Let $y \in Y$ be a singular point; then the fiber $\varphi_{H'}^{-1}(y) = (\rho \circ \sigma_Y)^{-1}(y)$ is reducible. But this contradicts the fact that the projection π_Z only contracts a single line to the point $y \in W$, and $\varphi_{H'}|Z' : Z' \longrightarrow Q$ is an isomorphism. Hence Y is a smooth curve, and $\varphi_{H'} : V' \longrightarrow W$ is the blow up with center $Y \subset W$. This proves (e).

(f) *The quadric W cannot be a cone.*

This follows immediately from the arguments we have just given in (e).

(g) *The inverse map* $W \longrightarrow V_5$ *is given by the linear system* $|\mathcal{O}_W(2) - Y|$ *consisting of sections of W by quadrics of* $\mathbf{P}^4$ *which contain the curve Y.*

Indeed, $\dim |\mathcal{O}_W(2) - Y| = 6$, and if $\sigma_Y : V'' \longrightarrow W$ is the blow up of Y, then the map $\varphi_{|\mathcal{O}_W(2) - Y|}$ when lifted to V'' becomes a birational morphism $\varphi'' : V'' \longrightarrow V$, the image of which is the 3-fold V of degree 5 in $\mathbf{P}^6$. It is quite elementary to check that $\varphi_{|\mathcal{O}_W(2) - Y|}$ contracts those lines of W which intersect Y in two points, and only these. These lines sweep out the two-dimensional quadric $Q \subset W$—the unique hyperplane section of W which contains Y.

The proper transform Z'' of Q in V'' is a nonsingular ruled surface, satisfying the criterion for contractibility, and $\varphi'' : V'' \longrightarrow V$ is a morphism onto a smooth Fano 3-fold of index 2, contracting Z'' onto a certain line Z.

(h) *All Fano 3-folds* $V_5 \subset \mathbf{P}^6$ *of index* 2 *are projectively equivalent.*

Since V_5 is embedded in $\mathbf{P}^6$ by means of a submultiple of the anticanonical sheaf, it is enough to show that all Fano varieties V_5 are isomorphic. Since each such variety is obtained from a pair (W, Y), where $W \subset \mathbf{P}^4$ is a smooth quadric, and $Y \subset W$ is a smooth rational cubic, by means of the construction described in (g) it is sufficient in turn to show that all such pairs are projectively equivalent in $\mathbf{P}^4$. Let (W, Y) and (W', Y') be two such pairs, and let Q and Q' be the corresponding hyperplane sections defined by Y and Y'. First of all, by means of a projective transformation one can take (W, Q) onto (W', Q'), and the question reduces to the equivalence of the curves Y and Y' on the smooth 2-dimensional quadric Q. The group Aut Q of automorphisms of Q is an extension

$$1 \to \mathrm{PGL}(2) \times \mathrm{PGL}(2) \to \mathrm{Aut}\, Q \to \{1, \tau\} \to 1,$$

where τ is the involution interchanging the factors in $\mathbf{P}^1 \times \mathbf{P}^1 \simeq Q$. We can assume, by using τ if necessary, that both Y and Y' belong to $|2s + f|$, where s and f are respectively the fibers of the first and second projection of $\mathbf{P}^1 \times \mathbf{P}^1$ onto $\mathbf{P}^1$. The group $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$ acts on $|2s + f|$, and the stabilizer G_Y of each smooth curve $Y \in |2s + f|$ is 1-dimensional, since the pair of ramification points of the second projection $Y \longrightarrow \mathbf{P}^1$ must remain invariant. Since $\dim G = 6$ and $\dim |2s + f| = 5$, the orbit of Y in $|2s + f|$ is everywhere dense. Hence (h) follows at once.

It is well known (see for example [28]) that the general section of the Grassmannian Gr(2, 5) of lines in $\mathbf{P}^4$ by 3 hyperplanes is a Fano variety $V_5 \subset \mathbf{P}^6$ of index 2. From (h) it follows immediately that any Fano 3-fold $V_5 \subset \mathbf{P}^6$ is obtained in this way. Our treatment of the case $d = 5$ is thus complete.

(6.6) COROLLARY. (i) *Pic* $V_5 \simeq \mathbf{Z}$ *and is generated by* $\mathcal{O}_V(1)$.
(ii) *The surface T of lines of V_5 is isomorphic to* $\mathbf{P}^2$.

PROOF. According to (4.2, iii, b), V_5 is a complete intersection of 3 hyperplanes in the Grassmannian Gr(2, 5). Since Pic Gr(2, 5) $\simeq \mathbf{Z} \cdot \mathcal{O}(1)$, by the Lefschetz theorem Pic $V_5 \simeq \mathbf{Z}$ with $\mathcal{O}_V(1)$ as a generator. This proves (i).

For the proof of (ii) let us show that the family of lines $\{Z \mid Z \subset V_5, Z \cap Z_0 \neq 0\}$ meeting any given line Z_0 is parametrized by a smooth irreducible rational curve Y_0. For $N_{Z_0} \simeq \mathcal{O}_{Z_0} \oplus \mathcal{O}_{Z_0}$, this was proved in (6.5, e). It ramains to consider the case when $N_{Z_0} \simeq \mathcal{O}_{Z_0}(-1) \oplus \mathcal{O}_{Z_0}(1)$ (see (5.2.1)). Let E_0 be the ruled surface in V_5 swept out by lines meeting Z_0, and let $\sigma_0 : V' \longrightarrow V_5$ be the blow up of Z_0. Set $Z_0' = \sigma_0^{-1}(Z_0)$, and consider the exact sequence

$$0 \to H^0(\mathcal{O}_{V'}(\sigma_0^* H - 2Z_0')) \to H^0(\mathcal{O}_{V'}(\sigma_0^* H - Z_0'))$$
$$\to H^0(\mathcal{O}_{Z_0'} \otimes \mathcal{O}_{V'}(\sigma_0^* H - Z_0')).$$

It is easily checked that the final group is 3-dimensional, and the middle one is 4-dimensional, so that $h^0(\mathcal{O}_{V'}(\sigma_0^* H - 2Z_0')) \geqslant 1$. The proper transform of any line Z meeting Z_0 has negative intersection number (equal to -1) with $\sigma_0^* H - 2Z_0'$. Hence E_0 must be a fixed component of the linear system $|H - 2Z_0|$. But since Pic $V_5 \simeq Z \cdot \mathcal{O}_{V_5}(H)$, any surface in $|H - 2Z_0|$ is irreducible, and hence $|H - 2Z_0|$ contains only E_0. Let $X_0 = E_0 \cap H$ for a sufficiently general hyperplane section H. Then, since E_0 is irreducible, X_0 is an irreducible curve. It has arithmetic genus 1 and a single double point $x_0 \in X_0$, $x_0 = Z_0 \cap H$. Through every point $x \in X_0$, $x \neq x_0$, there passes a single line of E_0, for otherwise x would be singular on E_0, and hence singular on X_0, which is impossible. Let E_0' be the proper transform of E_0 on V', and X_0' the proper transform of X_0; then X_0' is a smooth irreducible rational curve, which is a section of the ruled surface E_0'. Hence the base of this ruled surface is also a smooth irreducible rational curve.

From the assertion we have just proved it follows that T is an irreducible surface, and to every point $t \in T$ we have associated a smooth irreducible rational curve $Y_t \subset T$. Let us show that $(Y_{t_1} \cdot Y_{t_2}) = 1$ for all $t_1, t_2 \in T$. Since the family $\{Y_t \mid t \in T\}$ is parametrized by an irreducible and hence connected surface T, it is enough to check this for some arbitrarily chosen pair of points t_1, t_2. Let us choose t_1 and t_2 so that the corresponding lines $Z_1, Z_2 \subset V_5$ do not meet. We have to show that there exists a unique line $Z \subset V_5$ which meets both Z_1 and Z_2. For this note that the hyperplane section E_1 swept out by lines meeting Z_1 does not contain Z_2, since $Z_1 \cap Z_2 = \varnothing$. Hence $(E_1 \cdot Z_2) = 1$, and the point $z_2 = E_1 \cap Z_2$ cannot be a singular point of E_1. Since E_1 is ruled, there is a unique fiber of it passing through a nonsingular point of E_1, and this is the unique line which intersects Z_1 and Z_2.

Summarizing, we get that through each point $s \in T$ there passes a 1-dimensional family $\{Y_t \mid t \in T,\ Y_t \ni s\}$ of smooth rational curves, for which every curve Y_{t_1} not containing s for $t_1 \in T$ is a section. Hence this family is a linear pencil of smooth rational curves with the single base point s. It follows at once from this that $T \simeq \mathbf{P}^2$. The proof is complete.

(6.7) REMARK. We could have proved (4.2, iii) for $d = 5$ and 6 by projecting V_d into $\mathbf{P}^3$ from a normal rational curve X of degree $d - 3$. The normal sheaf $\mathcal{N}_X$ to X can easily be seen to be represented as an extension

$$0 \to \mathcal{O}_{\mathbf{P}^1}(d-5) \to \mathcal{N}_X \to \mathcal{O}_{\mathbf{P}^1}(d-3) \to 0$$

under the identification $X \simeq \mathbf{P}^1$. One can prove that

$$\mathcal{N}_X \simeq \mathcal{O}_{\mathbf{P}^1}(d-4) \oplus \mathcal{O}_{\mathbf{P}^1}(d-4)$$

for a sufficiently general curve X. For such a curve the projection $\pi_X : V_d \longrightarrow \mathbf{P}^3$ is not defined only on X, and if $\sigma_X : V' \longrightarrow V_d$ is the blow up of X, and $\sigma_X^{-1}(X) = X'$, then (see (2.11.2))

$$(\sigma_X^* H - X')^3 = 1,$$
$$(\sigma_X^* H - X')^2 X' = 2,$$
$$(\sigma_X^* H - X')^2 \sigma^* H = 3.$$

Hence the morphism $\varphi_{|\sigma_X^* H - X'|} : V' \longrightarrow \mathbf{P}^3$ is birational, $\varphi(X') = Q$ is quadric of $\mathbf{P}^3$, and the inverse map $\mathbf{P}^3 \longrightarrow V_d$ is given by some linear subsystem of the linear system of cubics $|\mathcal{O}_{\mathbf{P}^3}(3)|$.

One shows easily that in the case $d = 5$ the map $\mathbf{P}^3 \longrightarrow V_5$ is given by the linear system $|\mathcal{O}_{\mathbf{P}^3}(3) - Y|$, where $Y \subset Q$ is a smooth rational curve of degree 4 with $Y \sim 3s + f$. For $d = 6$, $\mathbf{P}^3 \longrightarrow V_6$ is given by the linear system $|\mathcal{O}_{\mathbf{P}^3}(3) - Y_1 - Y_2 - Y_3|$, where Y_1, Y_2 and Y_3 are pairwise disjoint lines on the quadric Q.

The proofs of all the assertions we have mentioned are simple, but as lengthy at the corresponding proofs of (6.3) and (6.5) in the text.

(6.8) PROOF OF (4.2, iii): *the cases $d = 4$ and* 3. In both cases the assertion follows immediately from (4.2, ii). For $d = 4$ one has to note only that from the exact sequence

$$0 \to I_V^2 \to S^2H^0(\mathcal{O}_V(1)) \to H^0(\mathcal{O}_V(2)) \to 0$$

(see (4.4.1)), where I_V^2 is the component of degree 2 of the ideal I_V defined in (4.4, iii), it follows at once that $\dim I_V^2 = 2$. Hence V_4 is the complete intersection of any two linearly independent quadrics in I_V^2. Conversely, from the adjunction formula for the canonical class of complete intersections we get that any smooth complete intersection $V_{2\cdot 2}$ in $\mathbf{P}^5$ and any smooth cubic V_3 in $\mathbf{P}^4$ are Fano 3-folds of index 2.

(6.9) PROOF OF (4.2, iv): *the case $d = 2$.* If $d = 2$, then by (1.12, iii) $\dim |H| = 3$, by (3.1) $|H|$ is without base points, and according to (2.2), $\deg \varphi_H = 2$, and hence $\varphi_H : V \longrightarrow \mathbf{P}^3$ is a morphism of degree 2. φ_H is finite, since H is ample; and since V is a smooth variety, φ_H has a smooth ramification divisor $D \subset \mathbf{P}^3$. The class of D is given by the formula

$$2H \sim -K_V = \varphi_{\mathcal{H}}^*(-K_{\mathbf{P}^3}) - \frac{1}{2}\varphi_{\mathcal{H}}^*(D).$$

It follows that V is a Fano variety with $r = 2$ and $d = 2$ if and only if D is a smooth quartic of P^3.

For an alternative representation of V note that the algebra $R(V) = \bigoplus_{n\geq 0} H^0(V, H^n)$ is not generated by $H^0(V, H)$; however, according to Proposition (6.12) below it is generated by $H^0(V, H) \oplus H^0(V, H^2)$. Furthermore, there is a natural inclusion

$$S^2H^0(V, \mathcal{H}) \to H^0(V, \mathcal{H}^2).$$

We have

$$\dim S^2H^0(V, \mathcal{H}) = \binom{3+2}{2} = 10$$

and $h^0(V, H^2) = 11$ (see (1.12)). Choose some element $x_4 \in H^0(V, H^2)$ not belonging to the image of $S^2H^0(V, H)$. Then the algebra $R(V)$ is gneerated by $H^0(V, H)$ and the element x_4, and if x_0, x_1, x_2, x_3 is a basis of $H^0(V, H)$, then $R(V) \simeq k[x_0, \ldots, x_4]/I_V$, where the grading of $k[x_0, \ldots, x_4]$ is given by assigning degrees $\deg x_i = 1$ for $i = 0, \ldots, 3$, and $\deg x_4 = 2$. The ideal I_V is generated by a single element, since $V = \operatorname{Proj} R(V)$ and $\dim \operatorname{Proj} R(V) = 3$. To find a generator of I_V, consider the exact sequence

$$0 \to I_V^4 \to S^2(S^2H^0(V, \mathcal{H}) \oplus kx_4) \to H^0(V, \mathcal{H}^4) \to 0.$$

According to (1.12, iii), $h^0(H^4) = 45$, and one easily computes

$$\dim S^2(S^2H^0(\mathcal{H}) \oplus kx_4) = 46.$$

Hence $\dim I_V^4 = 1$. Let $F_4(x_0, \ldots, x_4) \in I_V^4$ be some nonzero element. Clearly $I_V^n = 0$ for $n \leqslant 3$, and hence the element $F_4(x_0, \ldots, x_4) \in I_V$ of least degree is a generator of I_V.

Set $\mathbf{P}(x_0, \ldots, x_4) = \operatorname{Proj} k[x_0, \ldots, x_4]$, where $k[x_0, \ldots, x_4]$ is graded as indicated above. Then $F_4(x_0, \ldots, x_4) = 0$ is the equation of V in $\mathbf{P}(x_0, \ldots, x_4)$. Conversely, any smooth hypersurface $V' \subset \mathbf{P}(x_0, \ldots, x_4)$ of degree 4 is a Fano variety with $r = 2$ and $d = 2$. Indeed, the dualizing sheaf of smooth complete intersections in weighted projective space is easily computed, and we have $K_{V'} \simeq \mathcal{O}_{V'}(-2)$, and $-K_{V'}^3 = 16$. This completes our treatment of the case $d = 2$.

(6.10) PROOF OF (4.2, iv): *the case* $d = 1$. According to (3.1) the linear system $|H|$ has a single base point, and since $\dim |H| = 2$ (see (1.9, ii)), the rational map $\varphi_H : V \longrightarrow \mathbf{P}^2$ is undefined only at the base point $v_0 \in V$. Clearly every surface $H \in |H|$, and every curve $X \in |H_H|$ (that is, any fiber of φ_H) is irreducible, since $H^3 = (H \cdot X) = 1$, and H is ample.

To prove the existence of such Fano 3-folds V we study another representation for them. For this, note that the linear system $|H^2| = |K_V^{-1}|$ is without fixed components and base points. Indeed, there are no fixed components, since there are none in $|H|$, and obviously there are no base points outside the point v_0. Let us show that v_0 also cannot be a base point. Since $h^1(V, H) = 0$ (see (1.12)) and $h^1(H, H_H) = 0$, we have the exact sequences

$$H^0(\mathcal{H}^2) \to H^0(\mathcal{H}_H^2) \to 0,$$
$$H^0(\mathcal{H}_H^2) \to H^0(\mathcal{H}_X^2) \to 0,$$

where $H \in |H|$ is a smooth surface, and $X \in |H_H|$ is a smooth curve. Hence it is enough to show that the linear system $|H_X^2|$ on X is without base points. But X is an elliptic curve, and it is well known that a complete linear system of degree 2 on an elliptic curve cannot have base points.

According to (1.9), $h^0(K_V^{-1}) = 7$. Hence the morphism

$$\varphi_{\mathcal{K}_V^{-1}} : V \to \varphi_{\mathcal{K}_V^{-1}}(V) \subset \mathbf{P}^6$$

is defined. Set $\varphi = \varphi_{K_V^{-1}}$ and $W = \varphi(V)$. Then according to (2.2, ii), $\deg \varphi = 1$ or 2. But $\deg \varphi \neq 1$, since the restriction of φ to any elliptic curve $X \in |H_H|$ has degree 2. Hence $\deg \varphi = 2$, and $\deg W = -K_V^3/2 = 4$; that is, W satisfies the condition of Lemma (2.8). Furthermore, the morphism φ takes the fibers of $\varphi_H : V \longrightarrow \mathbf{P}^2$ into lines of W passing through the point $w_0 = \varphi(v_0)$. Hence W is a cone with vertex w_0 and the rational map φ_H factors through φ:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\varphi_{\mathcal{H}}"'] & W \arrow[dl] \\ \mathbf{P}^2 & \end{tikzcd} \qquad (6.10.1)$$

Hence the base of the cone W is isomorphic to $\mathbf{P}^2$, and since $\deg W = 4$, W is the cone over the Veronese surface $F_4 \subset \mathbf{P}^5$.

The closed subset of V at which φ is not smooth consists of the point v_0 and a nonsingular irreducible divisor $\widetilde{D} \subset V$ with $v_0 \notin \widetilde{D}$ and $(\widetilde{D} \cdot X) = 3$. Indeed, since $w_0 = \varphi(v_0)$ is a singular point of W, φ must be degenerate at v_0. Let us show that v_0 is an isolated component of the set of degeneracy of φ. Since H^2 is ample and W is a normal variety, φ is a finite morphism. It follows that the local ring $\mathcal{O}_{w_0}$ is isomorphic to the subring of invariants of the local ring $\mathcal{O}_{v_0}$ under a certain involution τ. Passing to the completions $\overline{\mathcal{O}_{v_0}}$ and $\overline{\mathcal{O}_{w_0}}$, we can assume that τ acts linearly on $\overline{\mathcal{O}_{v_0}} \simeq k[[x, y, z]]$. Of the 3 possible actions $(x, y, z) \longmapsto (-x, y, z)$, $(x, y, z) \longmapsto (-x, -y, z)$ and $(x, y, z) \longmapsto (-x, -y, -z)$, only the third can happen in our situation, since only in this case is the ring of invariants $\overline{\mathcal{O}^{\tau}_{v_0}}$ isomorphic to the local ring of an isolated singularity—the vertex of the cone on the Veronese surface. Using explicit formulas, it is now very simple to check that the morphism

$$\varphi : \operatorname{Spec} \mathcal{O}_{v_0} - \{v_0\} \to \operatorname{Spec} \mathcal{O}_{w_0} - \{w_0\}$$

is ramified.

Further, $\varphi \mid V - v_0 : V - v_0 \longrightarrow W - w_0$ is a finite morphism of degree 2 of smooth varieties. By Zariski's theorem on the purity of the branch locus, the degeneracy locus of φ is a divisor $\widetilde{D} \subset V - v_0$. As we have seen, the closure of $\widetilde{D}$ on V does not contain v_0; that is, $\widetilde{D} \subset V$ is closed, and is without singularities, since V and $W - w_0$ are smooth. The restriction $\varphi \mid X$ of φ to the elliptic curve X has 4 ramification points, one of which is v_0. Hence $(\widetilde{D} \cdot X) = 3$. Let $D = \varphi(\widetilde{D}) \subset W$. Then $D \not\ni w_0$, and since $\operatorname{Pic} W \simeq \mathbf{Z}$, $D \in |\mathcal{O}_W(3)|$. It follows from this that D, and hence $\widetilde{D}$, is irreducible.

Conversely, let $D \in |\mathcal{O}_W(3)|$ be some smooth divisor. Then $w_0 \notin D$, and since $\operatorname{Pic} W \simeq \mathbf{Z}$, D is connected, and hence irreducible. Let $\delta : \overline{W} \longrightarrow W$ be the blow up of the vertex w_0 of W. Then we have a fibration $\overline{\pi} : \overline{W} \longrightarrow \mathbf{P}^2$ with fibers $\mathbf{P}^1$, and the smooth surface $S \subset \overline{W}$, $S = \delta^{-1}(w_0)$, is a section of $\overline{\pi}$. In the commutative diagram

$$\begin{tikzcd} \overline{V} \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\varphi}_{\mathcal{H}}"'] & \overline{W} \arrow[dl, "\overline{\pi}"] \\ \mathbf{P}^2 & \end{tikzcd} \tag{6.10.2}$$

$\overline{V}$ is a double cover of $\overline{W}$ with smooth ramification divisor $D \cup S$ (where we write D for $\delta^{-1}(D)$, since δ is an isomorphism in a neighborhood of D). $\overline{V}$ exists, is uniquely determined by the ramification divisor, and is a smooth complete variety with a fibration over $\mathbf{P}^2$ with fibers elliptic curves. Indeed, let $U = \overline{W} - (D \cup S)$. It is enough to show that there exists a unique nontrivial unramified double cover $U' \longrightarrow U$ of U. For this it is in turn sufficient to show that $H^1(U, \mu_2) = \mathbf{Z}/2\mathbf{Z}$, where μ_2 is the sheaf of second roots of 1. Consider the Kummer exact sequence

$$1 \to H^0(U, \mathcal{O}_U^*)/H^0(U, \mathcal{O}_U^*)^2 \to H^1(U, \mu_2) \to \operatorname{Pic}_2 U \to 0, \tag{6.10.3}$$

where $\mathrm{Pic}_2 U \subset \mathrm{Pic}\ U$ is the subgroup of elements killed by 2. Note that $nD \not\sim mS$ for any integers n and m. Indeed, $D^3 > 0$ and $(D \cdot S) = 0$. Since D and S are irreducible divisors, this implies that $H^0(U, \mathcal{O}_U^*) \simeq k^*$.

Let $L = \bar{\pi}^* \mathcal{O}_{\mathbf{P}^2}(1)$; then $\mathrm{Pic}\ \overline{W} \simeq \mathbf{Z} \cdot \mathcal{O}_{\overline{W}}(S) \oplus \mathbf{Z} \cdot L$, and simple computations give $\mathcal{O}_{\overline{W}}(D) \simeq \mathcal{O}_{\overline{W}}(3S) \otimes L^6$. From this we have $\mathrm{Pic}\ U = \mathrm{Pic}\ \overline{W}/(\mathcal{O}_{\overline{W}}(D), \mathcal{O}_{\overline{W}}(S)) \simeq \mathbf{Z}/6\mathbf{Z}$. Hence $H^1(U, \mu_2) \simeq \mathrm{Pic}_2\ U \simeq \mathbf{Z}/2\mathbf{Z}$. $\overline{V}$ is a smooth variety, since the ramification divisor is smooth. Each fiber of $\bar{\varphi}_H$ is a double cover of the line with 4 ramification points, which cannot all coincide, since the fiber is an irreducible (possibly singular) elliptic curve.

Let T be the proper transform on $\overline{V}$ of S; then $T \simeq S \simeq \mathbf{P}^2$, and T is a section of the morphism $\bar{\varphi}_H : \overline{V} \longrightarrow \mathbf{P}^2$. Let us show that T satisfies the condition for contractibility to a nonsingular point. Let Z be a line on T; that is, a curve such that $\bar{\varphi}_H(Z)$ is a line on $\mathbf{P}^2$. We have

$$(2Z \cdot 2T) = (\bar{\varphi}^* \bar{\varphi}_*(Z) \cdot \bar{\varphi}^*(S)) = 2(\bar{\varphi}_*(Z) \cdot S),$$

where $\bar{\varphi} =_{\mathrm{def}} \bar{\varphi}_H$, and hence $2(Z \cdot T) = (\bar{\varphi}_*(Z) \cdot S)$. Let $F = \bar{\pi}^{-1} \circ \bar{\pi} \circ \bar{\varphi}(Z)$; then $\delta(F)$ is a quadratic cone, and hence $F \simeq \mathbf{F}_2$, with $(\bar{\varphi}(Z) \circ \bar{\varphi}(Z))_F = -2$. Hence

$$2(Z \cdot T) = (\bar{\varphi}_*(Z) \cdot S) = (F \cdot S \cdot S) = (\bar{\varphi}(Z) \cdot \bar{\varphi}(Z))_F = -2,$$

and so $(Z \cdot T) = -1$; hence T satisfies the condition for contractibility to a nonsingular point. In the diagram (6.10.2) above we can simultaneously perform the contraction of T and S and get the following commutative diagram:

$$\begin{tikzcd}
\overline{V} \arrow[rr, "\bar{\varphi}"] \arrow[dr, "\bar{\varphi}_{\mathcal{H}}"] \arrow[dd, "\sigma"'] & & \overline{W} \arrow[dl, "\bar{\pi}"'] \arrow[dd, "\delta"] \\
& \mathbf{P}^2 & \\
V \arrow[rr, "\varphi"'] \arrow[ur, dashed, "\varphi_{\mathcal{H}}"] & & W \arrow[ul, "\pi"]
\end{tikzcd} \tag{6.10.4}$$

where σ is the morphism contracting T, $\varphi : V \longrightarrow W$ is the double cover ramified in D and in the vertex w_0 of W, φ_H is the rational map induced by $\bar{\varphi}_H$ and undefined only at the point $v_0 = \sigma(T)$, and π is the composite of the projection of the cone W from its vertex onto the Veronese surface F_4 with the natural isomorphism $F_4 \xrightarrow{\sim} \mathbf{P}^2$. One easily computes the canonical divisors of the varieties entering in (6.10.4), and from these computations, which we omit, it follows that V is a Fano 3-fold with $r = 2$ and $d = 1$, with the lower triangle in diagram (6.10.4) being none other than diagram (6.10.1). Hence we have proved that the family of all Fano 3-folds with $r = 2$ and $d = 1$ is parametrized by the set $\{D\}$ of smooth divisors $D \in |\mathcal{O}_W(3)|$.

Now let us show that every Fano 3-fold with $r = 2$ and $d = 1$ has a realization (b) as in (4.2, iv). According to Proposition (6.12) below, the algebra $R(V) = \bigoplus_{n \geqslant 0} H^0(V, H^n)$ is generated by

$$H^0(V, \mathcal{H}) \oplus H^0(V, \mathcal{H}^2) \oplus H^0(V, \mathcal{H}^3).$$

We have natural inclusions:

$$\alpha : S^2H^0(V, \mathcal{H}) \to H^2(V, \mathcal{H}^2),$$
$$\beta : H^0(V, \mathcal{H}) \overset{s}{\otimes} H^0(V, \mathcal{H}^2) \to H^0(V, \mathcal{H}^3),$$

where $\overset{s}{\otimes}$ denotes the symmetric product. From (1.9, ii) we get

$$h^0(V, \mathcal{H}) = 3, \quad h^0(V, \mathcal{H}^2) = 7, \quad h^0(V, \mathcal{H}^3) = 14.$$

Trivial computations show that the cokernels of α and β are 1-dimensional. Let $x_3 \in H^0(V, H^2)$ be such that $x_3 \notin \operatorname{Im} \alpha$, and $x_4 \in H^0(V, H^3)$ such that $x_4 \notin \operatorname{Im} \beta$. Choosing a basis x_0, x_1, x_2 of $H^0(V, H)$, we have

$$R(V) \simeq k[x_0, \ldots, x_4]/I_V,$$

where the grading in the polynomial ring $k[x_0, \ldots, x_4]$ is defined by $\deg x_i = 1$ for $i = 0, 1, 2$, $\deg x_3 = 2$ and $\deg x_4 = 3$. By considerations of dimension the ideal I_V is principal, and its generator is in I_V^6. Indeed, in the exact sequence

$$0 \to I_V^6 \to S^2(S^3H^0(\mathcal{H}) \oplus (H^0(\mathcal{H}) \otimes kx_3) \oplus kx_4) \to H^0(\mathcal{H}^6) \to 0$$

the middle term has dimension 64, and $h^0(H^6) = 63$ by (1.9, ii). Hence $\dim I_V^6 = 1$, and any nonzero element $F_6(x_0, \ldots, x_4) \in I_V^6$ generates the ideal I_V.

As in the case $d = 2$, setting

$$\mathbf{P}(x_0, \ldots, x_4) = \operatorname{Proj} k[x_0, \ldots, x_4],$$

with the grading indicated above, we get that $F_6(x_0, \ldots, x_4) = 0$ is the equation of V in the weighted projective space $\mathbf{P}(x_0, \ldots, x_4)$. It is elementary to check that any smooth hypersurface of degree 6 in $\mathbf{P}(x_0, \ldots, x_4)$, where $\deg x_i = 1$ for $i = 0, 1, 2$, $\deg x_3 = 2$ and $\deg x_4 = 3$, is a Fano 3-fold of index 2 with $d = 1$. The proof is complete.

(6.11) COROLLARY. *If V is a Fano 3-fold with $r = 2$ and $d = 1$ or 2, then* $\operatorname{Pic} V \simeq \mathbf{Z}$, *with* H as a generator.

PROOF. It is known that the Lefschetz theorem for the Picard group remains valid for complete intersections of dimension $\geqslant 3$ in weighted projective space (I. V. Dolgačev, unpublished; see also [16]). From this, and from the fact that V can be represented as a hypersurface in weighted projective space, we get the required result.

In (6.9) and (6.10) we referred to the following result.

(6.12) PROPOSITION. (i) *If V is a Fano 3-fold with $r = 2$ and $d = 2$, then the graded k-algebra $\bigoplus_{n\geqslant 0} H^0(V, H^n)$ is generated by $H^0(V, H) \oplus H^0(V, H^2)$.*

(ii) *If V is a Fano 3-fold with $r = 2$ and $d = 1$, then the k-algebra $\bigoplus_{n \geqslant 0} H^0(V, H^n)$ is generated by $H^0(V, H) \oplus H^0(V, H^2) \oplus H^0(V, H^3)$.*

PROOF. Let $H \in |H|$ be a smooth surface. Considering the exact sequence

$$0 \to H^0(V, \mathcal{H}^{n-1}) \to H^0(V, \mathcal{H}^n) \to H^0(H, \mathcal{H}^n_H) \to 0$$

and arguing in the proof of Lemma (2.9), we reduce the proof of both assertions to the surface H and the sheaf $|\mathcal{H}_H|$. If $X \in |\mathcal{H}_H|$ is a smooth elliptic curve, then by the same method one reduces the proof to the curve X and the invertible sheaf $|\mathcal{H}_X|$.

Let us use the following particular case of a lemma of Castelnuovo (see [18], Theorem 2).

(6.13) LEMMA. *Let $\mathcal{L}$ be an invertible sheaf on an elliptic curve X such that* $\deg \mathcal{L} \geqslant 2$. *If $\mathcal{F}$ is a coherent sheaf on X such that $h^1(\mathcal{F} \otimes \mathcal{L}^{-1}) = 0$, then the natural map*

$$H^0(\mathcal{F}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L})$$

is surjective.

Let us apply this lemma first of all to case (i). Set $\mathcal{L} = \mathcal{H}_X$ and $\mathcal{F} = \mathcal{H}_X^m$, with $m \geqslant 2$ any integer. Then the conditions of the lemma are satisfied, and for $m \geqslant 2$ we get a surjection

$$H^0(\mathcal{H}_X^m) \otimes H^0(\mathcal{H}_X) \to H^0(\mathcal{H}_X^{m+1}).$$

This proves (i) of the proposition.

In case (ii) set $\mathcal{L} = \mathcal{H}_X^2$ and $\mathcal{F} = \mathcal{H}_X^m$ for $m \geqslant 3$. We get a surjection

$$H^0(\mathcal{H}_X^m) \otimes \mathcal{H}^0(\mathcal{H}_X^2) \to H^0(\mathcal{H}_X^{m+2}).$$

From this it follows that for any $n \geqslant 5$ the space $H^0(\mathcal{H}_X^n)$ is spanned by the $H^0(\mathcal{H}_X^p)$ for $p \leqslant 4$. It remains to show that $H^0(\mathcal{H}_X^4)$ is generated by $H^0(\mathcal{H}_X)$, $H^0(\mathcal{H}_X^2)$ and $H^0(\mathcal{H}_X^3)$. This is easily deduced from the Riemann-Roch theorem. Indeed, let $x_0 \in X$ be the point such that $\mathcal{H}_X = \mathcal{O}_X(x_0)$. Then the space $H^0(\mathcal{H}_X^m)$ is in a natural way identified with the space of functions on X which are regular outside x_0 and have a pole of order no more than m at x_0.

Using the Riemann-Roch theorem on X we can choose a basis for the spaces $H^0(\mathcal{H}_X^p)$ with $p \leqslant 4$, as follows:

$$\begin{aligned} \{1\} &\subset H^0(\mathcal{H}_X), \\ \{1, u\} &\subset H^0(\mathcal{H}_X^2), \\ \{1, u, v\} &\subset H^0(\mathcal{H}_X^3), \\ \{1, u, v, u^2\} &\subset H^0(\mathcal{H}_X^4), \end{aligned}$$

where u and v are functions having at x_0 poles of orders 2 and 3 respectively. It follows from this that the functions $1, u, v$ and u^2, which have poles of different orders at x_0, are linearly independent. Hence $H^0(\mathcal{H}_X^4)$ is generated by the spaces $H^0(\mathcal{H}_X^p)$ for $p \leqslant 3$, which proves (ii), and with it Proposition (6.12). This also completes the proof of Theorem (4.2).

(6.14) REMARK. On the Fano 3-folds V with $r = 2$ and $d = 1$ or 2 there also exist, as in the cases $d \geqslant 3$, 2-parameter families of "lines", that is, of curves $Z \subset V$ such that $(H \cdot Z) = 1$, for $H \in |H|$ a divisor. As in Proposition (5.1), one proves that this family is parametrized by a smooth projective surface. It would be interesting to compute the numerical invariants of these surfaces. It is possible that these invariants will turn out to be useful in the birational theory of the corresponding varieties.

§7. Hyperelliptic Fano 3-folds

(7.1) DEFINITION. A Fano 3-fold V of index $r = 1$ will be said to be *hyperelliptic* if its anticanonical map $\varphi_{K_V^{-1}}$ is a morphism and is of degree $\deg \varphi_{K_V^{-1}} = 2$.

Recall that, in the notation introduced above (see (1.8)), if $r = 1$ then $K_V^{-1} = H$. In the sequel we will write H for K_V^{-1}. If φ_H is a morphism, then by Bertini's theorem the linear system $|H|$ contains a smooth irreducible (since H is ample) surface H; that is, our hypothesis (1.14) is satisfied. According to (2.2, ii), $\deg \varphi_H = 1$ or 2. In this section we will study the case $\deg \varphi_H = 2$. Let W denote the image $\varphi_H(V)$ of the hyperelliptic 3-fold V; then $W \subset \mathbf{P}^{g+1}$, where $g + 1 = \dim |K_V^{-1}|$, and the condition of Lemma (2.8) is satisfied; indeed, $\deg \varphi_H \cdot \deg W = (-K_V)^3 = 2g - 2$ (see (1.6, i)). Hence $\deg W = g - 1$ and $\deg W = \operatorname{codim} W + 1$. Recall that the integer g is called the genus of V.

(7.2) THEOREM. *Let V be a hyperelliptic Fano variety, and let $\varphi_H : V \longrightarrow W \subset \mathbf{P}^{g+1}$ be the corresponding morphism of degree 2. Then W is nonsingular and V is uniquely determined by the pair (W, D), where $D \subset W$ is the ramification divisor of φ_H. The pair (W, D) belongs to one of the following families (and if D is a smooth divisor, then for each pair (W, D) there exists a Fano 3-fold V)*:

(i) *$W \simeq \mathbf{P}^3$, and D is a smooth hypersurface of degree 6; in this case V can be realized alternatively as a smooth hypersurface of degree 6 in the weighted projective space $\mathbf{P}(x_0, \ldots, x_4)$, where $\deg x_i = 1$ for $i = 0, 1 \ldots, 3$ and $\deg x_4 = 3$.*

(ii) *$W \simeq V_2$ is a smooth quadric in $\mathbf{P}^4$ and $D \in |\mathcal{O}_{V_2}(4)|$; that is, $D = V_2 \cap V_4$, where V_4 is a quartic of $\mathbf{P}^4$. In this case V can also be realized as a smooth complete intersection in the weighted projective space $\mathbf{P}(x_0, \ldots, x_5)$, where $\deg x_i = 1$ for $i = 0, \ldots, 4$, and $\deg x_5 = 2$: V is the intersection of a quadric cone and a hypersurface of degree 4*:

$$\begin{aligned} F_2(x_0, \ldots, x_4) &= 0, \\ F_4(x_0, \ldots, x_5) &= 0. \end{aligned} \tag{7.2.1}$$

(iii) *In the notation (2.4), $W \simeq \mathbf{P}_{\mathbf{P}^1}(E)$ is a rational scroll in the embedding*

$$\varphi_{\mathcal{M}} : W \to \mathbf{P}^{d_1+d_2+d_3+2},$$

and only the following possibilities occur:

$d_1 = d_2 = d_3 = 1$; *then $W \simeq \mathbf{P}^2 \times \mathbf{P}^1$ in its Segre embedding, and $D \in |M^4 \otimes L^{-2}|$, where $M \simeq p_1^*\mathcal{O}_{\mathbf{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(1)$, and $L \simeq p_2^*\mathcal{O}_{\mathbf{P}^1}(1)$, p_i denoting the projection of*

$\mathbf{P}^2 \times \mathbf{P}^1$ *onto the ith factor for* $i = 1$ *and* 2;

$d_1 = 2, d_2 = d_3 = 1$; *then* $W \subset \mathbf{P}^6$, $\deg W = 4$, *and* $D \in |\mathcal{O}_W(4)|$;

$d_1 = d_2 = d_3 = 2$; *then* $W \simeq \mathbf{P}^2 \times \mathbf{P}^1$ *and the embedding in* $\mathbf{P}^8$ *is given by the invertible sheaf* $\mathcal{M} \simeq p_1^*\mathcal{O}_{\mathbf{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(2)$, *and* $D \in |p_1^*\mathcal{O}_{\mathbf{P}^2}(4)|$; *in this case the 3-fold* V *is the product* $H \times \mathbf{P}^1$, *with* H *a smooth Del Pezzo surface with* $(K_H \cdot K_H) = 2$; *that is, a double plane with smooth ramification curve of degree* 4.

(7.3) Beginning of the Proof of Theorem (7.2). Let us prove first of all that W is smooth. From the ampleness of H it follows that $\varphi_H : V \longrightarrow W$ is a finite morphism, and in particular each fiber of φ_H is 0-dimensional. Suppose that $w \in W$ is a singular point; then, since V is nonsingular, there exists only one point $v \in V$ such that $\varphi_H(v) = w$. Since we have assumed that W is singular, by (2.8) it can only be a cone with 0- or 1-dimensional vertex. Consider first the case that $w \in W$ is an isolated singularity; that is, W is a cone with 0-dimensional vertex w. It follows from Lemma (2.8) that the base of the cone must be either a nonsingular scroll or the Veronese surface $F_4 \subset \mathbf{P}^5$. If the base of the cone is a scroll, then we can choose two planes $P_1, P_2 \subset W$, each of which contains the vertex w, and maps onto a certain line, a fiber of the scroll, on projection from w. It is clear that P_1 and P_2 do not intersect outside w. Let Q_1 and Q_2 be the closures of the open subsets $\varphi_H^{-1}(P_1 - w)$ and $\varphi_H^{-1}(P_2 - w)$ in V. Then the surfaces Q_1 and Q_2 only intersect in the single point v of V, where $\varphi_H(v) = w$. This is impossible on a smooth 3-fold V.

If the base of the cone W is the Veronese surface F_4, then W contains a hyperplane section E which passes through the vertex w and projects to a double conic on F_4—the image of a double line in $\mathbf{P}^2$ under the isomorphism $\mathbf{P}^2 \simeq F_4$. Let $H - v = \varphi_H^{-1}(E - w)$; then the surface $H - v$, and hence also its closure $H = \overline{H - v}$, is a double on V; that is, the invertible sheaf $\mathcal{O}_V(H)$ is divisible by 2 in Pic V, which contradicts the fact that V is of index 1.

Now consider the case that W is a cone with a 1-dimensional vertex $Z \subset W$. According to (2.8) the base of the cone W is a normal rational curve X of degree n in $\mathbf{P}^n$. Here also we can choose a hyperplane section $E \subset W$ containing the vertex Z, and projecting from Z into a point of X counted with multiplicity n. Arguing exactly as in the case of the cone over the Veronese surface, we get that V must have index $r = n > 1$. This contradicts the definition of hyperelliptic variety. Thus W cannot be a cone, and according to (2.8) it must be a nonsingular variety.

Let us show that V is uniquely defined by the pair (W, D), where $D \subset W$ is the ramification divisor of the morphism $\varphi_H : V \longrightarrow W$. Let $U = W - D$. As in (6.10), it is enough to show that $H^1(U, \mu_2) \simeq \mathbf{Z}/2\mathbf{Z}$. Note that in our case from the smoothness of D it follows that D is irreducible. Indeed, if $W = \mathbf{P}^3$ or $W = V_2 \subset \mathbf{P}^4$ this is obvious, since Pic $W \simeq \mathbf{Z}$. If W is a rational scroll, then $W \longrightarrow \mathbf{P}^1$ is a locally trivial fibration with fiber $\mathbf{P}^2$, and Pic $W \simeq \mathbf{Z} \oplus \mathbf{Z}$. We will show below that D cannot consist only of fibers of $W \longrightarrow \mathbf{P}^1$. Hence if D is reducible, any two of its components have nontrivial intersection in some fiber. This contradicts the smoothness of D.

From the irreducibility of D it follows that $H^0(U, \mathcal{O}_U^*) \simeq k^*$, and the Kummer exact sequence (6.10.3) gives the isomorphism $H^1(U, \mu_2) \simeq \mathrm{Pic}_2 U$. Since there does exist at least one cover $\varphi_H^{-1}(U) \longrightarrow U$, we have $H^1(U, \mu_2) \neq 0$. On the other hand, Pic $U =$ Pic $W/\mathbf{Z} \cdot \mathcal{O}_W(D)$ cannot have a subgroup of period 2 greater than $\mathbf{Z}/2\mathbf{Z}$, since Pic $W \simeq \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$. Hence $H^1(U, \mu_2) \simeq \mathbf{Z}/2\mathbf{Z}$, and the pair (W, D) determines V uniquely.

Recall that the ramification divisor is computed from the formula

$$K_V \sim \varphi_{\mathscr{H}}^* \left(K_W + \frac{1}{2} D\right). \tag{7.3.1}$$

From this it follows that if $W \simeq \mathbf{P}^3$ then $D \in |\mathcal{O}_{\mathbf{P}^3}(6)|$; if $W \simeq V_2 \subset \mathbf{P}^4$, then $D \in |\mathcal{O}_{V_2}(4)|$, and since Pic $V_2 \simeq$ Pic $\mathbf{P}^4$ under the natural restriction map, $D = V_2 \cap V_4$, with V_4 a quartic hypersurface.

Thus in (i) and (ii) it remains only to justify the realization of V as weighted complete intersections. (i) is analogous to the consideration in (6.9) of the double cover $V \longrightarrow \mathbf{P}^3$ ramified in a surface of degree 4, and we will not dwell on it. One shows easily, incidentally, that any cyclic cover of degree m of projective space $\mathbf{P}^n$ ramified in a hypersurface $F \subset \mathbf{P}^n$ can be realized as a hypersurface of degree deg F in a weighted projective space of dimension $n + 1$, where one coordinate has degree $(1/m)\deg F$, and the remainder degree 1.

Consider the case (ii). V is nothing other than the restriction to the quadric Q_2 of the double cover $Z \longrightarrow \mathbf{P}^4$ ramified in the quartic $V_4 \subset P^4$. Hence V can be given by a system of equations (7.2.1), where the first equation is that of V_2, and the second that of Z.

The idea behind the proof of (iii) is that the condition that the divisor D should be smooth, together with condition (7.3.1) above, puts extremely strong conditions on the rational scroll W. We will use the following result.

(7.4) LEMMA (M. Reid, unpublished). *Let $X = \mathbf{P}_{\mathbf{P}^1}(\mathcal{E})$, where $\mathcal{E} = \bigoplus_{i=1}^m \mathcal{O}_{\mathbf{P}^1}(d_i)$, with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m \geqslant 0$. Suppose that $d_1 > d_m$, and for every integer c with $d_1 > c \geqslant d_m$ define Y_c by $Y_c = \mathbf{P}_{\mathbf{P}^1}(\mathcal{E}_c)$, where $\mathcal{E}_c = \bigoplus_{d_i \leqslant c} \mathcal{O}_{\mathbf{P}^1}(d_i)$. We identify Y_c with a subvariety of X by means of the embedding defined by the projection $\mathcal{E} \longrightarrow \mathcal{E}_c$.*

If a and b are integers such that $H^0(X, \mathcal{M}^a \otimes \mathcal{L}^b) \neq 0$, then in order that every section $s \in H^0(X, \mathcal{M}^a \otimes \mathcal{L}^b)$ has a zero of order $\geqslant q$ on Y_c it is necessary and sufficient that the following inequality hold:

$$ac + b + (d_1 - c)(q - 1) < 0. \tag{7.4.1}$$

PROOF. Let $f: X \longrightarrow \mathbf{P}^1$ be the structural morphism; then for $a \geqslant 0$ we have natural isomorphisms

$$H^0(X, \mathcal{M}^a \otimes \mathcal{L}^b) \simeq H^0(\mathbf{P}^1, f_*(\mathcal{M}^a \otimes \mathcal{L}^b)) \simeq H^0(\mathbf{P}^1, S^a\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(b)),$$

where $S^a\mathcal{E}$ is the ath symmetric power of the sheaf $\mathcal{E}$. By hypothesis $H^0(X, \mathcal{M}^a \otimes \mathcal{L}^b) \neq 0$, so that $a \geqslant 0$. Set $\mathcal{E} = \mathcal{E}'_c \oplus \mathcal{E}_c$, where $\mathcal{E}'_c = \bigoplus_{d_i > c} \mathcal{O}_{\mathbf{P}^1}(d_i)$. Then

$$S^a\mathcal{E} = \bigoplus_{i=0}^{a} S^i\mathcal{E}'_c \otimes S^{a-i}\mathcal{E}_c.$$

The subvariety $Y_c \subset X$ is defined by the vanishing of sections in $H^0(\mathbf{P}^1, \mathcal{E}_c') \subset H^0(X, \mathcal{M})$. Hence in order that a section $s \in H^0(\mathbf{P}^1, S^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b))$ has a zero of order $\geqslant q$ on Y_c it is necessary and sufficient that s has degree $\geqslant q$ in the terms in $\mathcal{E}_c'$. In order that for this to hold for every section s it is necessary and sufficient that the degree of every line bundle occurring in the direct sum decomposition of $S^i\mathcal{E}_c' \otimes \mathcal{O}_{\mathbf{P}^1}(b)$ should be negative for every $i < q$. This is equivalent to demanding that the greatest of these degrees (occuring when $i = q - 1$) be negative; that is, that the inequality (7.4.1) should hold. The lemma is proved.

(7.5) Proof of (7.2, iii). First of all we have to compute the canonical class K_X of the scroll $X = \mathbf{P}_{\mathbf{P}^1}(\mathcal{E})$. This is easily computed from the following two exact sequences of sheaves (see for example [14]):

$$
\begin{gathered}
0 \to \Omega^1_{X/\mathbf{P}^1} \to f^*\mathcal{E} \otimes \mathcal{M}^{-1} \to \mathcal{O}_X \to 0, \\
0 \to f^*\Omega^1_{\mathbf{P}^1} \to \Omega^1_X \to \Omega^1_{X/\mathbf{P}^1} \to 0,
\end{gathered}
\tag{7.5.1}
$$

where Ω^1_Z denotes the sheaf of regular differentials on Z, and $\Omega^1_{X/\mathbf{P}^1}$ the sheaf of relative differentials. In the notation of (2.4) we have

$$\mathcal{K}_X \simeq \mathcal{M}^{-m} \otimes \mathcal{L}^{(\Sigma d_i - 2)}. \tag{7.5.2}$$

In our case $m = 3$, $X = W$ and $K_V^{-1} = \varphi^*M$, where $\varphi : V \longrightarrow W$ is the morphism defined by the anticanonical sheaf K_V^{-1}. From (7.5.2) and (7.3.1) we get

$$D \sim 4M - 2(d_1 + d_2 + d_3 - 2)L, \tag{7.5.3}$$

where $D \subset W$ is the ramification divisor of φ. Since W is nonsingular, $d_3 > 0$ (see (2.5, i)). First consider the case $d_1 = d_2 = d_3$. Then

$$W \simeq \mathbf{P}^2 \times \mathbf{P}^1, \quad \mathcal{M} \simeq p_1^*\mathcal{O}_{\mathbf{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(d_1), \quad \mathcal{L} \simeq p_2^*\mathcal{O}_{\mathbf{P}^1}(1).$$

From (7.5.3) we get

$$D \in |p_1^*\mathcal{O}_{\mathbf{P}^2}(4) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(4 - 2d_1)|.$$

Since D is an effective divisor, $4 - 2d_1 \geqslant 0$; and since $d_1 > 0$ there are only the two possibilities, $d_1 = 1$ or $d_1 = 2$. Both of these cases can occur. Indeed, if $d_1 = 1$ or 2 one sees easily that the linear system

$$|p_1^*\mathcal{O}_{\mathbf{P}^2}(4) \otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(4 - 2d_1)|$$

is without fixed components and base points, and is not composed of a pencil. Hence by Bertini's theorem almost all divisors D in this linear system are smooth and irreducible. Since $\mathcal{O}_W(D)$ is divisible by 2 in Pic W, there exists a smooth double cover $V \longrightarrow W$ ramified in D. It is clear that V is a hyperelliptic Fano 3-fold, and, as we have seen earlier, V is uniquely determined by the divisor D.

Thus the cases $d_1 = d_2 = d_3 = 1$ and $d_1 = d_2 = d_3 = 2$ of (7.2, iii) have been completely treated.

Now consider the case when $d_1 > d_3$. We use Lemma (7.4). Since the divisor D must be smooth it has to be irreducible, as has been shown. Hence the linear system $|4M - 2(d_1 + d_2 + d_3 - 2)L|$ cannot have fixed components, and in particular the surface Y_{d_2} (in the notation of (7.4)) cannot be a fixed component. Similarly, since D is smooth, it cannot contain Y_{d_3} with multiplicity $q \geqslant 2$. Substituting $a = 4$, $b = 4 - 2\Sigma_1^3 d_i$, $c = d_2$ and $q = 1$ in (7.4.1), and then the same values for a and b, with $c = d_3$ and $q = 2$, we find that the linear system $|4M + 2(2 - d_1 - d_2 - d_3)L|$, which contains a smooth divisor D, imposes the system of inequalities

$$\begin{aligned} d_1+d_3 &\leqslant 2+d_2, \\ d_1+2d_2 &\leqslant 4+d_3. \end{aligned} \tag{7.5.4}$$

It is easy to find all integer solutions of the system (7.5.4) which satisfy $d_1 \geqslant d_2 \geqslant d_3 > 0$. There are just 3:

$$(1, 1, 1), \qquad (2, 1, 1), \qquad (2, 2, 2).$$

Among these only (2, 1, 1) satisfies $d_1 > d_3$. Obviously $\mathcal{M}^4 \otimes \mathcal{L}^{-4}$ is divisible by 2 in Pic W, and it remains only to show that in the case $d_1 = 2$, $d_2 = d_3 = 1$ the linear system $|\mathcal{M}^4 \otimes \mathcal{L}^{-4}|$ contains a smooth divisor. Consider the sheaf $\mathcal{M} \otimes \mathcal{L}^{-1}$. We have

$$f_*(\mathcal{M} \otimes \mathcal{L}^{-1}) = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}.$$

Hence $\mathcal{M} \otimes \mathcal{L}^{-1}$ is generated by its global sections, and defines a birational morphism $\varphi_{\mathcal{M} \otimes \mathcal{L}^{-1}} : W \longrightarrow \mathbf{P}^3$ which contracts the surface $Y_{d_2} \simeq \mathbf{P}^1 \times \mathbf{P}^1$ onto some line in $\mathbf{P}^3$. It follows that the linear system $|\mathcal{M}^4 \otimes \mathcal{L}^4|$ is without fixed components and base points, and by Bertini's theorem almost all the divisors in it are smooth irreducible varieties. Note that $\varphi_{\mathcal{M} \otimes \mathcal{L}^{-1}}(D)$ is a surface of degree 4, having singularities at its points of intersection with the line $\varphi_{\mathcal{M} \otimes \mathcal{L}^{-1}}(Y_{d_2})$. Hence the sheaf $\varphi^*(\mathcal{M} \otimes \mathcal{L}^{-1})$ defines a morphism $V \longrightarrow \mathbf{P}^3$ having degree 2 at the general point. The quartic $\varphi_{\mathcal{M} \otimes \mathcal{L}^{-1}}(D) \subset \mathbf{P}^3$ is its ramification divisor. Theorem (7.2) is proved in full.

(7.6) COROLLARY. *If V is a hyperelliptic Fano 3-fold, then* Pic $V \simeq \mathbf{Z}$ *if and only if $W \simeq \mathbf{P}^3$ or $W \simeq V_2 \subset \mathbf{P}^4$.*

PROOF. If $W \simeq \mathbf{P}^3$ or $W \simeq V_2 \subset \mathbf{P}^4$, then by Theorem (7.2) V is a complete intersection in a weighted projective space, and hence Pic $V \simeq \mathbf{Z}$. In all the remaining cases Pic $W \simeq \mathbf{Z} \oplus \mathbf{Z}$, and since $\varphi : V \longrightarrow W$ induces an inclusion $\varphi^* :$ Pic $W \longrightarrow$ Pic V, we have Pic $V \neq \mathbf{Z}$. The proof is complete.

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