CURVES, K3 SURFACES, FANO THREEFOLDS

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(Notes by Anna Kazanova)

No Fano 3-fold in fact (too bad).

First let's talk about curves. Let X be a complex projective smooth curve, i.e. a compact Riemann surface. We will concentrate on genus $g \geq 2$. (If $g = 0$, then $X \simeq \mathbb{P}^1$ and if $g = 1$ then $X = \mathbb{C}/\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$).

- $g = 2$, there is a map $\phi : X \overset{2:1}{\to} \mathbb{P}^1$ with 6 branch points. In other words, $X = (F_6(X_0, X_1, Y)) \subset \mathbb{P}(1, 1, 3)$. After a change of coordinates the equation is of the form $Y^2 = G_6(X_0, X_1)$.
- $g = 3$. If X is not hyperelliptic then $X \hookrightarrow \mathbb{P}^2$ given by $F_4(X_0, X_1, X_2) = 0$ is a plane quartic.
- $g = 4$. Here $X \hookrightarrow \mathbb{P}^3$ is a codimension two complete intersection $F_2 = G_3 = 0$ (if not hyperelliptic).
- $g = 5$. $X \hookrightarrow \mathbb{P}^4$ is given by $(F_2 = G_2 = H_2 = 0)$.

Complete intersection property is not true in higher genus.
What is this embedding? Let X be a curve of genus g, then $\Gamma(X, \Omega_X) = \{\text{global holomorphic 1-forms on } X\}$. It is a complex vector space of dimension g, and $\Gamma(X, \Omega_X) = \langle \omega_1, \dots, \omega_g \rangle_{\mathbb{C}}$. Then we can define a map $\phi : X \to \mathbb{P}^{g-1}$ by $(\omega_1 : \cdots : \omega_g)$. It is called the canonical map, and there is the following theorem.

Theorem 0.1. *The map ϕ is embedding unless X is hyperelliptic.*

Note that $\deg(\phi(X)) = \#X \cap H$ where H is a general hyperplane in $\mathbb{P}^{g-1}$, $\deg(\phi(X)) = (2g - 2) = \{\ \# \text{ of zeroes of general } \omega \in \Gamma(\Omega)\}$.

For $g > 5$, $\phi(X) \subset \mathbb{P}^{g-1}$ is not complete intersection.

For $g = 6, 7, 8, 9$ Mukai gives an explicit description.

Eyal's note: In general, if $g \geq 2$, $\mathcal{M}_g$ = moduli space of curves of genus g. So for the case $g = 3$ we have $(\mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)) \setminus \Delta)\big/PGL(3)$ is Zariski open in $\mathcal{M}_3$. Here Δ = discriminant (locus of singular curves).

Recap: The Grassmannian $G(r, n) = \{r \text{ dimensional subspaces of } \mathbb{C}^n\} \to \mathbb{P}(\wedge^r \mathbb{C}^n)) \simeq \mathbb{P}^{\binom{n}{r}-1}$. Dimension $\dim G(r, n) = r(n - r)$.

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- $g=6$. Take Grassmanian $G(2,5)$. Then $X=X_{1^4,2}=H_1\cap\cdots\cap H_4\cap Q\subset G(2,5)\subset\mathbb{P}^9$. ($H$ = hyperplane section, Q= quadric section.) $X\subset\mathbb{P}^5=H_1\cap\cdots\cap H_4$ is the canonical embedding.
- $g=8$. Then $X=X_{1^7}=H_1\cap\cdots\cap H_7\subset G(2,6)\subset\mathbb{P}^{14}$.
- $g=7$. Then $X=X_{1^7}=H_1\cap\cdots\cap H_7\subset OGr_+(5,10)\subset\mathbb{P}^{15}$. Here OGr_+ = orthogonal Lagrangian Grassmanian. Take $V=\mathbb{C}^{2n}$, $q=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$, then OGr_+ is one of the two connected components of $\{L\subset V\,|\,\dim L=1/2\dim V, q|_L=0\}$.
- $g=9$, then $X=X_{1^5}\subset SG(3,6)\subset\mathbb{P}^{13}$ where SG = symplectic Lagrangian Grassmanian, defined as above but with the symmetric form q replaced by the alternating form $\omega=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$.

How? Why?

Recall X a variety, L a line bundle. Look at $\Gamma(X,L)=\langle s_0,\ldots,s_N\rangle_{\mathbb{C}}$. Assume that those sections do not vanish simultaneously, we get a map $\phi: X\to\mathbb{P}^N$ defined by $(s_0:\cdots:s_N)$. Equivalently, $\Gamma(X,L)\to L_p$ should be surjective for all $p\in X$.

Similarly, if E is a vector bundle on X of rank r. Assume E is globally generated: the map

$$\Gamma(X,E)\to E_p \tag{1}$$

is surjective for all $p\in X$. Then we get a map $X\hookrightarrow G(r,\Gamma(X,E)^*)$ defined by $p\mapsto E_p^*\subset\Gamma(X,E)^*$ dual to the surjection (1).

Let $L=\det E=\wedge^r E$. Assume that $\wedge^r\Gamma(E)\to\Gamma(L)$ is a surjection (so the dual map $\Gamma(L)^*\hookrightarrow\wedge^r\Gamma(E)^*$ is injective). Then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_E} & G(r,\Gamma(X,E)^*)\\ {\scriptstyle\phi_L}\downarrow & & \downarrow\text{Plucker}\\ \mathbb{P}(\Gamma(L)^*) & \xrightarrow{\text{linear}} & \mathbb{P}(\wedge^r\Gamma(E)^*)\end{array}$$

If we are lucky, it is Cartesian, i.e. $X=\mathbb{P}(\Gamma(L)^*)\cap G(r,\Gamma(E)^*)$.

Who is E?

Idea: If X curve such that $g\le 9$, then X lies on a K3 surface S. Why do we care about lying on a $K3$? Mukai classified vector bundles on a K3. (paper from Tata institute on the webpage).

Situation we will use: Let S be a K3, L a line bundle on S, C a smooth curve on a K3 given by a global section of L. Also we want to assume that $\mathrm{Pic}(S)=\mathbb{Z}L$. We consider vector bundles E over S such that 1) $\det E=L$ and 2) E is uniquely determined by its topological data, i.e. rank, $c_1(E) = L$ and $c_2(E)$. The second condition is achieved by insisting that E is stable in the sense of Mumford and choosing rank and $c_2(E)$ appropriately.

Look at $\chi(End(E)) = \sum(-1)^i \dim H^i(EndE)$. By Riemann –Roch $\chi(End(E)) = r^2\chi(\mathcal{O}_S) + (r-1)c_1(E)^2 - 2rc_2(E)$. Also $\chi(End(E)) = 1-0+1 = 2$ (because $h^0(End(E)) = 1$ by stability, $h^2 = h^0 = 1$ by Serre duality (using $End(E)$ is self dual and $\omega_S \simeq \mathcal{O}_S$), and $h^1(End(E)) = 0$ if E is rigid (no nontrivial deformations)). So $2 = 2r^2 + (r-1)L^2 - 2rc_2$. By adjunction $L^2 = 2g-2$, thus we get the equation

$$rc_2 = (r-1)(r+g).$$

Eg. if $g = 8$, then $C \to G(2,6)$, so $C \subset S \to G(2,6)$. E is a vector bundle on S of rank 2, $\det E = L$, and $C \in |L|$, $L^2 = 2g-2 = 14$. Then $2c_2 = 1(2+g) = 10$, $r = 2$, thus $c_2 = 5$. So E is a vector bundle on a K3 surface S with rank 2, $c_1(E)^2 = 14$, $c_2(E) = 5$. The K3 S containing C is not uniquely determined by C, so it is not obvious from this construction that $E|_C$ is uniquely determined by C, but one can show that it is.